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RESEARCH PAPER

FRACTIONAL CALCULUS ON TIME SCALES WITH TAYLOR'S THEOREM

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Abstract

We present a definition of the Riemann-Liouville fractional calculus for arbitrary time scales through the use of time scales power functions, unifying a number of theories including continuum, discrete and fractional calculus. Basic properties of the theory are introduced including integrability conditions and index laws. Special emphasis is given to extending Taylor's theorem to incorporate our theory.

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1. Introduction

Fractional calculus (FC), the theory of integrals and derivatives of non-integer order, is a field of research with a history dating back to Abel, Riemann and Liouville (see [31] for a historical summary). Indeed, the most famous and extensively studied formulation,

$$I^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, f(t)\, dt, \tag{1.1}$$

is called the Riemann-Liouville fractional integral in their honour. The corresponding fractional derivative is obtained by composition of fractional

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integral by a integer order derivative. The definitions of fractional integrals and derivatives are not unique, and there has been a large number of alternative approaches to FC, see for example [17], [27] and [32]. Each definition has its own advantages for applications. For details, see [29], [31], [34], [37], [39], [40] and [43].

In parallel to these developments, there have been also attempts to define analogues of FC opearors with respect to discrete calculus and quantum calculus. In regards to the former, there are four major streams in the literature, as [19], [26], [30] and [22]. It is the last which is of relevance to this paper, where the fractional summation of order $\alpha > 0$ is defined as

$$\underset{a}{\overset{t}{S}}{}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=a}^{t} \frac{\Gamma(t-k+\alpha)}{\Gamma(t-k)} f(k), \tag{1.2}$$

and fractional differences are obtained by composing (1.2) with the backwards difference. Therefore the expression can be thought as an analogue to (1.1).

The study of q-analogues of fractional calculus was initiated by [2] and [1], both using the q-analogue of Cauchy's formula obtained in [3] to obtain an analogue to the Riemann-Liouville fractional calculus. As with the continuum and discrete cases, alternative formulations exist, for example [35]. For the presentation here, we shall consider the definition given in [1], where $I_q^0 f(x) := f(x)$ and for $\alpha > 0$,

$$I_q^{\alpha} f(x) := \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - tq)_q^{\alpha-1} f(t)\, d(t;q). \tag{1.3}$$

As before, the q-fractional derivative is obtained by composition with an integer q-derivative. For an introduction to quantum calculus, see [28].

This paper concerns an unification of the three cases through time scales calculus. It does so by considering an axiomatic approach to the integral kernels from (1.1), (1.2) and (1.3). This is similar to [5] attempting to unify (1.1) with (1.2), but the axioms we posit are stronger and some of the results are not in agreement. Independently, [18] have managed to unify (1.2) with (1.3) via the (q,h)-time scale. A third approach, via the Laplace transform, is considered in [9], which is useful when one has the necessary techniques to invert the transform. The notation and concepts of time scales calculus are described only briefly. Readers who are unfamiliar with the theory are encouraged to consult the introductory text book [14].

The paper is structured as follows. In Section **2**, we give a brief summary of the time scales calculus, with emphasis on the Lebesgue ∇-integral. In Section **3**, we present our axioms for a time scales analogue of the power functions and illustrate how the previous example satisfy them. Then we define the Riemann-Liouville fractional calculus using the power functions as integral kernels. Important properties such as integrability and an index law are derived. In Section **4**, we prove a very general fractional Taylor type theorem.

2. Review of time scale calculus

A time scale $\mathbb{T}$ is a closed non-empty subset of $\mathbb{R}$. Examples include $\mathbb{R}$, $\mathbb{Z}$ and the *quantum time scale* $\overline{q^{\mathbb{Z}}} := \{0, 1, q, q^{-1}, q^2, \ldots\}$ for a fixed q, $0 < q < 1$. We define the *right and left jump operators* as $\sigma(t) := \inf\{s \in \mathbb{T} \,|\, s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} \,|\, s < t\}$. We define the *right and left graininess* as $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, respectively. By convention, $\inf\{\emptyset\} = \sup\{\mathbb{T}\}$ and $\sup\{\emptyset\} = \inf\{\mathbb{T}\}$ for these operators. We say that a point is *left-dense* if $\rho(t) = t$ and *left-scattered* if $\rho(t) \neq t$. Similar are the definitions for *right-dense* and *right-scattered*. A point which is both left and right scattered is *discrete*. If $\inf \mathbb{T} = a_0 > -\infty$, we define $\mathbb{T}_\kappa := \mathbb{T} \setminus a_0$, otherwise $\mathbb{T}_\kappa := \mathbb{T}$.

The three cases illustrated in the introduction correspond to three different time scales: continuum calculus corresponds to taking $\mathbb{R}$ as one's time scale and discrete calculus corresponds to taking $\mathbb{Z}$ as one's time scale. Similarly, the quantum calculus corresponds to the quantum time scale. The left jump operator is $\rho(t) = t$, $\rho(t) = t - 1$ and $\rho(t) = qt$ respectively.

For a time scale $\mathbb{T}$, and $f : \mathbb{T} \to \mathbb{R}$, we define $f^\rho(t) = f(\rho(t))$ and $f^\sigma(t) = f(\sigma(t))$. Further f called *ld-continuous* if it is continuous at every left-dense point, and $f(t+)$ exists for every right-dense $t \in \mathbb{T}$.

Definition **2.1**. A subset $I \subset \mathbb{T}$ is a *time scale interval* if it is of the form $I = (a, b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$. For a time scale interval I, a function $f : I \to \mathbb{R}$ is said to be *left dense absolutely continuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$ whenever a disjoint finite collection of sub-time scale intervals $(a_k, b_k] \cap \mathbb{T} \subset I$, for $1 \leq k \leq n$ satisfies $\sum_{k=1}^n |b_k - a_k| < \delta$.

Definition **2.2**. For a function $f : \mathbb{T} \to \mathbb{R}$ and a point $t \in \mathbb{T}_\kappa$, we define $f^\nabla(t)$ to be the number such that for all $\epsilon > 0$ there exists a neighborhood $U \subset \mathbb{T}$ of t which satisfies $\left|f^\rho(t) - f(\tau) - f^\nabla(t)[\tau - \rho(t)]\right| \leq \epsilon\,|\tau - \rho(t)|$ for all $\tau \in U$.

If $f^\nabla(t)$ is defined for all $t \in \mathbb{T}_\kappa$, then the function obtained is called the *nabla derivative* of f and is defined on $\mathbb{T}_\kappa$. The space of ld-continuous functions is denoted $C_{\mathrm{ld}}(\mathbb{T})$ and the set of ld-absolutely continuous functions $AC_{\mathrm{ld}}(\mathbb{T})$. For $n \geq 1$, the space of function with ld-continuous f^∇, $f^{\nabla^2}, \ldots,$

f^{∇^n} is denoted $C^n_{\mathrm{ld}}(\mathbb{T})$. The space of functions with ld-continuous $f^{\nabla}, \ldots,$ $f^{\nabla^{n-1}}$ and ld-absolutely continuous f^{∇^n} is denoted $AC^n_{\mathrm{ld}}(\mathbb{T})$.

