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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 15, NUMBER 4 (2012)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS USING ITERATIVE METHOD

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Abstract

The purpose of this paper is to obtain solutions for both linear and nonlinear initial value problems (IVPs) for fractional transport equations and fractional diffusion-wave equations using the iterative method.

MSC 2010: Primary 26A33; Secondary 31B10

Key Words and Phrases: Caputo fractional derivative, Riemann-Liouville fractional integral, fractional transport equations, fractional diffusion-wave equations, iterative method

1. Introduction

Many phenomena in the physical, chemical and biological sciences as well as in technologies are governed by differential equations. In recent years, fractional differential equations have attracted researchers' interest due to their applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electroanalytical chemistry, fractional multipoles, etc. (see [10], [23], [31]). Consider the general fractional partial differential equation

$$D_t^{\alpha} u(x,t) = \sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + A(u(x,t))$$

$$m-1 < \alpha \le m,\ 3 < \delta_j \le 4,\ 1 < \beta_j \le 2,\ 0 < \gamma_j \le 1, \quad m \in N,$$

pp. 684–699 , DOI: 10.2478/s13540-012-0046-8

where $0 \leq t \leq T, x = (x_1, x_2, ...x_n) \in R^n, a_j, b_j, c_j$ are nonnegative real constants and $A(u(x,t))$ is linear or nonlinear in $u(x,t)$ and $u_x(x,t)$. Note that $D_t^{\mu} u(x,t)$ is the μ^{th}-order Caputo partial fractional derivative of function $u(x,t)$ with respect to "t". These equations appear in many interesting physical processes such as transportation, diffusion of heat, propagation of waves, deflection and vibration of plates and membranes. We study the following fractional transport equation

$$D_t^{\alpha} u(x,t) = \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + A(u(x,t)), \;\; 0 < \alpha \leq 1, \;\; 0 < \gamma_j \leq 1, \quad (1.1)$$

the fractional diffusion-wave equation of fourth order

$$D_t^{\alpha} u(x,t) = \sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + A(u(x,t)) \quad (1.2)$$

and fractional diffusion-wave equation of second order

$$D_t^{\alpha} u(x,t) = \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + A(u(x,t)). \quad (1.3)$$

Note that for wave propagation in beams and modeling formation of grooves on a flat surface because of grain require fourth-order space derivative terms in their formulations [29], [35]. Both second and fourth order diffusion-wave equations appear in diffusion of heat, deflection and vibration of plates and membranes and propagation of waves. Also such equations appear in relaxation phenomena in complex viscoelastic material [15], propagation of mechanical waves in viscoelastic media [24], [25], [26], non-Markovian diffusion process with memory [27], electromagnetic acoustic and mechanical responses [28]. Roman and Alemany [33] investigated a continuous time random walks on fractals by studying diffusion-wave equations. Transport phenomena is governed by fractional transport equations.

These types of equations are solved by various methods such as Adomain decomposition method [1], [2], [11],[13],[22], homotopy perturbation method [18], [19], [20], [30], variational iteration method [17], [21], an iterative method [4], [6], [8], [9], [12], finite element method [14], finite difference method [16], finite sine transform method [3], method of images and Fourier transform [27], as well as Green's function method [34].

We need to recall the definition of the Caputo partial fractional derivative that uses the Riemann-Liouville fractional integral.

Definition **1.1**. The (left sided) Riemann-Liouville fractional integral of order μ ($\mu > 0$) of a function $u(x,t) \in C_{\alpha}, \; \alpha \geq -1$ is denoted by $I_t^{\mu} u(x,t)$ and defined as (see e.g. [23], [31])

$$I_t^{\mu}u(x,t)=\frac{1}{\Gamma(\mu)}\int_0^t(t-\tau)^{\mu-1}u(x,\tau)d\tau,\ \ t>0.$$

Definition **1.2**. The (left sided) Caputo partial fractional derivative of a function $u(x,t)\in C_l^m$, w.r.t. "t", denoted by $D_t^{\mu}u(x,t)$, is defined as (see [23], [31])

$$D_t^{\mu}u(x,t)=\begin{cases}\frac{\partial^m}{\partial t^m}u(x,t), & \mu=m,\ m\in N,\\ I_t^{m-\mu}\frac{\partial^m}{\partial t^m}u(x,t), & m-1<\mu<m,\ m\in N,\end{cases}$$

where $I_t^{\mu}u(x,t)$ is Riemann-Liouville fractional integral of order μ, $\ \mu>0$.

