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RESEARCH PAPER

NUMERICAL STUDIES FOR THE VARIABLE-ORDER NONLINEAR FRACTIONAL WAVE EQUATION

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Abstract

In this paper, the explicit finite difference method (FDM) is used to study the variable order nonlinear fractional wave equation. The fractional derivative is described in the Riesz sense. Special attention is given to study the stability analysis and the convergence of the proposed method. Numerical test examples are presented to show the efficiency of the proposed numerical scheme.

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Key Words and Phrases: variable order fractional calculus, nonlinear fractional wave equation, explicit finite difference method, convergence and stability analysis

1. Introduction

In the recent years, the fractional calculus as theory allowing operations of integration and differentiation of arbitrary (real or complex value, including fractional) order has enjoyed much attention. It enable us to consider the order of the fractional integrals and derivatives as a function of the time or some other space variables. Samko and Ross [17], first proposed the concept of variable order operator and investigated the properties of variable order integration and differentiation operators of Riemann-Liouville type. In this respect, the order of the operator is allowed to vary either

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as a function of the independent variable of integration or differentiation, or as a function of some other variables. The concept of a variable order operator is a product of much more recent developments and is yet less widely known. The order of the derivative/integral is allowed to vary over the domain of interest, as in the modeling systems with evolving dynamics. Such systems include deformation of viscoelastic materials ([6], [16]), and the mechanics of a variable viscoelastic oscillators [5]. Similar to the state of affairs in fractional calculus, multiple definitions of a variable order derivative have been suggested ([7], [12]), each preferred for different reasons [17]. In this paper we use the generalized Riesz variable-order fractional derivative ([1]-[4], [18], [19], [22]-[25]).

The main aim of this paper is to introduce the numerical solution of the variable-order nonlinear fractional wave equation using the finite difference method. This method is used to solve many of similar problems to the proposed problem, for example, the variable-order nonlinear cable equation [1], the variable-order nonlinear Stokes' first problem for a heated generalized second grade fluid [2], the variable-order anomalous sub-diffusion equation [4], the space-time Riesz-Caputo fractional advection-diffusion equation [19], the fractional partial differential equations with Riesz space fractional derivatives [24], the variable-order fractional advection-diffusion with a nonlinear source term [25], and others ([8]-[11], [20], [21]).

Definition **1.1**. The generalized Riesz variable-order fractional derivative is defined by the following form

$$ {}_x\bar{R}^{\alpha(x,t)}u(x,t) = -(-\frac{d^2}{dx^2})^{\frac{\alpha(x,t)}{2}}u(x,t) $$

$$ = \frac{-\sec(\alpha(x,t)\pi/2)}{\Gamma(2-\alpha(x,t))}\left(\rho\frac{d^2}{d\theta^2}\int\limits_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}+\sigma\frac{d^2}{d\theta^2}\int\limits_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x}, $$

where $1<\alpha(x,t)\leq 2$, $X_a\leq x\leq X_b$, $0<t\leq T$, $\rho\geq 0$, $\sigma\geq 0$, and $\rho+\sigma=1$.

Let us note the following variable-structure differential form [13]

$$ D_x{}^{\alpha(x,t)}u(x,t) = \lim_{h\to 0} h^{-\alpha(x,t)}\sum_{j=0}^{[\frac{x-X_a}{h}]}(-1)^j\begin{pmatrix}\alpha(x,t)\\ j\end{pmatrix}u(x-jh,t), $$

where $(-1)^j\begin{pmatrix}\alpha(x,t)\\ j\end{pmatrix} = \dfrac{\Gamma(j-\alpha(x,t))}{\Gamma(-\alpha(x,t))\Gamma(j+1)}$.

Now using the relationship between the Grünwald-Letnikov and Riemann-Liouville fractional derivatives [15], we obtain

$$\frac{1}{\Gamma(2-\alpha(x,t))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x} =$$
$$\lim_{h\to 0} h^{-\alpha(x,t)}\sum_{j=0}^{[\frac{x-X_a}{h}]}(-1)^j\begin{pmatrix}\alpha(x,t)\\ j\end{pmatrix}u(x-jh,t),$$

and

$$\frac{1}{\Gamma(2-\alpha(x,t))}\left(\frac{d^2}{d\theta^2}\int_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x} =$$
$$\lim_{h\to 0} h^{-\alpha(x,t)}\sum_{j=0}^{[\frac{X_b-x}{h}]}(-1)^j\begin{pmatrix}\alpha(x,t)\\ j\end{pmatrix}u(x+jh,t).$$

For more details about the variable-order fractional calculus definitions and its properties, see ([14], [15]).