2.1. Summary of the Lebesgue ∇-integral

Integration on time scales is traditionally interpreted as the inverse of differentiation. A time scale analogue of the Riemann integration was introduced in [23], and developed further in [24] and [10]. This paper is mainly concerned with unification of the continuum case with the discrete and quantum cases. Just as the existing theory for the Riemann-Liouville fractional calculus is through the Lebesgue integral, we shall use the theory of the Lebesgue ∇-integral (for details, see [11], [12], [13], [15] and [16]).

To review the Lebesgue theory of time scales, we fix an interval $[a, b]$ with $b > a$. Without loss of generality, we will assume $a, b \in \mathbb{T}$, and denote $\widetilde{\mathbb{T}} = \mathbb{T} \cap [a, b]$. Let $\mathcal{F}_2$ be the semi-ring of subsets of form $(a', b'] \cap \mathbb{T}$ for $a' \leq b'$ and $a', b' \in \widetilde{\mathbb{T}}$. We apply the convention $(t, t] = \emptyset$. We define the set function $m_2 : \mathcal{F}_2 \to [0, \infty]$ by $(a', b'] \cap \mathbb{T} \mapsto b' - a'$. The measure obtained by the Carathéodory extension, μ_∇, is called the *Lebesgue ∇-measure*. The σ-algebra of all m_2^*-measurable subsets of $\mathbb{T}$ is denoted by $\mathcal{M}(m_2^*)$. The Lebesgue ∇-integral is the integral of $f : \mathbb{T} \to \mathbb{R}$ obtained by this measure. We write

$$\int_E f(\tau)\, \nabla\tau,$$

for $E \in \mathcal{M}(m_2^*)$ and $a \notin E$. If $E = (a', b'] \cap \mathbb{T}$, then we shall write

$$\int_{a'}^{b'} f(\tau)\, \nabla\tau,$$

instead. Given E, we denote I_E as the index set of the left-scattered points of E (that is for $i \in I_E$, $t_i \in E$ is left-scattered). This set is known to be countable. We also define $\|f\| = \int_a^b |f(\tau)|\, \nabla\tau$, and L_1 as set of all $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ with $\|f\| < \infty$. The connection between the Lebesgue integral on $\mathbb{R}$ and the Lebesgue ∇-integral is expressed in the following formula, which will shall use later.

THEOREM **2.1**. *Let $E \subset \widetilde{\mathbb{T}}$ be a ∇-measurable set such that $a \notin E$. If $f : E \to \mathbb{R}$ is a ∇-measurable function, then*

$$\int_E f(s)\, \nabla s = \int_E f(s)\, ds + \sum_{i \in I_E} f(t_i)\, \nu(t_i). \tag{2.1}$$

Finally, we shall give the fundamental theorem of calculus for the Lebesgue ∇-integral.

THEOREM **2.2**. *A function $f : \mathbb{T} \to \mathbb{R}$ is ld-absolutely continuous on $\widetilde{\mathbb{T}}$ if and only if*

- *f is ∇-differentiable μ_∇-a.e. on $(a,b] \cap \mathbb{T}$ and $f^\nabla \in L_1$, and*
- *for every $t \in \widetilde{\mathbb{T}}$,*

$$f(t) = f(a) + \int_a^t f^\nabla(\tau)\,\nabla\tau. \tag{2.2}$$

3. Definitions and basic properties

As stated above, this paper is aimed at unifying the fractional calculus from the Riemann-Liouville perspective, and extending Taylor's theorem to incorporate it. First, we will prove the Cauchy formula for multiple integration. We begin by recalling the *time scale monomials*.

DEFINITION **3.1**. Let $\mathbb{T}$ be a time scale. Define $\hat{h}_0(t,s) = \hat{g}_0(t,s) = 1$ for all $t, s \in \mathbb{T}$. Then for $k = 1, 2, \dots$ define

$$\hat{h}_{k+1}(t,s) := \int_s^t \hat{h}_k(\tau,s)\,\nabla\tau, \quad \hat{g}_{k+1}(t,s) := \int_s^t \hat{g}_k(\rho(\tau),s)\,\nabla\tau. \tag{3.1}$$

The monomials are related by the relation $\hat{g}_n(t,s) = (-1)^n \hat{h}_n(s,t)$ for $t \in \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$ and $s \in \mathbb{T}^{\kappa^n}$, proved in [6]. They are used in the Taylor theorem for the nabla calculus.

THEOREM **3.1** ([6]). *Let $\alpha \in \mathbb{T}_{\kappa^{n-1}}$, f a function which is n times nabla differentiable on $\mathbb{T}_{\kappa^n}$, and f^{∇^n} being ld-continuous over $\mathbb{T}$. Then*

$$f(x) = \sum_{k=0}^{n-1} \hat{h}_k(x,\alpha)\, f^{\nabla^k}(\alpha) + \int_\alpha^x \hat{h}_{n-1}(x,\rho(\tau))\, f^{\nabla^n}(\tau)\,\nabla\tau. \tag{3.2}$$

We are interested in generalizing the Riemann-Liouville fractional integral. Let us now prove a Cauchy formula for multiple integration.

THEOREM **3.2** (Cauchy Integral Formula). *Let $\mathbb{T}$ be a time scale, and let $\alpha \in \mathbb{T}_\kappa$, $x \in \mathbb{T}^{\kappa^n}$ satisfying $\alpha < x$. Suppose $\phi : \mathbb{T} \to \mathbb{R}$ is an ld-continuous function. Then*

$$\int_\alpha^x \nabla t_1 \int_\alpha^{t_1} \nabla t_2 \cdots \int_\alpha^{t_{n-1}} \phi(t_n)\,\nabla t_n = \int_\alpha^x \hat{h}_{n-1}\big(x,\rho(t)\big)\,\phi(t)\,\nabla t. \tag{3.3}$$

P r o o f. The case $n = 1$ is trivial. Assume it is true for n, then for $n+1$

$$\int_\alpha^x \hat{h}_n(x,\rho(t))\,\phi(t)\,\nabla t = \int_\alpha^x (-1)^n \hat{g}_n(\rho(t),x)\,\phi(t)\,\nabla t.$$

We shall denote $\phi^{(-1)}(t) = \int_\alpha^t \phi(\tau)\nabla\tau$ with $\phi^{(-1)}(\alpha) = 0$. Because the integrand is ld-continuous the integration by parts formula [14, Theorem 8.47(v)] is valid. We obtain

$$\Big[(-1)^n \hat{g}_n(t,x)\,\phi^{(-1)}(t)\Big]_{t=\alpha}^{t=x} - \int_a^x (-1)^n \hat{g}_n^{\nabla}(t,x)\,\phi^{(-1)}(t)\,\nabla t$$
$$= \int_\alpha^x (-1)^{n-1}\hat{g}_{n-1}(\rho(t),x)\,\phi^{(-1)}(t)\,\nabla t = \int_\alpha^x \hat{h}_{n-1}(x,\rho(t))\,\phi^{(-1)}(t)\,\nabla t.$$

Then by assumption the formula is true for n,

$$\int_\alpha^x \hat{h}_{n-1}(x,\rho(t))\,\phi^{(-1)}(t)\nabla t = \int_\alpha^x \nabla t_1 \int_\alpha^{t_1} \nabla t_2 \cdots \int_\alpha^{t_{n-1}} \phi^{(-1)}(t_n)\,\nabla t_n$$
$$= \int_\alpha^x \nabla t_1 \int_\alpha^{t_1} \nabla t_2 \cdots \int_\alpha^{t_{n-1}} \nabla t_n \int_\alpha^{t_n} \phi(t_{n+1})\,\nabla t_{n+1},$$

as required. □

An alternate proof of (3.3) can be obtained with (3.2) under the stronger conditions of $\alpha \in \mathbb{T}_{\kappa^{n-1}}$ by using $\phi(t) = f^{\nabla^n}(t)$ and manually calculating $\phi^{(-n)}(x)$. Later, we will remove the restriction of ld-continuity.