Note that

$$I_t^{\mu}D_t^{\mu}u(x,t)=u(x,t)-\sum_{k=0}^{m-1}\frac{\partial^k u}{\partial t^k}u(x,0)\frac{t^k}{k!},\ \ m-1<\mu\leq m, m\in N,$$

and

$$I_t^{\mu}\,t^{\nu}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}\,t^{\nu+\mu}\,.$$

Now we discuss the iterative method developed by Daftardar-Gejji and Jafari in [8]. It is also called new iterative method. Consider the functional equation

$$u(x,t)=f(x,t)+L(u(x,t))+N(u(x,t)), \tag{1.4}$$

where $f(x,t)$ is a known function. Note that L and N are linear and nonlinear operators from a Banach space $B\to B$, respectively. Suppose a solution $u(x,t)$ of the equation (1.4) has the series form

$$u(x,t)=\sum u_i(x,t)=u_0+u_1+u_2+... \tag{1.5}$$

The above series solution converges absolutely and uniformly to a unique solution of equation (1.4) (for details, see Daftardar-Gejji and Jafari [8]). Since L is linear,

$$L\bigg(\sum_{i=0}^{\infty}u_i(x,t)\bigg)=L(u_0(x,t))+L(u_1(x,t))+L(u_2(x,t))+... \tag{1.6}$$

Decompose nonlinear operator N as in [8]:

$$N\left(\sum_{i=0}^{\infty}u_i\right)=N(u_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i-1}u_j\right)\right\}. \tag{1.7}$$

From equations (1.5) - (1.7), the equation (1.4) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f(x,t) + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\}, \tag{1.8}$$

$$\begin{aligned} u_0 + u_1 + \ldots = {} & f + L(u_0) + L(u_1) + L(u_2) + \ldots + N(u_0) + \\ & [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \\ & [N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)] + \ldots \end{aligned}$$

We define the iterations

$$\begin{aligned} u_0 &= f \\ u_1 &= L(u_0) + N(u_0) \\ u_2 &= L(u_1) + [N(u_0+u_1) - N(u_0)] \\ u_3 &= L(u_2) + [N(u_0+u_1+u_2) - N(u_0+u_1)] \\ u_{m+1} &= L(u_m) + N(u_0+\ldots+u_m) - N(u_0+\ldots+u_{m-1}),\ m = 0,1,2,\ldots \end{aligned}$$

The rest of the paper is organized as follows: Iterative solution of IVP for general fractional partial differential equation is obtained using the iterative method in Section **2**. Some illustrative examples for fractional transport equations, fractional diffusion equations of second and fourth order and fractional wave equations of second and fourth order are discussed in the last Section **3**.

2. Fractional initial value problems

Consider the general fractional partial differential equations

$$D_t^{\alpha} u(x,t) = \sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t) + \sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t) + \sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t) + A(u(x,t)); \tag{2.1}$$

$$m-1 < \alpha \leq m,\ 3 < \delta_j \leq 4,\ 1 < \beta_j \leq 2,\ 0 < \gamma_j \leq 1,\ m \in N$$

with the initial conditions

$$\frac{\partial^k}{\partial t^k} u(x,0) = h_k(x),\ \ 0 \leq k \leq m-1,\ m = 1,2. \tag{2.2}$$

Applying I_t^{α} to (2.1) on both sides,

$$\begin{aligned} I_t^{\alpha} D_t^{\alpha} u(x,t) = {} & I_t^{\alpha}\left(\sum_{j=1}^{n} a_j D_{x_j}^{\delta_j} u(x,t)\right) + I_t^{\alpha}\left(\sum_{j=1}^{n} b_j D_{x_j}^{\beta_j} u(x,t)\right) \\ & + I_t^{\alpha}\left(\sum_{j=1}^{n} c_j D_{x_j}^{\gamma_j} u(x,t)\right) + I_t^{\alpha}\Big(A(u(x,t))\Big) \end{aligned}$$

and using initial conditions (2.2), we get

$$u(x,t)=\sum_{k=0}^{m-1}h_k(x)\frac{t^k}{k!}+I_t^{\alpha}\left(\sum_{j=1}^{n}a_jD_{x_j}^{\delta_j}u(x,t)\right)+I_t^{\alpha}\left(\sum_{j=1}^{n}b_jD_{x_j}^{\beta_j}u(x,t)\right)$$
$$+\ I_t^{\alpha}\left(\sum_{j=1}^{n}c_jD_{x_j}^{\gamma_j}u(x,t)\right)+I_t^{\alpha}\Big(A(u(x,t))\Big). \quad (2.3)$$