In this paper we consider the variable order nonlinear fractional wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = B(x,t)_x\bar{R}^{\alpha(x,t)}u(x,t) + f(u,x,t), \tag{1.1}$$

with initial conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, $X_a \le x \le X_b$, and boundary conditions $u(X_a,t) = u_a(t)$, $u(X_b,t) = u_b(t)$, $0 \le t \le T$, where $1 < \alpha(x,t) \le 2$, and $B(x,t) > 0$.

We assume that the function $f(u,x,t)$ is nonlinear and satisfies the Lipschitz condition, i.e.

$$|f(u_1,x,t) - f(u_2,x,t)| \le L|u_1 - u_2|, \quad \text{for any constant } L > 0.$$

And for simplicity, let us assume $A(x,t) = -\sec(\alpha(x,t)\pi/2)B(x,t)$.

Then by using the generalized of the variable-order Riesz fractional derivative, we can rewrite Eq. (1.1) in the following form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{A(x,t)}{\Gamma(2-\alpha(x,t))}\left(\rho\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right. \tag{1.2}$$
$$\left. + \sigma\frac{d^2}{d\theta^2}\int_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha(x,t)-1}}\right)_{\theta=x} + f(u,x,t).$$

2. FDM for the variable-order nonlinear fractional wave equation

In this section, we use the FDM to find the numerical solution of Eq. (1.1). So, pick integers m, $n > 0$, and define the space step size $h = \frac{X_b - X_a}{m}$, and the time step size $\tau = \frac{T}{n}$. Define $x_i = X_a + ih$, for $i = 0, 1, ..., m$, $t_j = j\tau$, for $j = 0, 1, ..., n$, and $u_i^j = u(x_i, t_j)$, $f_i^j = f(u_i^j, x_i, t_j)$, $A_i^j = A(x_i, t_j)$.

From the shifted Grünwald formula ([14], [15]) for the $\alpha-$fractional derivative approximation we have

$$\frac{1}{\Gamma(2-\alpha(x_i,t_j))}\left(\frac{d^2}{d\theta^2}\int_{X_a}^{\theta}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}}\right)_{\theta=x_i} \approx h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{\alpha_{i+1}^j}u(x_k,t_j), \tag{2.3}$$

$$\frac{1}{\Gamma(2-\alpha(x_{i+1},t_j))}\left(\frac{d^2}{d\theta^2}\int_{\theta}^{X_b}\frac{u(\xi,t)d\xi}{(\theta-\xi)^{\alpha-1}}\right)_{\theta=x_{i+1}} \approx h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{\alpha_{i-1}^j}u(x_k,t_j), \tag{2.4}$$

where $\omega_k^{\alpha_i^j} = (-1)^k \begin{pmatrix} \alpha_i^j \\ k \end{pmatrix}$, $\alpha_i^j = \alpha(x_i, t_j)$.

The coefficients $\omega_k^{\alpha(x,t)} = (-1)^k \begin{pmatrix} \alpha(x,t) \\ k \end{pmatrix}$, for $1 < \alpha(x,t) \leq 2$, and $k = 0, 1, ...,$ satisfy the following conditions:

(i) $\omega_0^{\alpha(x,t)} = 1$;

(ii) $\omega_1^{\alpha(x,t)} < 0$ and $\omega_k^{\alpha(x,t)} > 0$ for all $k > 1$;

(iii) $\omega_1^{\alpha(x,t)} + \omega_2^{\alpha(x,t)} + ... = -\omega_0^{\alpha(x,t)}$.

Substituting from (2.3) and (2.4) into (1.2), and using the second center finite difference to approximate $\frac{\partial^2 u(x,t)}{\partial t^2}$ yields

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\tau^2} \tag{2.5}$$

$$= \rho A_i^j h^{-\alpha_{i+1}^j}\sum_{k=0}^{i+1}\omega_{i+1-k}^{\alpha_{i+1}^j}u_k^j + \sigma {A_i}^j h^{-\alpha_{i-1}^j}\sum_{k=i-1}^{m}\omega_{k-(i-1)}^{\alpha_{i-1}^j}u_k^j + f_i^j,$$

for $i = 1, 2, ..., m-1$, $\quad j = 0, 1, ..., n-1$.

Putting $r_i^j = \frac{\tau^2}{h^{\alpha_i^j}}$, we then obtain the following explicit difference approximation

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} + \rho A_i^j r_{i+1}^j \sum_{k=0}^{i+1}\omega_{i+1-k}^{\alpha_{i+1}^j}u_k^j + \sigma A_i^j r_{i-1}^j \sum_{k=i-1}^{m}\omega_{k-(i-1)}^{\alpha_{i-1}^j}u_k^j + \tau^2 f_i^j, \tag{2.6}$$

i.e.

$$\begin{aligned} u_i^{j+1} &= 2u_i^j - u_i^{j-1} + \rho A_i^j r_{i+1}^j (\omega_{i+1}^{\alpha_{i+1}^j} u_0^j + ... + \omega_1^{\alpha_{i+1}^j} u_i^j + \omega_0^{\alpha_{i+1}^j} u_{i+1}^j) \\ &\quad + \sigma A_i{}^j r_{i-1}^j (\omega_0^{\alpha_{i-1}^j} u_{i-1}^j + \omega_1^{\alpha_{i-1}^j} u_i^j + ... + \omega_{m-(i-1)}^{\alpha_{i-1}^j} u_m^j) + \tau^2 f_i^j, \end{aligned} \tag{2.7}$$

where $i = 1, 2, ..., m-1, \quad j = 0, 1, ..., n-1$.