Definition **3.2**. For a time scale $\mathbb{T}$, a collection of functions $\{\hat{h}_\alpha(\cdot,\cdot) : \mathbb{T}^2 \to \mathbb{R}\}_{\alpha>-1}$ are called *time scale power functions*, if they satisfy:

P1: For all $\alpha > -1$, $\hat{h}_\alpha(t,s)$ is a positive ld-continuous function in both variables when $t > s$ and $\hat{h}_\alpha(t,s) \equiv 0$ whenever $t \le s$.

P2: Whenever $\alpha \in \mathbb{N}_0$ and $t \ge s$, $\hat{h}_\alpha(t,s)$ corresponds with the nabla time scale monomials.

P3: For all $\alpha, \beta > -1$ one has

$$\int_s^t \hat{h}_\alpha(t,\rho(\tau))\,\hat{h}_\beta(\tau,s)\,\nabla\tau = \hat{h}_{\alpha+\beta+1}(t,s), \tag{3.4}$$

for $t, s \in \widetilde{\mathbb{T}}$ satisfying $s < t$.

Importantly, $\mathbb{R}$, $h\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$ all have functions which satisfy these axioms.

Example **3.1**. Let $\mathbb{T} = \mathbb{R}$. Then for $\nu > -1$, the functions

$$\hat{h}_\nu(t,s) = \frac{(t-s)^\nu}{\Gamma(\nu+1)}, \quad \text{for } t > s, \tag{3.5}$$

and $\hat{h}_\nu(t,s) = 0$ for $t \le s$, satisfy the requirements. The first two properties are trivial. As for (3.4), after a suitable change of variable we see that

$$\int_s^t \frac{(t-\tau)^\alpha}{\Gamma(\alpha+1)} \frac{(\tau-s)^\beta}{\Gamma(\beta+1)}\, d\tau = \frac{B(\alpha+1,\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\,(t-s)^{\alpha+\beta+1},$$

and the result follows from the properties of the beta function.

Example **3.2**. Let $h > 0$ and let $\mathbb{T} = h\mathbb{Z}$. We shall use the definition of the h-gamma function [20], where if $t \neq 0, -h, -2h, \ldots$,

$$\Gamma_h(t) := \lim_{n\to\infty} \frac{n!\,h^n (nh)^{t/h-1}}{t(t+h)\ldots(t+(n-1)/h)} = \int_0^\infty x^{t-1} e^{-x^k/k}\, dx. \tag{3.6}$$

Then we define

$$\hat{h}_\nu(t,s) = \frac{\Gamma_h(t-s+\nu h)}{\Gamma(\nu+1)\,\Gamma_h(t-s)}, \tag{3.7}$$

when $t > s$ and $\hat{h}_\nu(t,s) = 0$ otherwise. By [20, Proposition 4], the h-gamma function satisfies the identity $\Gamma_h(t) = h^{t/h-1}\Gamma(t/h)$. Hence by choosing $\tilde{t}, \tilde{s} \in \mathbb{Z}$ such that $\tilde{t} = th$ and $\tilde{s} = sh$, (3.7) can be written as

$$\hat{h}_\nu(t,s) = \frac{h^\nu\,\Gamma(\tilde{t}-\tilde{s}+\nu h)}{\Gamma(\nu+1)\,\Gamma(\tilde{t}-\tilde{s})}\,. \tag{3.8}$$

The proof of (3.4) is due to the identity

$$\frac{\Gamma(\tilde{t}-\tilde{s}+\alpha+\beta+1)}{\Gamma(\tilde{t}-\tilde{s})\Gamma(\alpha+\beta+2)} = \sum_{k=0}^{\tilde{t}-\tilde{s}-1} \binom{\tilde{t}-\tilde{s}-1}{k} \frac{\Gamma(\alpha+\tilde{t}-\tilde{s}-k)\Gamma(\beta+1+k)}{\Gamma(\tilde{t}-\tilde{s})\Gamma(\alpha+\tilde{t}-\tilde{s})\Gamma(\beta+1)}\,.$$

The result holds after the substitution $k = \tau - \tilde{s} - 1$. For details, see Ref [22].

Example **3.3**. Fix a $0 < q < 1$, and let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$. Denote

$$(t-s)_q^\nu := t^\nu \prod_{i=0}^\infty \frac{1-(s/t)q^i}{1-(s/t)q^{i+\nu}}, \quad \nu > -1. \tag{3.9}$$

Denote the q-gamma function as $\Gamma_q(\nu)$ [28, Section 21]. Then we let

$$\hat{h}_\nu(t,s) = \frac{(t-s)_q^\nu}{\Gamma_q(\nu+1)}, \quad \text{for } t > s, \tag{3.10}$$

and $\hat{h}_\nu(t,s) = 0$ for $t \le s$. We show (3.4) in two cases. Suppose that $s = 0$, then after the substitution $\tau = t\theta$,

$$\int_0^t \frac{(t-q\tau)_q^\alpha}{\Gamma_q(\alpha+1)} \frac{\tau^\beta}{\Gamma_q(\beta+1)}\, \nabla\tau = \frac{t^{\alpha+\beta+1}}{\Gamma_q(\alpha+1)\,\Gamma_q(\beta+1)} \int_0^1 (1-q\theta)_q^\alpha\, \theta^\beta\, \nabla\theta. \tag{3.11}$$

The result holds from the properties of the q-beta function. For details, see Ref. [7].

Suppose now $s \neq 0$, then there exists n such that $s = tq^n$. By the properties of the basic hypergeometric function, the left hand side of (3.4) is equal to

$$\frac{t(1-q)}{\Gamma_q(\alpha+1)\,\Gamma_q(\beta+1)} \sum_{k=0}^{n-1} q^k (t - tq^{k+1})_q^{\alpha}\,(tq^k - tq^n)_q^{\beta}$$
$$= t^{\alpha+\beta+1} \frac{(1-q)\,(1-q)_q^{\alpha}\,(1-q^n)_q^{\beta}\,{}_2\phi_1(q^{1-n}, q^{\alpha+1}; q^{-\beta-n-1}; q, q)}{\Gamma_q(\alpha+1)\,\Gamma_q(\beta+1)}$$
$$= t^{\alpha+\beta+1}\,(1-q)^{\alpha+\beta+1}\,\frac{(1-q^n)_q^{\alpha+\beta+1}}{(1-q)_q^{\alpha+\beta+1}},$$

completing the proof. See Ref. [1] for details.