Equation (2.3) has the form of equation (1.4) with

$$f(x,t)=\sum_{k=0}^{m-1}h_k(x)\,\frac{t^k}{k!}\,,$$
$$L(u(x,t))=I_t^{\alpha}\left(\sum_{j=1}^{n}a_jD_{x_j}^{\delta_j}u(x,t)\right)+I_t^{\alpha}\left(\sum_{j=1}^{n}b_jD_{x_j}^{\beta_j}u(x,t)\right)$$
$$+\ I_t^{\alpha}\left(\sum_{j=1}^{n}c_jD_{x_j}^{\gamma_j}u(x,t)\right),$$
$$N(u(x,t))=I_t^{\alpha}\Big(A(u(x,t))\Big),$$

and can be solved by the iterative method.

3. Illustrative examples

In this section, we discuss some illustrative examples for time fractional transport equation, time fractional diffusion equations of second and fourth order, time fractional wave equations of second and fourth order.

I. Time fractional transport equation:

Example **3.1**. Consider time fractional transport equation

$$D_t^{\alpha}u(x,t)+\nabla u(x,t)=0,\ t>0,\ x=(x_1,x_2,...,x_n)\in R^n \quad (3.1)$$

with the initial condition

$$u(x,0)=\exp\left(\sum_{k=1}^{n}x_k\right). \quad (3.2)$$

The initial value problem (3.1)-(3.2) is a special case of IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t)=e^{(x_1+x_2+...+x_n)}-I_t^{\alpha}\Big(D_{x_1}+D_{x_2}+...+D_{x_n}\Big)u(x,t).$$

Applying the iterative method, we get

$$u_0 = e^{(x_1+x_2+...+x_n)},\ u_1 = L(u_0) = e^{(x_1+x_2+...+x_n)} \frac{-nt^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = L(u_1) = e^{(x_1+x_2+...+x_n)} \frac{(-nt^\alpha)^2}{\Gamma(2\alpha+1)} \quad \text{and so on.}$$

In general, we have

$$u_i = e^{(x_1+x_2+...+x_n)} \frac{(-nt^\alpha)^i}{\Gamma(i\alpha+1)},\quad i = 0, 1, 2, ...$$

The solution of initial value problem (3.1)-(3.2) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = \exp\left(\sum_{k=1}^{n} x_k\right) \sum_{i=0}^{\infty} \frac{(-nt^\alpha)^i}{\Gamma(i\alpha+1)} = \exp\left(\sum_{k=1}^{n} x_k\right) E_\alpha(-nt^\alpha).$$

Example **3.2**. Consider nonlinear, nonhomogeneous time fractional transport equation

$$D_t^\alpha u(x,t) + (u_x(x,t))^2 = 2,\quad t > 0,\ x \in R \tag{3.3}$$

with the initial condition

$$u(x,0) = x \tag{3.4}$$

The initial value problem (3.3)-(3.4) is a special case of IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = x + I_t^\alpha(2) + I_t^\alpha\Big(-(D_x u(x,t))^2\Big).$$

Applying the iterative method, we get

$$u_0 = x,\ u_1 = L(u_0) + N(u_0) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = L(u_1) + N(u_0+u_1) - N(u_0) = 0,\ u_3 = 0, ...$$

The solution of the initial value problem (3.3)-(3.4) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = x + \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

II. Second order time fractional diffusion equation:

Example **3.3**. Consider the time fractional diffusion equation

$$D_t^\alpha u(x,t) = D_{x_1}^2 u(x,t) + D_{x_2}^2 u(x,t) - u(x,t), \tag{3.5}$$

$$0 < \alpha \le 1,\ \beta = 2,\ t > 0,\ x = (x_1, x_2) \in R^2,$$

with the initial condition

$$u(x,0) = \sin x_1 \sin x_2 + \cos x_1 \cos x_2. \tag{3.6}$$

The Cauchy problem (3.5)-(3.6) is a special case of IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = \sin x_1 \sin x_2 + \cos x_1 \cos x_2 + I_t^{\alpha}(D_{x_1}^2 + D_{x_2}^2 - 1)u(x,t).$$