Let $U^{j+1} = (u_{m-1}^{j+1}, u_{m-2}^{j+1}, ..., u_1^{j+1})^T, \quad U^{j-1} = (u_{m-1}^{j-1}, u_{m-2}^{j-1}, ..., u_1^{j-1})^T$,

$$B^j = \begin{pmatrix} \rho A_{m-1}^j r_m^j \omega_0^{\alpha_m^j} u_m^j + \rho A_{m-1}^j r_m^j \omega_m^{\alpha_m^j} u_0^j \\ \rho A_{m-2}^j r_{m-1}^j \omega_{m-1}^{\alpha_{m-1}^j} u_0^j \\ \vdots \\ \rho A_2^j r_3^j \omega_3^{\alpha_3^j} u_0^j \\ \rho A_1^j r_2^j \omega_2^{\alpha_2^j} u_0^j \end{pmatrix} + \begin{pmatrix} \sigma A_{m-1}^j r_{m-2}^j \omega_2^{\alpha_{m-2}^j} u_m^j \\ \sigma A_{m-2}^j r_{m-3}^j \omega_3^{\alpha_{m-3}^j} u_m^j \\ \vdots \\ \sigma A_2^j r_1^j \omega_{m-1}^{\alpha_1^j} u_m^j \\ \sigma A_1^j r_0^j \omega_m^{\alpha_0^j} u_m^j + \sigma A_1^j r_0^j \omega_0^{\alpha_0^j} u_0^j \end{pmatrix},$$

$F^j = (\tau^2 f(u_{m-1}^j, x_{m-1}, t_j), ..., \tau^2 f(u_1^j, x_1, t_j))^T$,

$$P^j = \rho \begin{pmatrix} A_{m-1}^j r_m^j \omega_1^{\alpha_m^j} & A_{m-1}^j r_m^j \omega_2^{\alpha_m^j} & A_{m-1}^j r_m^j \omega_3^{\alpha_m^j} & \dots \\ A_{m-2}^j r_{m-1}^j \omega_0^{\alpha_{m-1}^j} & A_{m-2}^j r_{m-1}^j \omega_1^{\alpha_{m-1}^j} & A_{m-2}^j r_{m-1}^j \omega_2^{\alpha_{m-1}^j} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & A_1^j r_2^j \omega_0^{\alpha_2^j} & A_1^j r_2^j \omega_1^{\alpha_2^j} \end{pmatrix} + \sigma \begin{pmatrix} A_{m-1}^j r_{m-2}^j \omega_1^{\alpha_{m-2}^j} & A_{m-1}^j r_{m-2}^j \omega_0^{\alpha_{m-2}^j} & 0 & \dots \\ A_{m-2}^j r_{m-3}^j \omega_2^{\alpha_{m-3}^j} & A_{m-2}^j r_{m-3}^j \omega_1^{\alpha_{m-3}^j} & A_{m-2}^j r_{m-3}^j \omega_0^{\alpha_{m-3}^j} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & A_1^j r_0^j \omega_3^{\alpha_0^j} & A_1^j r_0^j \omega_2^{\alpha_0^j} & A_1^j r_0^j \omega_1^{\alpha_0^j} \end{pmatrix} + 2\, I_{m-1}.$$

Then the system (2.7) can be written in the matrix form

$$U^{j+1} = P^j U^j - U^{j-1} + B^j + F^j. \tag{2.8}$$

3. The convergence and the stability analysis

To study the convergence of the system (2.8), we denote by $\bar{U}^{j+1} = (\bar{u}^{j+1}_{m-1}, \bar{u}^{j+1}_{m-2}, ..., \bar{u}^{j+1}_{1})^T$ the exact solution of this system at time level $t = t_{j+1}$. Define the error vector at time $t = t_{j+1}$ as

$$e^{j+1} = \bar{U}^{j+1} - U^{j+1}, \quad e^0 = \mathbf{0}.$$

We assume that $\bar{A} \geq A(x,t)$, and $\bar{\alpha} \geq \alpha(x,t)$.