Example **3.4**. The (q,h)-time scale [18] allows one to generalize (3.7) and (3.10). Choose a $t_0 \in (0,\infty)$, then for $q > 1$ define

$$\mathbb{T}_{(q,h)}^{t_0} := \left\{ t_0\, q^k + [k]_q\, h \,|\, k \in \mathbb{Z} \right\} \cup \left\{ \frac{h}{1-q} \right\}$$

and for $q = 1$ define $\mathbb{T}_{(1,h)}^{t_0} := \{t_0 + k\,h \,|\, k \in \mathbb{Z}\} \cup \{-\infty\}$. The left jump operator satisfies $\rho^k(t) = q^{-k}t + [-k]_q\, h$ for $k \in \mathbb{N}$. Observe that when $q = 1$ we have $\mathbb{T}_{(q,h)}^{t_0} = h\mathbb{Z} + t_0$, that is the discrete set with step size h and shifted by t_0. Similarly, setting $q > 1$ and $h = 0$ we can obtain the quantum time scale, again translated by t_0. For simplicity, we shall assume $t_0 = 0$. Let us now define the power function for this time scale. When $q = 1$, we define the power function as in (3.7). When $q > 1$, we define $\tilde{t} := t + hq/(1-q)$ and $\tilde{s} := s + hq/(1-q)$. Then

$$\hat{h}_{\alpha}(t,s) := \frac{\tilde{t}^{\alpha}}{\Gamma_{q^{-1}}(\alpha+1)} \prod_{j=0}^{\infty} \frac{1 - \tilde{s}\tilde{t}^{-1}q^{-j}}{1 - \tilde{s}\tilde{t}^{-1}q^{-\alpha-j}}\,. \tag{3.12}$$

We shall consider (3.4). The only case not covered in the previous work is for $q > 1$ and $s > h/(1-q)$. We see that there exists a positive integer m such that $s = \rho^m(t)$. By [18, Lemma 1] and the standard properties of the q-binomial we have

$$\hat{h}_{\alpha+\beta+1}(t,s) = (-1)^{m-1}\,\nu(t)^{\alpha+\beta+1}\,q^{m-1+\binom{m-1}{2}} \begin{bmatrix} -\alpha-\beta-2 \\ m-1 \end{bmatrix}_q .$$

Meanwhile the right hand side is equal to

$$\sum_{k=0}^{m-1} q^{-k-k\alpha+\beta-\beta m} \nu(t)^{\alpha+\beta+1} \begin{bmatrix} \alpha+k \\ \alpha \end{bmatrix}_q \begin{bmatrix} \beta+m-k-1 \\ \beta \end{bmatrix}_q .$$

The two equations are equal after applying the summation formula for the q-binomial. For details see Ref. [18].

Definition **3.3**. For $\alpha \geq 0$, $c \in \widetilde{\mathbb{T}}$ and a function $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ we define the *fractional* ∇*-integral of order* α as $({}_c\nabla_t^0 f)(t) = f(t)$ and

$$({}_c\nabla_t^{-\alpha} f)(t) := \int_c^t \hat{h}_{\alpha-1}(t, \rho(\tau))\, f(\tau)\, \nabla\tau, \tag{3.13}$$

for $\alpha > 0$ and $t \in (c, b] \cap \mathbb{T}$.

From the above work, we see that when $\mathbb{T} = \mathbb{R}$ we obtain the Riemann-Liouville fractional integral (1.1), when $\mathbb{T} = \mathbb{Z}$ we have Gray-Zhang fractional summation (1.2), and when $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ we obtain the Agarwal q-fractional integral (1.3). Further Example **3.2** encapsulates discrete time scales with constant graininess, while Example **3.4** considers more general constructs. However, finding power functions for other time scales is still a matter of ongoing research.

Lemma **3.1**. *Let* $\alpha > -1$, $\beta > 0$, *then* ${}_c\nabla_t^{-\beta} \hat{h}_\alpha(t, c) = \hat{h}_{\alpha+\beta}(t, c)$.

Proof. An immediate consequence of (3.4). We see that

$${}_c\nabla_t^{-\beta} \hat{h}_\alpha(t, c) = \int_c^t \hat{h}_{\beta-1}(t, \rho(\tau))\, \hat{h}_\alpha(\tau, c)\, \nabla\tau = \hat{h}_{\alpha+\beta}(t, c),$$

as required. $\square$

Theorem **3.3**. *Let* $\alpha > 0$, *then for* $f \in L_1$, *the fractional* ∇*-integral exists almost everywhere.*

Proof. We shall use a well known argument. By observing that $\hat{h}_{\alpha-1}(t, \rho(\tau))$ is zero over for $a < t \leq \rho(\tau)$,

$$\int_a^b \hat{h}_{\alpha-1}(t, \rho(\tau))\, \nabla t = \int_{\rho(\tau)}^b \hat{h}_{\alpha-1}(t, \rho(\tau))\, \nabla t = \hat{h}_\alpha(b, \rho(\tau)).$$

Therefore

$$\int_a^b \nabla\tau \int_a^b \hat{h}_{\alpha-1}(t, \rho(\tau))\; |f(\tau)|\; \nabla t = \int_a^b \hat{h}_\alpha(b, \rho(\tau))\; |f(\tau)|\; \nabla\tau,$$

which is bounded by $\hat{h}_\alpha(b, a)\; \|f\|$. So then the function

$$t \mapsto \int_a^b \hat{h}_{\alpha-1}(t, \rho(\tau))\, f(\tau)\, \nabla\tau,$$

is ∇-integrable by the Fubini-Tonelli theorem. Hence $({}_c\nabla_t^{-\alpha} f)(t)$ exists almost everywhere and is also ∇-integrable. $\square$

THEOREM **3.4**. *If $f \in L_1$, then given $\alpha \geq 1$, $({}_c\nabla_t^{-\alpha} f)(t)$ is ld-continuous over $t \in \widetilde{\mathbb{T}}$.*

Proof. Let $x, y \in \widetilde{\mathbb{T}}$ such that $a \leq x < y \leq b$. Suppose that y is left-dense. Then

$$\left|({}_c\nabla_y^\alpha f)(y) - ({}_c\nabla_x^\alpha f)(x)\right| \leq \int_c^x \left|\hat{h}_{\alpha-1}(y, \rho(\tau)) - \hat{h}_{\alpha-1}(x, \rho(\tau))\right| |f(\tau)| \, \nabla\tau$$
$$+ \int_x^y \hat{h}_{\alpha-1}(y, \rho(\tau)) \, |f(\tau)| \, \nabla\tau. \tag{3.14}$$

Observe that the second integral of (3.14) is bounded above by $\|f\| \, \hat{h}_\alpha(y, x)$, which approaches 0 as $x \to y$. As for the first integral, observe that $\hat{h}_{\alpha-2}(\cdot, s)$ positive implies that $\hat{h}_{\alpha-1}(\cdot, s)$ is non-decreasing, thus

$$\left|\hat{h}_{\alpha-1}(y, \rho(\tau)) - \hat{h}_{\alpha-1}(x, \rho(\tau))\right| \leq 2\hat{h}_{\alpha-1}(b, a).$$