Applying the iterative method, we get

$$u_0 = \sin x_1 \sin x_2 + \cos x_1 \cos x_2, \quad u_1 = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\frac{-3t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2 = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\frac{(-3t^{\alpha})^2}{\Gamma(2\alpha+1)} \quad \text{and so on.}$$

In general, we have

$$u_i = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\,\frac{(-3t^{\alpha})^i}{\Gamma(i\alpha+1)}, \quad i = 0, 1, 2, ...$$

The solution of Cauchy problem (3.5)-(3.6) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\, E_{\alpha}(-3t^{\alpha})\,.$$

Example **3.4**. Consider the nonlinear time fractional diffusion equation

$$D_t^{\alpha}u(x,t) - D_x^2 u(x,t) = u^2(x,t) - (u_x^2(x,t)), \tag{3.7}$$
$$0 < \alpha \le 1,\ \beta = 2,\ t > 0,\ x \in R,$$

with the initial condition

$$u(x,0) = e^x. \tag{3.8}$$

The Cauchy problem (3.7)-(3.8) is a special case of IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = e^x + I_t^{\alpha}\Big(D_x^2 u(x,t)\Big) + I_t^{\alpha}\Big(u^2(x,t) - (u_x^2(x,t))\Big).$$

Applying the iterative method, we get

$$u_0 = e^x,\ u_1 = e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ u_2 = e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \quad \text{and so on.}$$

In general, we have

$$u_i = e^x\,\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\,,\ \ i = 0, 1, 2, ...$$

The solution of the Cauchy problem (3.7)-(3.8) is

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)=e^x\sum_{i=0}^{\infty}\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}=e^xE_\alpha(t^\alpha)\,.$$

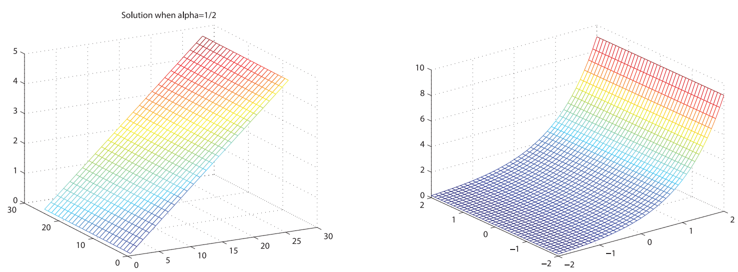

Figure 1: Graphical representation of the solution of Example 3.2 and Example 3.4, respectively.

EXAMPLE **3.5**. Consider the following time fractional nonlinear Schrödinger equation

$$iD_t^\alpha u(x,t)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u=0, \tag{3.9}$$

for $0<\alpha\le 1, t>0$, with the initial condition

$$u(x,0)=e^{ix}. \tag{3.10}$$

The initial value problem (3.9)-(3.10) is solved by the Adomain decomposition method by Rida et al. in [32] which is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t)=e^{ix}-I_t^\alpha\left(|u|^2u\right)-I_t^\alpha\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right).$$

Applying the iterative method, we get

$$u_0=e^{ix},\ u_1=\frac{i}{2}e^{ix}\frac{t^\alpha}{\Gamma(\alpha+1)},\ u_2=\frac{-1}{4}e^{ix}\frac{(t^\alpha)^2}{\Gamma(2\alpha+1)}\ \text{ and so on.}$$

In general, we have

$$u_n=\frac{1}{2^n}e^{ix}\frac{(it^\alpha)^n}{\Gamma(n\alpha+1)},\ \ n=0,1,2,...$$

The solution of initial value problem (3.9)-(3.10) is

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=e^{ix}\,E_\alpha\left(\frac{it^\alpha}{2}\right). \tag{3.11}$$

REMARK **3.1**. Observe that the solution of the initial value problem (3.9)-(3.10) obtained by the Adomain decomposition method [32] is exactly the same as the solution (3.11) obtained above.