THEOREM **3.1**. *The system (2.8) is convergent and* $|u^j_i - \bar{u}^j_i| = O(\tau^2 + h)$ *for any* i, j, *when*

$$\tau^2 \leq \frac{2h^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho+\sigma)}.$$

P r o o f. From (2.8), we have $e^{j+1} = P^j e^j - e^{j-1} + F^j_e + \tau^2 O(\tau^2 + h)$, where

$$F^j_e = (\tau^2 f(u^j_{m-1}, x_{m-1}, t_j) - \tau^2 f(\bar{u}^j_{m-1}, x_{m-1}, t_j), ..., \tau^2 f(u^j_1, x_1, t_j)$$
$$-\tau^2 f(\bar{u}^j_1, x_1, t_j))^T \leq (\tau^2 L^j_{m-1} e^j_{m-1}, ..., \tau^2 L^j_1 e^j_1)^T = \Delta F^j e^j,$$

where $\Delta F^j = \mathrm{diag}(\tau^2 L^j_{m-1}, ..., \tau^2 L^j_1)$. Noting that $|L^j_i| \leq L$, for any i, j, we have

$$\| P^j \|_\infty = \max_{1\leq i\leq m-1}(|\sigma A^j_i r^j_{i-1}\omega_{m-(i-1)-1}^{\alpha^j_{i-1}}| + ... + |\sigma A^j_i r^j_{i-1}\omega_3^{\alpha^j_{i-1}}| + |\rho A^j_i r^j_{i+1}\omega_0^{\alpha^j_{i+1}}$$
$$+\sigma A^j_i r^j_{i-1}\omega_2^{\alpha^j_{i-1}}| + |\rho A^j_i r^j_{i+1}\omega_1^{\alpha^j_{i+1}} + \sigma A^j_i r^j_{i-1}\omega_1^{\alpha^j_{i-1}} + 2| + |\rho A^j_i r^j_{i+1}\omega_2^{\alpha^j_{i+1}}$$
$$+\sigma A^j_i r^j_{i-1}\omega_0^{\alpha^j_{i-1}}| + |\rho A^j_i r^j_{i+1}\omega_3^{\alpha^j_{i+1}}| + ... + |\rho A^j_i r^j_{i+1}\omega_i^{\alpha^j_{i+1}}|).$$

For given h, we choose τ to satisfy $\rho A^j_i r^j_{i+1}\omega_1^{\alpha^j_{i+1}} + \sigma A^j_i r^j_{i-1}\omega_1^{\alpha^j_{i-1}} + 2 \geq 0$, i.e.,

$$\tau^2 \leq -2/((\rho A^j_i \omega_1^{\alpha^j_{i+1}})/h^{\alpha^j_{i+1}} + (\sigma A^j_i \omega_1^{\alpha^j_{i-1}})/h^{\alpha^j_{i-1}}), \quad \text{for any } i,\ j.$$

From

$$\frac{2h^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho+\sigma)} \leq \frac{-2}{((\rho A^j_i \omega_1^{\alpha^j_{i+1}})/h^{\alpha^j_{i+1}} + (\sigma A^j_i \omega_1^{\alpha^j_{i-1}})/h^{\alpha^j_{i-1}})},$$

we have that the condition $\tau^2 \leq -2/((\rho A^j_i \omega_1^{\alpha^j_{i+1}})/h^{\alpha^j_{i+1}} + (\sigma A^j_i \omega_1^{\alpha^j_{i-1}})/h^{\alpha^j_{i-1}})$ holds when $\tau^2 \leq \frac{2h^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho+\sigma)}$. Under this condition, we get

$$\| P^j \|_\infty = 2 + \rho A^i_k r^j_{k+1} \sum_{i=0}^{i+1} \omega_i^{\alpha^j_{k+1}} + \sigma A^j_k r^j_{k-1} \sum_{i=0}^{m-(k-1)} \omega_i^{\alpha^j_{k-1}}$$
$$= 2 + \rho A^j_k r^j_{k+1} \sum_{i=k+2}^{\infty} \omega_i^{\alpha^j_{k+1}} + \sigma A^j_k r^j_{k-1} \sum_{i=m-(k-2)}^{\infty} \omega_i^{\alpha^j_{k-1}} \leq 2.$$

This is true, by using the property (iii) of $\omega_i^{\alpha_i^j}$ and since $\sum_{i=k+2}^{\infty}\omega_i^{\alpha_{k+1}^j}$, $\sum_{i=m-(k-2)}^{\infty}\omega_i^{\alpha_{k-1}^j} \leq 0$. So, when $\tau^2 \leq \frac{2h^{\bar{\alpha}}}{A\bar{\alpha}(\rho+\sigma)}$, and for any constant $c > 0$ independent of h, τ, we obtain

$$\begin{aligned}\| e^{j+1} \|_\infty &\leq \| P^{j+1} + \Delta F^{j+1} \|_\infty \| e^j \|_\infty + \| e^{j-1} \|_\infty + c(\tau^2 + h)\\ &\leq (2 + \tau^2 L) \| e^j \|_\infty + \| e^{j-1} \|_\infty + c(\tau^2 + h).\end{aligned}$$

