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SHORT SURVEY

AN HISTORICAL PERSPECTIVE ON FRACTIONAL CALCULUS IN LINEAR VISCOELASTICITY

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Abstract

The article provides an historical survey of the early contributions on the applications of fractional calculus in linear viscoelasticty. The period under examination covers four decades, since 1930's up to 1970's, and authors are from both Western and Eastern countries. References to more recent contributions may be found in the bibliography of the author's book.

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Key Words and Phrases: linear viscoelasticity, fractional derivatives and integrals

1. The first generation of pioneers of fractional calculus in viscoelasticity

Linear viscoelasticity is certainly the field of the most extensive applications of fractional calculus, in view of its ability to model hereditary phenomena with long memory. During the twentieth century a number of authors have (implicitly or explicitly) used the fractional calculus as an empirical method of describing the properties of viscoelastic materials.

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In the first half of that century the early contributors were: Gemant [10, 11] in USA, Scott-Blair [33, 34, 35] in England, Gerasimov [14] and Rabotnov [24] in the former Soviet Union.

In 1950 Gemant [13] published a series of 16 articles entitled Frictional Phenomena in Journal of Applied Physics since 1941 to 1943, which were collected in a book of the same title. In his eighth chapter-paper dated 1942, Gemant [12], p. 220, referred to his previous articles [10, 11] for justifying the necessity of fractional differential operators to compute the shape of relaxation curves for some elasto-viscous fluids. Thus, the words *fractional* and *frictional* were coupled, presumably for the first time, by Gemant.

Scott-Blair used the fractional calculus approach to model the observations earlier made by Nutting [21, 22, 23], that the stress relaxation phenomenon could be described by fractional powers of time. He noted that time derivatives of fractional order would simultaneously model the observations of Nutting on stress relaxation and those of Gemant on frequency dependence. It is quite instructive to cite some words by Scott-Blair quoted in the 1979 article by Stiassnie [37]: *I was working on the assessing of firmness of various materials (e.g. cheese and clay by experts handling them) these systems are of course both elastic and viscous but I felt sure that judgments were made not on an addition of elastic and viscous parts but on something in between the two so I introduced fractional differentials of strain with respect to time.* Further, in the same letter Scott-Blair added: *I gave up the work eventually, mainly because I could not find a definition of a fractional differential that would satisfy the mathematicians.*

The 1948 papers by Gerasimov and Rabotnov were published in Russian, so their contents remained unknown to the majority of Western scientists up to the translation into English of the treatises by Rabotnov [25, 28]. Whereas Gerasimov explicitly used a fractional derivative to define his model of viscoelasticity (akin to the Scott-Blair model), Rabotnov preferred to use the Volterra integral operators with weakly singular kernels that could be interpreted in terms of fractional integrals and derivatives. After the appearance of the books by Rabotnov it has became common to speak about Rabotnov's theory of hereditary solid mechanics. The relation between Rabotnov's theory and the models of fractional viscoelasticity has been briefly recalled in the recent paper by Rossikhin and Shitikova [31]. According to these Russian authors, Rabotnov could express his models in terms of the operators of the fractional calculus, but he considered these operators only as some mathematical abstraction.

2. The second generation of pioneers of fractional calculus in viscoelasticity

In the late sixties, formerly Caputo [5, 6, 7], then Caputo and Mainardi [8, 9], explicitly suggested that derivatives of fractional order (nowadays usually referred to as of Caputo type) could be successfully used to model the dissipation in seismology and in metallurgy.

In this respect the present author likes to recall a correspondence carried out between himself (then as a young post-doc student) and the Russian Academician Rabotnov, related to two courses on Rheology held at CISM (International Centre for Mechanical Sciences, Udine, Italy) in 1973 and 1974, where Rabotnov [26, 27] was an invited speaker but without participating. Rabotnov recognized the relevance of the 1971 review paper by Caputo and Mainardi [9] writing in his unpublished 1974 CISM Lecture Notes: *That's way it was of great interest for me to know the paper of Caputo and Mainardi from the University of Bologna published in 1971. These authors have obtained similar results independently without knowing the corresponding Russian publications.....* Then he added: *The paper of Caputo and Mainardi contains a lot of experimental data of different authors in support of their theory. On the other hand a great number of experimental curves obtained by Postnikov and his co-workers as also by foreign authors can be found in numerous papers of Shermergor and Meshkov.* Unfortunately, the eminent Russian scientist did not cite the 1971 paper by Caputo and Mainardi (presumably for reasons independently from his willing) in the Russian and English editions of his later book [28].

Nowadays, several articles (originally in Russian) published in the years since 1966 up to 1973 by Shermergor, Meshkov and their associated researchers have been re-printed in English in Journal of Applied Mechanics and Technical Physics (English translation of Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki), see e.g. [15], [17, 18, 19, 20], [39]. available at http://www.springerlink.com/. On this respect we cite the recent review papers by Rossikhin [30], Rossikhin and Shitikova [32], where the works of the Russian scientists on fractional viscoelasticity are examined.

The beginning of the modern applications of fractional calculus in linear viscoelasticity is generally attributed to the 1979 PhD thesis by Bagley [1] under supervision of Prof. Torvik, followed by a number of relevant papers, e.g. [2, 3, 4] and [38]. However, for the sake of completeness, one would recall also the 1970 PhD thesis of Rossikhin under the supervision of Prof. Meshkov, see [29], and the 1971 PhD thesis of the author under the supervision of Prof. Caputo, summarized in [9].

To date, applications of fractional calculus in linear and nonlinear viscoelasticity have been considered by a great and increasing number of authors to whom we have tried to refer in the huge (but not exhaustive) bibliography of our recent book [16]. Furthermore, there the reader can find an accessible introduction to fractional calculus and related special functions.

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