III. Fourth order time fractional diffusion equation:

EXAMPLE **3.6**. Consider the fourth order time fractional parabolic equation

$$D_t^\alpha u(x,t) = D_x^4 u(x,t) + D_x^2 u(x,t) + u(x,t),\ 0 < \alpha \le 1, x \in R, t > 0, \quad (3.12)$$

with the initial condition

$$u(x,0) = \cosh x\,. \quad (3.13)$$

The Cauchy problem (3.12)-(3.13) is a special case of the IVP (2.1)-(2.2). It is equivalent to integral equation

$$u(x,t) = \cosh x + I_t^\alpha \left(D_x^4 u(x,t) + D_x^2 u(x,t) + u(x,t)\right).$$

Applying the iterative method, we get

$$u_0 = \cosh x,\ \ u_1 = \cosh x \frac{3t^\alpha}{\Gamma(\alpha+1)},\ \ u_2 = \cosh x \frac{(3t^\alpha)^2}{\Gamma(2\alpha+1)} \ \text{ and so on.}$$

In general, we have

$$u_i = \cosh x \frac{(3t^\alpha)^i}{\Gamma(i\alpha+1)},\ \ i = 0,1,2,...$$

The solution of the IVP (3.12)-(3.13) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i = \cosh x \sum_{i=0}^{\infty} \frac{(3t^\alpha)^i}{\Gamma(i\alpha+1)} = \cosh x\ E_\alpha(3t^\alpha).$$

EXAMPLE **3.7**. Consider the fourth order time fractional KdV equation. It is nonlinear equation of parabolic type

$$D_t^\alpha u(x,t) + D_x^4 u(x,t) - u(x,t) D_x^2 u(x,t) + u u_x(x,t) = 0,\ x \in R, t > 0, \quad (3.14)$$

with the initial condition

$$u(x,0) = e^x. \quad (3.15)$$

The Cauchy problem (3.14)-(3.15) is a special case of the IVP (2.1)-(2.2). It is equivalent to integral equation

$$u(x,t) = e^x + I_t^\alpha\Big(- D_x^4 u(x,t)\Big) + I_t^\alpha\Big(u(x,t) D_x^2 u(x,t) - u(x,t) u_x(x,t)\Big).$$

Applying the iterative method, we get

$$u_0 = e^x,\ u_1 = -e^x \frac{t^\alpha}{\Gamma(\alpha+1)},\ u_2 = e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \ \text{ and so on.}$$

In general, we have

$$u_i = e^x \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)}, \quad i = 0, 1, 2, ...$$

The solution of the Cauchy problem (3.14)-(3.15) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = e^x \sum_{i=0}^{\infty} \frac{(-t^\alpha)^i}{\Gamma(i\alpha+1)} = e^x\, E_\alpha(-t^\alpha).$$

IV. Second order time fractional wave equation:

EXAMPLE **3.8**. Consider the 2-space dimension time fractional wave equation

$$D_t^\alpha u(x,t) = D_{x_1}^2 u(x,t) + D_{x_2}^2 u(x,t) - 2u(x,t), \tag{3.16}$$
$$1 < \alpha \le 2,\ \beta = 2,\ t > 0,\ \ x = (x_1, x_2) \in R^2,$$

with the initial conditions

$$u(x,0) = \sin x_1 \sin x_2 + \cos x_1 \cos x_2, \tag{3.17}$$
$$u_t(x,0) = 0. \tag{3.18}$$

The initial value problem (3.16)-(3.18) is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = \sin x_1 \sin x_2 + \cos x_1 \cos x_2 + I_t^\alpha D_{x_1}^2 u(x,t) + I_t^\alpha D_{x_2}^2 u - 2I_t^\alpha u(x,t).$$

Applying the iterative method, we get

$$u_0 = \sin x_1 \sin x_2 + \cos x_1 \cos x_2,\ \ u_1 = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\frac{-4t^\alpha}{\Gamma(\alpha+1)}$$

$$u_2 = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\frac{(-4t^\alpha)^2}{\Gamma(2\alpha+1)} \quad \text{and so on.}$$

In general, we have

$$u_i = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\frac{(-4t^\alpha)^i}{\Gamma(i\alpha+1)}, \quad i = 0, 1, 2, ...$$

The solution of the initial value problem (3.16)-(3.18) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = (\sin x_1 \sin x_2 + \cos x_1 \cos x_2)\ E_\alpha(-4t^\alpha).$$