Let $s = (2 + \tau^2 L)$, $q = c(\tau^2 + h)$, then

$$\begin{aligned}&\| e^{j+1} \|_\infty \leq s \| e^j \|_\infty + \| e^{j-1} \|_\infty + q\\ &\leq s(s \| e^{j-1} \|_\infty + \| e^{j-2} \|_\infty + q) + \| e^{j-1} \|_\infty + q\\ &= (s^2 + 1) \| e^{j-1} \|_\infty + s \| e^{j-2} \|_\infty + q(s + 1)\\ &\leq (s^2 + 1)(s \| e^{j-2} \|_\infty + \| e^{j-3} \|_\infty + q) + s \| e^{j-2} \|_\infty + q(s + 1)\\ &= (s^3 + 2s) \| e^{j-2} \|_\infty + (s^2 + 1) \| e^{j-3} \|_\infty + q(s^2 + s + 2)\\ &\leq ... \leq c_k \| e^{j-k} \|_m + c_{k-1} \| e^{j-k-1} \|_m + q(c_{N-2} + c_{N-3} + c_{N-4} + c_{N-5}),\end{aligned}$$

such that $c_k = sc_{k-1} + c_{k-2}$, $k = 2, 3, ..., N$; $c_1 = s$, $c_0 = 1$. Then,

$$\begin{aligned}&\| e^{j+1} \|_\infty \leq c_{N-1} \| e^1 \|_m + c_{N-2} \| e^0 \|_m + q(c_{N-2} + c_{N-3} + c_{N-4} + c_{N-5})\\ &\leq c_{N-1} \| e^0 \|_m + c_{N-2} \| e^0 \|_m + q(c_{N-2} + c_{N-3} + c_{N-4} + c_{N-5})\\ &\leq c_{N-1} \| e^0 \|_m + c_{N-2} \| e^0 \|_m + c(\tau^2 + h)(c_{N-2} + c_{N-3} + c_{N-4} + c_{N-5})\\ &\leq c \| e^0 \|_m + O(\tau^2 + h),\end{aligned}$$

since, from the second initial condition we have $\| e^1 \|_m \leq \| e^0 \|_m$, and $c = c_{N-1} + c_{N-2}$. □

The stability of the system (2.8), is given in the following theorem.

THEOREM **3.2**. *Let W^{j+1} and U^{j+1} be two different numerical solutions of (2.8) with initial values given by W^0 and U^0, respectively. If $\tau^2 \leq \frac{2h^{\bar{\alpha}}}{A\bar{\alpha}(\rho+\sigma)}$, then the system (2.8) is stable and*

$$|W^{j+1} - U^{j+1}| \leq c|W^0 - U^0|,$$

for any j, and for any constant $c > 0$, independent of h, τ.

P r o o f. Let $W^{j+1} - U^{j+1} = \varepsilon^{j+1}$. From (2.8) we have $\varepsilon^{j+1} = P^j\varepsilon^j - \varepsilon^{j-1} + F_\varepsilon^j$, where

$$\begin{aligned}F_\varepsilon^j &= (\tau^2 f(u_{m-1}^j, x_{m-1}, t_j) - \tau^2 f(w_{m-1}^j, x_{m-1}, t_j), ..., \tau^2 f(u_1^j, x_1, t_j)\\ &-\tau^2 f(w_1^j, x_1, t_j))^T \leq (\tau^2 L_{m-1}^j \varepsilon_{m-1}^j, ..., \tau^2 L_1^j \varepsilon_1^j)^T = \Delta F^j \varepsilon^j.\end{aligned}$$

From Theorem **3.1**, we have $\| P^j + \Delta F^j \|_\infty \leq (2+\tau^2 L)$ when $\tau^2 \leq \frac{2h^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho+\sigma)}$, then

$$\begin{aligned}\| \varepsilon^{j+1} \|_\infty &\leq \| P^j + \Delta F^j \|_\infty \| \varepsilon^j \|_\infty + \| \varepsilon^{j-1} \|_\infty \\ &\leq (2+\tau^2 L) \| \varepsilon^j \|_\infty + \| \varepsilon^{j-1} \|_\infty .\end{aligned}$$