So $\left|\hat{h}_{\alpha-1}(y, \rho(\tau)) - \hat{h}_{\alpha-1}(x, \rho(\tau))\right| |f(\tau)|$ is dominated by $2\hat{h}_{\alpha-1}(b, a) \, |f(\tau)|$. Applying the dominated convergence theorem, we see

$$\lim_{x \to y-} \int_c^x \left|\hat{h}_{\alpha-1}(y, \rho(\tau)) - \hat{h}_{\alpha-1}(x, \rho(\tau))\right| |f(\tau)| \, \nabla\tau$$
$$= \int_c^x \lim_{x \to y-} \left|\hat{h}_{\alpha-1}(y, \rho(\tau)) - \hat{h}_{\alpha-1}(x, \rho(\tau))\right| |f(\tau)| \, \nabla\tau = 0.$$

So, $({}_c\nabla_x^\alpha f)(x)$ is left-continuous at y.

Choose $a \leq x < y \leq b$. Suppose now x is right-dense. As before, $\left|{}_c\nabla_y^\alpha f(y)\right| \leq \|f\| \, \hat{h}_\alpha(y, c)$, and by ld-continuity of $\hat{h}_\alpha$,

$$\lim_{y \to x+} \left|{}_c\nabla_y^\alpha f(y)\right| \leq \lim_{y \to x+} \|f\| \, \hat{h}_\alpha(y, c) < \infty,$$

as required. □

The fractional integrals satisfy an index law. To prove this, we shall need a time scale analogue of Dirichlet's theorem. A corollary of this will be another proof of the Cauchy formula (3.3).

LEMMA **3.2**. *Let $c, d \in \widetilde{\mathbb{T}}$ satisfy $c < d$. Suppose that $f : \widetilde{\mathbb{T}} \times \widetilde{\mathbb{T}} \to \mathbb{R}$ be a function such that*

$$\int_c^d \nabla\tau \int_c^\tau |f(\tau, \theta)| \, \nabla\theta < \infty. \tag{3.15}$$

Then the order of integration can be changed and

$$\int_c^d \nabla\tau \int_c^\tau f(\tau, \theta) \, \nabla\theta = \int_c^d \nabla\theta \int_{\rho(\theta)}^d f(\tau, \theta) \, \nabla\tau. \tag{3.16}$$

P r o o f. From Bohner and Guseinov [12], we see that the region in (3.15) corresponds to the region $E = \{(t,s) \in \mathbb{T}^2 \,|\, c < t \le d, c < s \le t\}$. We define the extension of f by $F(\tau,\theta) = f(\tau,\theta)$ for $(\tau,\theta) \in E$ and $F(\tau,\theta) = 0$ elsewhere. Then by (3.15) and the Fubini-Tonelli theorem

$$\int_c^d \nabla\tau \int_c^\tau |f(\tau,\theta)| \, \nabla\theta = \int_E |F(\tau,\theta)| \, d\mu_\nabla \otimes \mu_\nabla, \tag{3.17}$$

where $\mu_\nabla \otimes \mu_\nabla$ is the product measure.

To prove (3.16) we shall use (2.1). In this section, τ and θ will be dummy variables for time scale integrals and t and s for Lebesgue integration and summations. We shall denote $I_{x,y} = I_{(x,y]\cap\mathbb{T}}$. The left hand side expands to

$$\begin{aligned} &\int_{(c,d]\cap\mathbb{T}} dt \int_{(c,t]\cap\mathbb{T}} f(t,s)\, ds + \sum_{j\in I_{c,d}} \nu(t_j) \int_{(c\cdot t_j]\cap\mathbb{T}} f(t_j,s)\, ds \\ &\quad + \int_{(c,d]\cap\mathbb{T}} \sum_{i\in I_{c,\tau}} \nu(s_i)\, f(t,s_i)\, dt + \sum_{j\in I_{c,d}} \nu(t_j) \sum_{i\in I_{c,t_j}} \nu(s_i)\, f(t_j,s_i). \end{aligned} \tag{3.18}$$

By the observation that the set of left-scattered points of $(c,d]\cap\mathbb{T}$ has a Lebesgue measure zero, the right hand side of (3.16) expands to

$$\int_c^d \nabla\theta \int_{\rho(\theta)}^d f(\tau,\theta)\, \nabla\tau = \sum_{i\in I_{c,d}} \nu(s_i)^2\, f(s_i,s_i) + \int_c^d \nabla\theta \int_\theta^d f(\tau,\theta)\, \nabla\tau. \tag{3.19}$$

We apply (2.1) to the double integral in (3.19), and after standard manipulations, we see that (3.16) is true if and only if the following four equations are true:

$$\int_{(c,d]\cap\mathbb{T}} dt \int_{(c,t]\cap\mathbb{T}} f(t,s)\, ds = \int_{(c,d]\cap\mathbb{T}} ds \int_{(s,d]\cap\mathbb{T}} f(t,s)\, ds, \tag{3.20}$$

$$\int_{(c,d]\cap\mathbb{T}} \sum_{i\in I_{c,t}} \nu(s_i)\, f(t,s_i)\, dt = \sum_{i\in I_{c,d}} \nu(s_i) \int_{(s_i,d]\cap\mathbb{T}} f(t,s_i)\, dt, \tag{3.21}$$

$$\sum_{j\in I_{c,d}} \nu(t_j) \int_{(c,t_j]\cap\mathbb{T}} f(t_j,s)\, ds = \int_{(c,d]\cap\mathbb{T}} \sum_{j\in I_{s,d}} \nu(t_j)\, f(t_j,s)\, ds, \tag{3.22}$$

$$\begin{aligned} \sum_{j\in I_{c,d}} \nu(t_j) \sum_{i\in I_{c,t_j}} \nu(s_i)\, f(t_j,s_i) &= \sum_{i\in I_{c,d}} \nu(s_i) \sum_{j\in I_{s_i,d}} \nu(t_j)\, f(t_j,s_i) \\ &\quad + \sum_{i\in I_{c,d}} \nu(s_i)^2\, f(s_i,s_i). \end{aligned} \tag{3.23}$$

As (3.20) is a standard Lebesgue integral, from the definition of F we see that

$$\int_{(c,d]\cap\mathbb{T}} dt \int_{(c,t]\cap\mathbb{T}} f(t,s)\,ds = \int_{(c,d)} dt \int_{(c,t)} F(t,s)\,ds. \tag{3.24}$$

The result is obtained after applying the Dirichlet theorem for $\mathbb{R}$. The proof of (3.21) and (3.22) are similar so we shall only prove the former. Let $\chi_i(t)$ be the characteristic function over $(s_i, b]\cap\mathbb{T}$. Then,

$$\sum_{i\in I_{c,d}} \int_{(s_i,b]\cap\mathbb{T}} \nu(s_i)\ |f(t,s_i)|\ dt = \sum_{i\in I_{c,d}} \int_{[c,d]} \nu(s_i)\,\chi_i(t)\ |f(t,s_i)|\ dt. \tag{3.25}$$