EXAMPLE **3.9**. Consider the 2-space dimension nonlinear time fractional wave equation

$$D_t^\alpha u(x,t) - u(x,t)\Delta u(x,t) + 2(u(x,t))^2 - 4u(x,t) = 0, \tag{3.19}$$

for $1 < \alpha \leq 2,\ \beta = 2,\ t > 0,\ \ x = (x_1, x_2) \in R^2,$ with the initial conditions

$$u(x,0) = e^{x_1+x_2}, \tag{3.20}$$

$$u_t(x,0) = 0. \tag{3.21}$$

The initial value problem (3.28)-(3.30) is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = e^{x_1+x_2} + I_t^{\alpha}\Big(4u\Big) + I_t^{\alpha}\Big(uD_{x_1}^2 u + uD_{x_2}^2 u - 2u^2\Big).$$

Applying the iterative method, we get

$$u_0 = e^{x_1+x_2},\ u_1 = e^{x_1+x_2}\frac{4t^{\alpha}}{\Gamma(\alpha+1)},\ u_2 = e^{x_1+x_2}\frac{(4t^{\alpha})^2}{\Gamma(2\alpha+1)} \text{ and so on.}$$

In general, we have

$$u_i = e^{x_1+x_2}\,\frac{(4t^{\alpha})^i}{\Gamma(i\alpha+1)},\ \ i = 0, 1, 2, ...$$

The solution of the initial value problem (3.28)-(3.30) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = e^{x_1+x_2}\ \sum_{i=0}^{\infty}\frac{(4t^{\alpha})^i}{\Gamma(i\alpha+1)} = e^{x_1+x_2}\ E_{\alpha}(4t^{\alpha}).$$

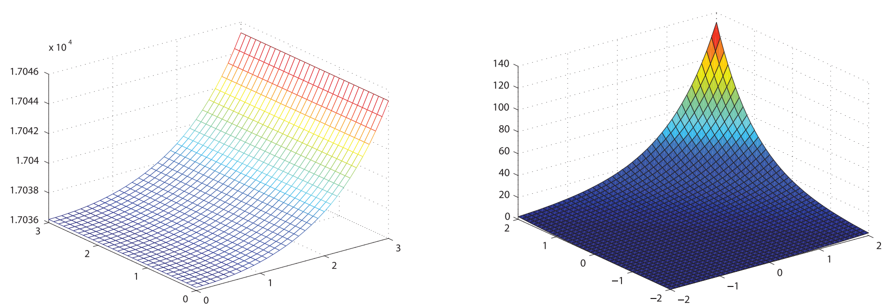

Figure 2: Graphical representation of solution of Example 3.7 and Example 3.9, respectively.

V. Fourth order time fractional wave equation:

Example **3.10**. Consider the fourth order time fractional hyperbolic equation

$$D_t^{\alpha}u(x,t) = 3D_x^4 u(x,t) + 3D_x^2 u(x,t) + 3u(x,t), x \in R,\ t > 0, \tag{3.22}$$

with the initial conditions

$$u(x,0) = \sinh x, \tag{3.23}$$

$$u_t(x,0) = 0. \tag{3.24}$$

The initial value problem (3.22)-(3.24) is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = \sinh x + 3I_t^{\alpha}\left(D_x^4 u(x,t) + D_x^2 u(x,t) + u(x,t)\right).$$

Applying the iterative method, we get

$$u_0 = \sinh x,\ u_1 = \sinh x \frac{9t^{\alpha}}{\Gamma(\alpha+1)},\ u_2 = \sinh x \frac{(9t^{\alpha})^2}{\Gamma(2\alpha+1)} \ \text{ and so on.}$$

In general, we have

$$u_i = \sinh x\, \frac{(9t^{\alpha})^i}{\Gamma(i\alpha+1)},\ i = 0,1,2,...$$

The solution of the IVP (3.22)-(3.24) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i = \sinh x \sum_{i=0}^{\infty} \frac{(9t^{\alpha})^i}{\Gamma(i\alpha+1)} = \sinh x\ E_{\alpha}(9t^{\alpha}).$$

EXAMPLE **3.11**. Consider the fourth order time fractional nonlinear hyperbolic equation

$$D_t^{\alpha} u(x,t) = 4D_x^4 u(x,t) - u(x,t) D_x u(x,t) - (D_x u(x,t))^2,\ x \in R,\ t > 0, \tag{3.25}$$

with the initial conditions

$$u(x,0) = e^{-x}, \tag{3.26}$$

$$u_t(x,0) = 0. \tag{3.27}$$

The initial value problem (3.25)-(3.27) is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = e^{-x} + 4I_t^{\alpha}\left(D_x^4 u(x,t)\right) + I_t^{\alpha}\left(-u(x,t)D_x u(x,t) - (D_x u(x,t))^2\right).$$