Let $s = (2+\tau^2 L)$, then

$$\begin{aligned}\| \varepsilon^{j+1} \|_\infty &\leq s \| \varepsilon^j \|_\infty + \| \varepsilon^{j-1} \|_\infty \\ &\leq s(s \| \varepsilon^{j-1} \|_\infty + \| \varepsilon^{j-2} \|_\infty) + \| \varepsilon^{j-1} \|_\infty \\ &= (s^2+1) \| \varepsilon^{j-1} \|_\infty + s \| \varepsilon^{j-2} \|_\infty \\ &\leq (s^2+1)(s \| \varepsilon^{j-2} \|_\infty + \| \varepsilon^{j-3} \|_\infty) + s \| \varepsilon^{j-2} \|_\infty \\ &= (s^3+2s) \| \varepsilon^{j-2} \|_\infty + (s^2+1) \| \varepsilon^{j-3} \|_\infty \\ &\leq ... \leq c_k \| \varepsilon^{j-k} \|_m + c_{k-1} \| \varepsilon^{j-k-1} \|_m,\end{aligned}$$

such that $c_k = sc_{k-1} + c_{k-2},\ k = 2,3,...,N;\ c_1 = s,\ c_0 = 1$. Then,

$$\| \varepsilon^{j+1} \|_\infty \leq c_{N-1} \| \varepsilon^1 \|_m + c_{N-2} \| \varepsilon^0 \|_m$$

$$\leq c_{N-1} \| \varepsilon^0 \|_m + c_{N-2} \| \varepsilon^0 \|_m \leq c \| \varepsilon^0 \|_m,$$

since, from the second initial condition we have $\| \varepsilon^1 \|_m \leq \| \varepsilon^0 \|_m$, and $c = c_{N-1} + c_{N-2}$.

Therefore, for the system (2.8), if there is a perturbation in U^0, the small change would not cause large error in the numerical solution. Then the system (2.8) is stable when $\tau^2 \leq \dfrac{2h^{\bar{\alpha}}}{\bar{A}\bar{\alpha}(\rho+\sigma)}$. □

4. Numerical examples

In this section, we study the numerical solution of the linear and non-linear variable-order wave equation using the proposed scheme (2.7). This study is to achieve the numerical and theoretical analysis introduced in the previous sections of the paper.

Example **4.1**. We consider the following variable-order linear fractional wave equation [11]

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -0.5\cos(\alpha(x,t)\pi/2)_x \bar{R}^{\alpha(x,t)} u(x,t) + f(u,x,t), \quad 0 \leq x \leq 8,$$

with $\alpha(x,t) = 1.5 + 0.5e^{-(xt)^2-1}$, where

$$f(u,x,t) = \frac{2u}{t^2+1} - (t^2+1)\left(\frac{16x^{2-\alpha(x,t)}}{\Gamma(3-\alpha(x,t))} + \frac{6x^{3-\alpha(x,t)}}{\Gamma(4-\alpha(x,t))}\right),$$

with zero boundary conditions and the initial conditions

$$u(x,0) = \phi(x) = x^2(8-x), \quad u_t(x,0) = \psi(x) = 0.$$

The exact solution of this problem is given by

$$u(x,t) = x^2(8-x)(t^2+1).$$

We study this problem with respect to $\rho = 1$, and $\sigma = 0$.

A comparison between the exact and the numerical solutions using the proposed method with $h = 0.0416$ and $\tau = 0.0052$ at $t = 0.0156$ is presented in Table 4.1.

x	Numerical solution	Exact solution	Absolute error
0	0	0	0
0.4167	01.31662254	01.31669332	0.00007078
0.8333	04.97712099	04.97739015	0.00026916
1.2500	10.54744515	10.54801575	0.00057060
1.6667	17.59354364	17.59449541	0.00095176
2.0833	25.68136517	25.68275439	0.00138923
2.5000	34.37685847	34.37871799	0.00185953
2.9167	43.24597227	43.24831149	0.00233922
3.3333	51.85465539	51.85746015	0.00280480
3.7500	59.76885642	59.77208925	0.00323283
4.1667	66.55452425	66.55812407	0.00359982
4.5833	71.77760759	71.78148989	0.00388231
5.0000	75.00405517	75.00811199	0.00405682
5.4167	75.79981573	75.80391566	0.00409989
5.8333	73.73083812	73.73482615	0.00398803
6.2500	68.36307093	68.36676875	0.00369779
6.6667	59.26246312	59.26566874	0.00320568
7.0833	45.99496310	45.99745139	0.00248824
7.5000	28.12652002	28.12804199	0.00152199
7.9167	05.22308236	05.22336582	0.00028346

Table 4.1: A comparison between the exact and the numerical solutions using the proposed method with $h = 0.0416$ and $\tau = 0.0052$ at $t = 0.0156$.

The numerical comparison for Example 4.1 between the exact solution and the numerical solution obtained using the explicit FDM with $h = 0.04$ and $\tau = 0.005$ and the final time $t = 0.5$, is given in Fig. 4.1. Also, the behavior of the numerical solution of the proposed problem with different values of the space and the time with $h = 0.04$ and $\tau = 0.005$ is given in the surface Fig. 4.2. From these figures, we can observe that the system exhibits wave behaviors that the solution continuously depends on the space and time variables.