From (3.17), we can deduce that (3.25) is finite, thus we can interchange the sum and integral. From the identity $\sum_{i\in I_{c,d}} \nu(s_i)\,\chi_i(t) = \sum_{i\in I_{c,t}} \nu(s_i)$, we obtain (3.21). Finally, the proof of (3.23) is clear from the observation that

$$\sum_{j\in I_{c,d}} \nu(t_j) \sum_{i\in I_{c,t_j}} \nu(s_i)\, f(t_j,s_i) = \sum_{j\in I_{c,d}} \sum_{i\in I_{(c,t_j)}} \nu(s_i)\, f(t,s_i)\,\nu(t_j) + \sum_{j\in I_{c,d}} \nu(t_j)\,\nu(s_j)\, f(t_j,s_i),$$

and the proof is complete. $\square$

Corollary **3.1** (Cauchy Formula, Lebesgue version). *Let $\mathbb{T}$ be a time scale, and $\alpha, x\in\widetilde{\mathbb{T}}$ with $\alpha < x$. Let $\phi\in L_1$. Then (3.3) is valid.*

P r o o f. It is easy to see that the left hand side of (3.3) is well-defined. Observe that $\left|\int_\alpha^x \hat{h}_1(x,\rho(t))\,\phi(t)\,\nabla t\right| \le \int_\alpha^x |x-\rho(t)|\ |\phi(t)|\ \nabla t \le |b-a|\ \|\phi\| < \infty$. Indeed with induction on n,

$$\begin{aligned}\left|\int_\alpha^x \hat{h}_n(x,\rho(t))\,\phi(t)\,\nabla t\right| &\le \int_\alpha^x \left|\int_{\rho(t)}^x \hat{h}_{n-1}(\tau,\rho(t))\,\nabla\tau\right| |\phi(t)|\ \nabla t\\ &\le |b-a| \int_\alpha^x \hat{h}_{n-1}(x,\rho(t))\ |\phi(t)|\ \nabla t < \infty,\end{aligned}$$

hence the right hand side is also well-defined. This also guarantees that we can change the order of integration. As before (3.3) is trivial for $n=1$. Assume it is true for n, then

$$\begin{aligned}\int_\alpha^x \nabla t_1 \int_\alpha^{t_1} \nabla t_2 \ldots \int_\alpha^{t_n} \phi(t_{n+1})\,\nabla t_{n+1} &= \int_\alpha^x \nabla t_1 \int_\alpha^{t_1} \hat{h}_{n-1}(t_1,\rho(t))\,\phi(t)\,\nabla t\\ = \int_\alpha^x \phi(t)\,\nabla t \int_{\rho(t)}^x \hat{h}_{n-1}(t_1,\rho(t))\,\nabla t_1 &= \int_\alpha^x \hat{h}_n(x,\rho(t))\,\phi(t)\,\nabla t,\end{aligned}$$

as required. $\square$

THEOREM **3.5** (Index Law). *Let $\alpha, \beta > 0$ and suppose that $f \in L_1$. Then for μ_∇-almost all $t \in \widetilde{\mathbb{T}}_\kappa$,*

$$({}_c\nabla_t^{-\alpha}\,{}_c\nabla_t^{-\beta} f)(t) = ({}_c\nabla_t^{-(\alpha+\beta)} f)(t). \tag{3.26}$$

P r o o f. By (3.16) we see

$$\begin{aligned}({}_c\nabla_t^{-\alpha}\,{}_c\nabla_t^{-\beta} f)(t) &= \int_c^t \hat{h}_{\alpha-1}(t,\rho(\tau))\,\nabla\tau \int_c^\tau \hat{h}_{\beta-1}(\tau,\rho(\theta))\,f(\theta)\,\nabla\theta \\ &= \int_c^t f(\theta)\,\nabla\theta \int_{\rho(\theta)}^t \hat{h}_{\alpha-1}(t,\rho(\tau))\,\hat{h}_{\beta-1}(\tau,\rho(\theta))\,\nabla\tau,\end{aligned}$$

and the result follows. □

Fractional integrals compose with the classical ∇-derivatives.

THEOREM **3.6**. *Let $t \in \widetilde{\mathbb{T}}_\kappa$, and $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ be ld-absolutely continuous. For $\alpha > 0$ one has*

$$({}_c\nabla_t^{-\alpha} f)(t) = ({}_c\nabla_t^{-(\alpha+1)} f^\nabla)(t) + f(c)\,\hat{h}_\alpha(t,c). \tag{3.27}$$

P r o o f. Because f is ld-absolutely continuous, there exists $f^\nabla \in L_1$, and so the fractional integral exists for all $t \in \widetilde{\mathbb{T}}_\kappa$. We have ${}_c\nabla_t^{-(\alpha+1)} f^\nabla(t) = {}_c\nabla_t^{-\alpha}\,{}_c\nabla_t^{-1} f^\nabla(t) = {}_c\nabla_t^{-\alpha}[f(t)-f(c)] = {}_c\nabla_t^{-\alpha} f(t) - f(c)\,\hat{h}_\alpha(t,c)$, as required. □

DEFINITION **3.4**. Let $\alpha > 0$, $n = \lfloor\alpha\rfloor + 1$ and $f : \widetilde{\mathbb{T}} \to \mathbb{R}$. For $c, t \in \widetilde{\mathbb{T}}_{\kappa^n}$ with $c < t$, we define the *Riemann-Liouville fractional ∇-derivative of order α* to be the expression

$$({}_c\nabla_t^\alpha f)(t) := (\nabla^n\,{}_c\nabla_t^{-(n-\alpha)} f)(t) = \left[\int_c^t \hat{h}_{n-\alpha-1}(t,\rho(\tau))\,f(\tau)\,\nabla\tau\right]^{\nabla^n}, \tag{3.28}$$

if it exists.

DEFINITION **3.5**. Let $\alpha > 0$, $m = \lceil\alpha\rceil$ and $c, t \in \widetilde{\mathbb{T}}_{\kappa^m}$ with $c < t$. For $f : \widetilde{\mathbb{T}} \to \mathbb{R}$, we define the *Caputo fractional ∇-derivative of order α* is defined as

$$({}_c^C\nabla_t^\alpha f)(t) := ({}_c\nabla_t^{-(m-\alpha)} f^{\nabla^m})(t) = \int_c^t \hat{h}_{m-\alpha-1}(t,\rho(\tau))\,f^{\nabla^m}(\tau)\,\nabla\tau, \tag{3.29}$$

if it exists.

In this paper, the notation $({}_c\nabla_t^\alpha f)(c)$ will denote the limit of $({}_c\nabla_t^\alpha f)(t)$ as $t \to c+$. When c is right scattered, this limit is clearly zero. It is easy to see that the Riemann-Liouville fractional ∇-derivative generalizes the standard ∇-derivative. Indeed choosing $\alpha = k$, then $n = k+1$ and for $f \in AC_{ld}^k(\widetilde{\mathbb{T}})$, we have $({}_c\nabla_t^\alpha f)(t) = \left[\int_c^t f(\tau)\,\nabla\tau\right]^{\nabla^{k+1}} = f^{\nabla^k}(t)$.