Applying the iterative method, we get

$$u_0 = e^{-x},\ u_1 = \frac{e^{-x} 4t^{\alpha}}{\Gamma(\alpha+1)},\ u_2 = e^{-x}\frac{(4t^{\alpha})^2}{\Gamma(2\alpha+1)} \ \text{ and so on.}$$

In general, we have

$$u_i = e^{-x}\, \frac{(4t^{\alpha})^i}{\Gamma(i\alpha+1)},\ \ i = 0,1,2,...$$

The solution of IVP (3.22)-(3.24) is

$$u(x,t) = \sum_{i=0}^{\infty} u_i = e^{-x} \sum_{i=0}^{\infty} \frac{(4t^{\alpha})^i}{\Gamma(i\alpha+1)} = e^{-x}\ E_{\alpha}(4t^{\alpha}).$$

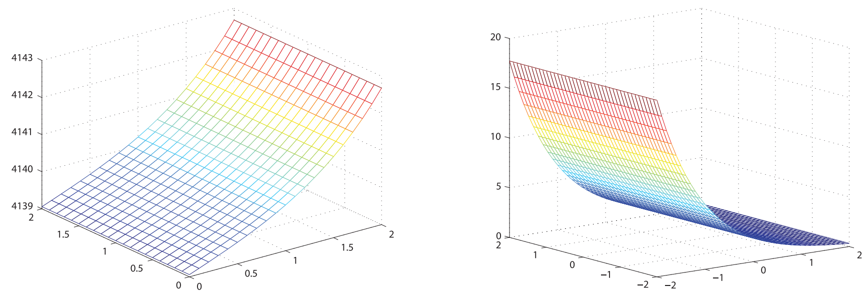

Figure 3: Graphical representation of solution of Example 3.10 and Example 3.11, respectively.

Example **3.12**. Consider the 2-dimension fourth-order fractional wave equation

$$D_t^{\alpha} u(x,t) = -2\Big(D_{x_1}^4 u(x,t) + D_{x_1}^4 u(x,t) \Big), \tag{3.28}$$

for $\alpha \in (1,2)$, $t > 0$, $x = (x_1, x_2) \in R^2$, with the initial conditions

$$u(x,0) = \cos x_1 \cos x_2, \tag{3.29}$$

$$u_t(x,0) = 0. \tag{3.30}$$

The initial value problem (3.28)-(3.30) is a special case of the IVP (2.1)-(2.2). It is equivalent to the integral equation

$$u(x,t) = \cos x_1 \cos x_2 - 2I_t^{\alpha}\Big(D_{x_1}^4 u + D_{x_2}^4 u \Big).$$

Applying the iterative method, we get

$$u_0 = \cos x_1 \cos x_2, \;\; u_1 = -4 \cos x_1 \cos x_2 \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \;\; u_2 = \cos x_1 \cos x_2 \frac{(-4t^{\alpha})^2}{\Gamma(2\alpha+1)},$$

and so on. In general, we have

$$u_n = \cos x_1 \cos x_2 \frac{(-4t^{\alpha})^n}{\Gamma(n\alpha + 1)}, \quad n = 0, 1, 2, \ldots$$

The solution of initial value problem (3.28)-(3.30) is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \cos x_1 \cos x_2 \sum_{n=0}^{\infty} \frac{(-4t^{\alpha})^n}{\Gamma(n\alpha+1)} = \cos x_1 \cos x_2 \, E_{\alpha}(-4t^{\alpha}).$$

Remark **3.2**. It is interesting to note that, solution of initial value problem (3.28)-(3.30) obtained above is same as that of solution obtained by Jafari et al., using the Adomain decomposition method in [22].

MATLAB has been used for graphical computations of the solutions in the present paper.

Acknowledgement

The first author is grateful to the University Grants Commission, New Delhi, India for financial support under the grant No.F.4-1/2006(BSR)/11-78/2008(BSR).

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Received: March 30, 2012
Revised: August 3, 2012

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Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **15**, No 4 (2012), pp. 684–699;
DOI: 10.2478/s13540-012-0046-8