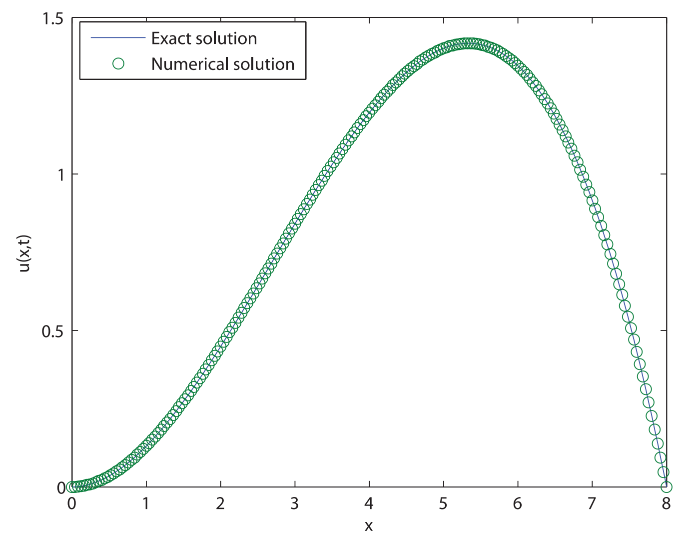

Fig. 4.1: Numerical comparison for Example 4.1 between the exact solution and the numerical solution obtained using the explicit FDM with $h = 0.04$, $\tau = 0.005$ and $t = 0.5$.

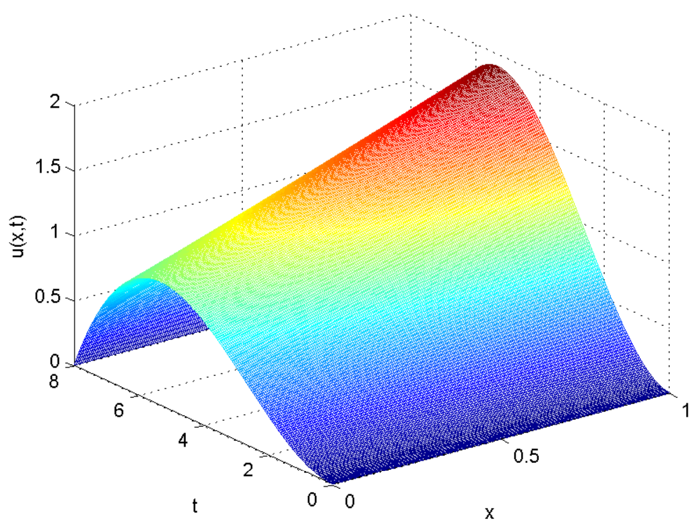

Fig. 4.2: The behavior of the numerical solution of Example 4.1 using the proposed FDM with $h = 0.04$ and $\tau = 0.005$.

When we take h and τ such that $\tau^2 > \frac{2h^{\bar{\alpha}}}{A\bar{\alpha}(\rho+\sigma)}$, the stability condition is not satisfied, the results showed in Fig. 4.3. From this figure, we note that the numerical solution is unstable. Where we take $h = 0.042$ and $\tau = 0.25$ and $t = 5.75$.

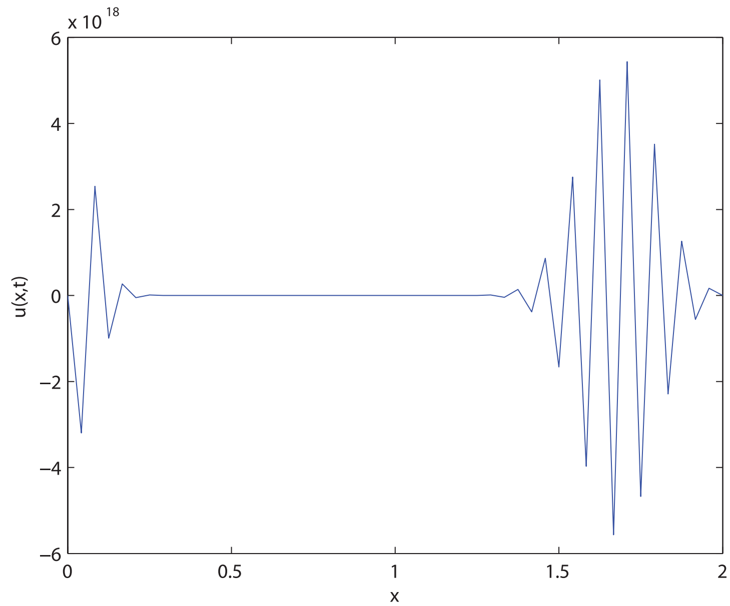

Fig. 4.3: The behavior of the numerical solution of Example 4.1 using the proposed FDM with $h = 0.04$ and $\tau = 0.25$ and $t = 5.75$.