THEOREM **3.7**. *Let $\alpha > 0$, $n = \lceil\alpha\rceil$ and suppose that $\hat{h}_\beta(t,s) \to 1$ as $\beta \to 0+$ for all $s, t \in \widetilde{\mathbb{T}}$. Let $f \in AC_{ld}^{n+1}(\widetilde{\mathbb{T}})$. Then*

$$\lim_{\alpha\to n} ({}_c^C\nabla_t^\alpha f)(t) = f^{\nabla^n}(t). \tag{3.30}$$

P r o o f. We have $f^{\nabla^{n+1}}$ ld-absolutely continuous over $\mathbb{T}\cap[a,t]$ for each $t \in \widetilde{\mathbb{T}}_{\kappa^n}$, thus the fractional integral of order $n-\alpha+1$ is defined. Then by (3.27), ${}_c\nabla_t^{-(n-\alpha+1)} f^{\nabla^{n+1}}(t) = {}_c\nabla_t^{-(n-\alpha)} f^{\nabla^n}(t) - f^{\nabla^n}(c)\,\hat{h}_{n-\alpha}(t,c) = ({}_c^C\nabla_t^\alpha f)(t) - f^{\nabla^n}(c)\,\hat{h}_{n-\alpha}(t,c)$. Hence

$$({}_c^C\nabla_t^\alpha f)(t) = f^{\nabla^n}(c)\,\hat{h}_{n-\alpha}(t,c) + \int_c^t \hat{h}_{n-\alpha}(t,\rho(\tau))\,f^{\nabla^{n+1}}(\tau)\,\nabla\tau.$$

Because $\hat{h}_{n-\alpha-1}(\cdot,\rho(\cdot))$ is positive, $\hat{h}_{n-\alpha}(t,\rho(\tau))$ is non-decreasing in t and non-increasing in τ. So

$$\left|\hat{h}_{n-\alpha}(t,\rho(\tau))\,f^{\nabla^{n+1}}(\tau)\right| \le \hat{h}_{n-\alpha}(b,a)\,\left|f^{\nabla^{n+1}}(\tau)\right|,$$

which is integrable due to ld-absolute continuity. The dominated convergence theorem then allows limit passage

$$\lim_{\alpha\to n}\int_c^t \hat{h}_{n-\alpha}(t,\rho(\tau))\,f^{\nabla^{n+1}}(\tau)\,\nabla\tau = \int_c^t \lim_{\alpha\to n}\hat{h}_{n-\alpha}(t,\rho(\tau))\,f^{\nabla^{n+1}}(\tau)\,\nabla\tau,$$

and the result holds. □

4. Taylor theorems

Extending Taylor's theorem to fractional derivatives has a long history dating back to

$$f(x+h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)}(D_a^{m+r}f)(x),$$

established by Riemann (see [25] for a detailed investigation). For further works, see [4], [8], [27], [33], [36], [38], [41] and [42]. This section is dedicated to constructing a time scale analogue of the expression obtained in [21]. We shall start with a simple Taylor theorem for the Caputo fractional ∇-derivative.

THEOREM **4.1**. *Let $\alpha > 0$ and set $n = \lceil\alpha\rceil$ and $c \in \widetilde{\mathbb{T}}_{\kappa^n}$. Suppose that $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ is a function such that $({}_c^C\nabla_x^\alpha f)(x)$ exists and is integrable of order α. Then*

$$f(x) = \sum_{k=0}^{n-1} f^{\nabla^k}(c)\,\hat{h}_k(x,c) + A_f(x), \tag{4.1}$$

where the remainder term $A_f(x)$ has integral representation

$$A_f(x) = \int_c^x \hat{h}_{\alpha-1}(x,\rho(\tau))\,({}_c^C\nabla_\tau^\alpha f)(\tau)\,\nabla\tau. \tag{4.2}$$

P r o o f. By applying (3.16) and (3.4) to (4.2) we obtain

$$\begin{aligned}
&\int_c^x \hat{h}_{\alpha-1}(x,\rho(\tau))\,({}_c^C\nabla_\tau^\alpha f)(\tau)\,\nabla\tau \\
&\quad = \int_c^t f^{\nabla^n}(\theta)\nabla\theta \int_{\rho(\theta)}^t \hat{h}_{\alpha-1}(t,\rho(\tau))\,\hat{h}_{n-\alpha-1}(\tau,\rho(\theta))\,\nabla\tau \\
&\quad = \int_c^t f^{\nabla^n}(\theta)\,\hat{h}_{n-1}(t,\rho(\theta))\,\nabla\theta.
\end{aligned} \tag{4.3}$$

Applying (3.2) to (4.3) we obtain

$$\int_c^t f^{\nabla^n}(\theta)\,\hat{h}_{n-1}(t,\rho(\theta))\,\nabla\theta = f(t) - \sum_{k=0}^{n-1} f^{\nabla^k}(c)\,\hat{h}_k(t,c), \tag{4.4}$$

as required. □

EXAMPLE **4.1**. Choosing $\mathbb{T} = \mathbb{Z}$, we obtain

$$f(t) = \sum_{k=0}^{n-1} \frac{\Gamma(t-a+k)}{k!\Gamma(t-a)}\nabla^k f(a) + \frac{1}{\Gamma(\alpha)} \sum_{\tau=a+1}^{t} \frac{\Gamma(t-\tau+\alpha)}{\Gamma(t-\tau+1)}\, {}_a\nabla_\tau^\alpha f(\tau), \tag{4.5}$$

the expression in [4].

It should be mentioned that the formula in [5] corresponding to (4.1) contains a superfluous term. (4.1) can be generalized to integrating the Caputo fractional ∇-derivative.

THEOREM **4.2**. *Let $0 < \alpha < \beta$ and let $n := \lceil\alpha\rceil$, $m := \lceil\beta\rceil$. Suppose that $f \in AC^m(\widetilde{\mathbb{T}})$. Then for $n = m$,*

$$({}_c^C\nabla_x^\alpha f)(x) = B_f(x), \tag{4.6}$$

and if $m > n$,

$$({}_c^C\nabla_x^\alpha f)(x) = \sum_{k=0}^{m-n-1} f^{\nabla^k}(c)\hat{h}_{k+n-\alpha}(x,c) + B_f(x), \tag{4.7}$$

where

$$B_f(x) = ({}_c\nabla_x^{-(\beta-\alpha)}\, {}_c^C\nabla_x^{\beta} f)(x). \tag{4.8}$$

P r o o f. We shall begin with (4.6). Applying (3.16) and (3.4) we obtain

$$({}_c\nabla_x^{-(\beta-\alpha)}{}_c^C\nabla_x^{\beta} f)(x) = \int_c^x f^{\nabla^m}(\theta)\, \hat{h}_{m-\alpha-1}(x,\rho(\theta))\, \nabla\theta. \tag{4.9}$$