Example **4.2**. In this example we consider the following variable-order fractional nonlinear wave equation [11]

$$\frac{\partial^2 u(x,t)}{\partial t^2} =_x \bar{R}^{\alpha(x,t)} u(x,t) + f(u,x,t), \quad -a \leq x \leq a,$$

with $\alpha(x,t) = 1.5 + 0.25\cos(x)\sin(2t)$, where $f(u,x,t) = -u^2 + 6xt(x^2 - t^2) + x^6t^6$, the boundary conditions are $u(-a,t) = -u(a,t) = a^3t^3$, and the initial conditions are $u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = 0$.

The exact solution of this problem at $\alpha = 2$ is given by $u(x,t) = x^3t^3$. We study this problem with respect to $\rho = 1$, and $\sigma = 0$.

The numerical solution using the explicit FDM of Example 4.2 with different values of time t, ($t = 0.05$, 0.5, 1.0) at $h = 0.08$ and $\tau = 0.0052$ is given in Fig. 4.4. Also, the behavior of the exact solution at $\alpha = 2$ (top), numerical solution (bottom) of Example 4.2 using the proposed FDM with $h = 0.08$ and $\tau = 0.0052$ is given in Fig. 4.5.

From these figures, we can see that the numerical solution takes the same behavior of the exact solution.

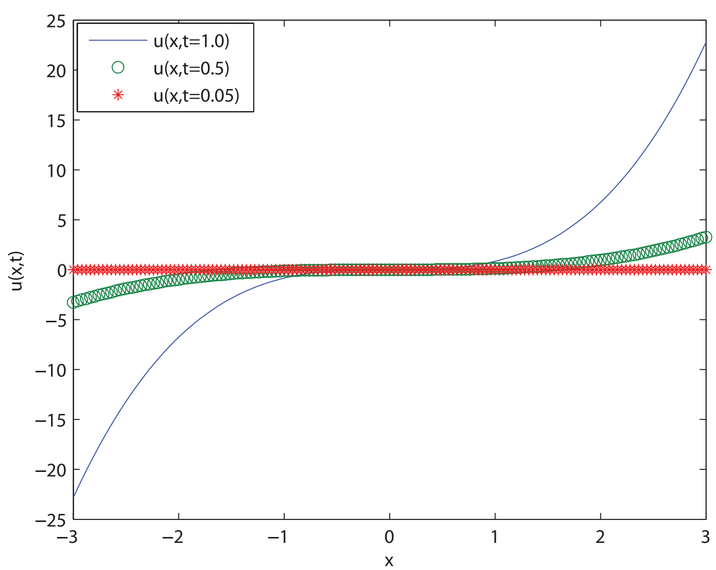

Fig. 4.4: The behavior of the numerical solution of Example 4.2 using the proposed FDM with different values of t, at $h = 0.0416$ and $\tau = 0.0052$ at different values of t.

5. Conclusions

In this article a variable-order nonlinear fractional wave equation is investigated. An explicit FDM for this equation has been described and demonstrated. The convergence and the stability of the obtained numerical scheme are established. Numerical examples have been presented to demonstrate the effectiveness of the method. The numerical results are in excellent agreement with the exact solution and coincide with the theoretical analysis introduced in this paper. Summarizing our results, we can say that the FDM in its general form gives reasonable calculations, easy to use and can be applied to solve other kinds of variable-order nonlinear fractional differential equations. All results are obtained by using *MATLAB 8*.

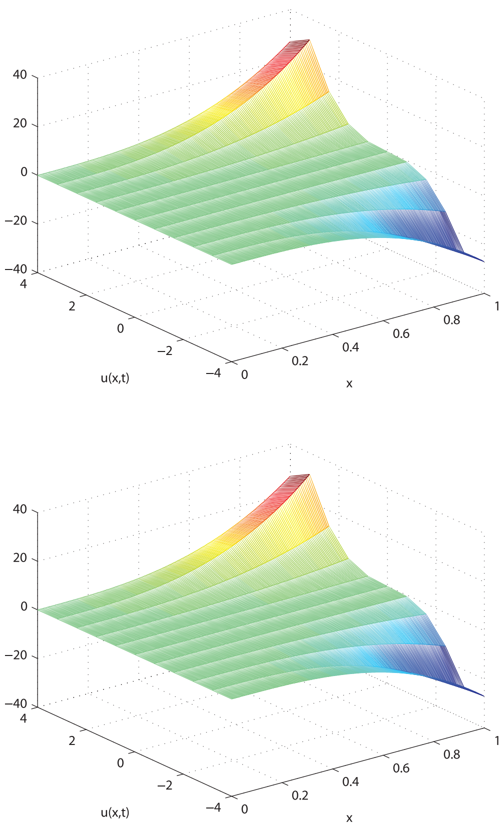

Fig. 4.5: The behavior of the exact solution at $\alpha = 2$ (top) numerical solution (bottom) of Example 4.2 using the proposed FDM with $h = 0.08$ and $\tau = 0.0052$.

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