If $m = n$, then (4.9) is equal to (4.6). Suppose that $m > n$, then by (3.4),

$$\hat{h}_{m-\alpha-1}(x,\rho(\theta)) = \int_{\rho(\theta)}^x \hat{h}_{n-\alpha-1}(x,\rho(\tau))\, \hat{h}_{m-n-1}(\tau,\rho(\theta))\, \nabla\tau. \tag{4.10}$$

Thus (4.9) is equal to

$$\int_c^x \hat{h}_{n-\alpha-1}(x,\rho(\tau))\, \nabla\tau \int_c^\tau \hat{h}_{m-n-1}(\tau,\rho(\theta))\, [f^{\nabla^n}(\theta)]^{\nabla^{m-n}}\, \nabla\theta, \tag{4.11}$$

after applying (3.16). Then by (3.2), (4.11) is equal to

$$\int_c^t \hat{h}_{n-\alpha-1}(t,\rho(\tau)) \left[f^{\nabla^n}(\tau) - \sum_{k=0}^{m-n-1} \hat{h}_k(\tau,c)\, f^{\nabla^k}(c) \right] \nabla\tau$$
$$= ({}_c^C\nabla_t^{\alpha} f)(t) - \sum_{k=0}^{m-n-1} \hat{h}_{k+n-\alpha}(x,c)\, f^{\nabla^k}(c),$$

as required. □

THEOREM **4.3**. *For $m \in \mathbb{N}$, let $0 = \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_m$ be an increasing sequence and define $\nu_k := \lceil \mu_{k+1} - \mu_k \rceil$. Let $c \in \widetilde{\mathbb{T}}_{\kappa^{\lceil \mu_m \rceil}}$. Define the operator $D^{(\mu_0)} f(x) = f(x)$ and $D^{(\mu_{k+1})} f(x) = {}_c^C\nabla_x^{\mu_{k+1}-\mu_k} D^{(\mu_k)} f(x)$. Suppose that $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ has ld-continuous $(D^{(\mu_m)} f)(t)$ and for each $0 \leq i \leq m-1$, $D^{(\mu_i)} f$ has ld-continuous nabla derivatives up to order $\nu_i - 1$. Then*

$$f(x) = \sum_{i=0}^{m-1} \sum_{k=0}^{\nu_i - 1} \left[D^{(\mu_i)} f \right]^{\nabla^k} (c)\, \hat{h}_{\mu_i+k}(x,c) + C_f(x), \tag{4.12}$$

where the remainder term $C_f(x)$ has integral representation

$$C_f(x) = \int_c^x \hat{h}_{\mu_m-1}(x,\rho(\tau))\, (D^{(\mu_m)} f)(\tau)\, \nabla\tau. \tag{4.13}$$

P r o o f. We prove by induction on m. Suppose $m = 1$, then $\nu_0 = \lceil \mu_1 \rceil$ and $(D^{(\mu_1)} f)(x) = ({}_c^C\nabla_x^{\mu_1} f)(x)$. By (4.1) we see $C_f(x) = A_f(x) = f(x) - \sum_{k=0}^{\nu_0-1} f^{\nabla^k}(c) \hat{h}_k(x,c)$.

For the $m+1$ case, we repeat the previous argument

$$\begin{aligned} C_f(x) &= ({}_c\nabla_x^{-\mu_m}\, {}_c\nabla_x^{-(\mu_{m+1}-\mu_m)}\, {}_c^C\nabla_x^{\mu_{m+1}-\mu_m}\, D^{(\mu_m)} f)(x) \\ &= {}_c\nabla_x^{-\mu_m}\left[(D^{(\mu_m)} f)(x) - \sum_{k=0}^{\nu_m-1} (D^{(\mu_m)} f)^{\nabla^k}(c)\, \hat{h}_k(x,c)\right] \\ &= ({}_c\nabla_x^{-\mu_m}\, D^{(\mu_m)} f)(x) - \sum_{k=0}^{\nu_m-1} (D^{(\mu_m)} f)^{\nabla^k}(c)\, \hat{h}_{\mu_m+k}(x,c). \end{aligned}$$

Assuming the formula holds for m, expand ${}_c\nabla_x^{-\mu_m}\, D^{(\mu_m)} f$ to obtain

$$\begin{aligned} f(x) &- \sum_{i=0}^{m-1}\sum_{k=0}^{\nu_i-1} (D^{(\mu_i)} f)^{\nabla^k}(c)\, \hat{h}_{\mu_i+k}(x,c) - \sum_{k=0}^{\nu_m-1} (D^{(\mu_m)} f)^{\nabla^k}(c)\, \hat{h}_{\mu_m+k}(x,c) \\ &= f(x) - \sum_{i=0}^{m}\sum_{k=0}^{\nu_i-1} (D^{(\mu_i)} f)^{\nabla^k}(c)\, \hat{h}_{\mu_i+k}(x,c), \end{aligned}$$

as required. □

EXAMPLE **4.2**. Suppose we choose $\mu_0 < \ldots < \mu_m$ to satisfy $0 < \mu_{k+1} - \mu_k \leq 1$ for $k = 0, \ldots, m-1$. Then $\nu_k = 1$, and (4.12) is equal to

$$f(x) = \sum_{i=0}^{m-1} (D^{(\mu_i)} f)(c)\, \hat{h}_{\mu_i}(x,c) + \int_c^x \hat{h}_{\mu_m-1}(x,c)\, (D^{(\mu_m)} f)(\tau)\, \nabla\tau.$$

If $\mathbb{T} = \mathbb{R}$ and $c = 0$, then we obtain the expression from [21],

$$f(x) = \sum_{i=0}^{m-1} \frac{D^{(\mu_i)} f(0)}{\Gamma(\mu_i+1)}\, x^{\mu_i} + \frac{1}{\Gamma(\mu_m)} \int_0^x (x-\tau)^{\mu_m-1}\, D^{(\mu_m)} f(\tau)\, d\tau.$$

EXAMPLE **4.3**. For $0 < \alpha \leq 1$, choose $\mu_k = k\alpha$. Thus $\mu_{k+1} - \mu_k = \alpha$ and $\nu_k = 1$. Denote

$$({}_c^C\nabla_x^\alpha)^n f(x) := \underbrace{{}_c^C\nabla_x^\alpha \cdot \cdots \cdot {}_c^C\nabla_x^\alpha}_{n} f(x).$$

Then (4.12) is equal to

$$f(x) = \sum_{i=0}^{m-1} ({}_c^C\nabla_x^\alpha)^i f(c)\, \hat{h}_{i\alpha}(x,c) + \int_c^x \hat{h}_{m\alpha-1}(x,\rho(\tau))\, ({}_c^C\nabla_x^\alpha)^m f(\tau)\, \nabla\tau.$$

If $\mathbb{T} = \mathbb{R}$, then we obtain an equivalent formula to that in [33],

$$f(x) = \sum_{i=0}^{m-1} \frac{({}^C D^\alpha)^i f(c)\, (x-c)^{i\alpha}}{\Gamma(i\alpha+1)} + \frac{1}{\Gamma(m\alpha)} \int_c^x (x-c)^{m\alpha-1}\, ({}^C D^\alpha)^m f(\tau)\, d\tau.$$

To prove a Riemann-Liouville version of (4.12), we need the following.

LEMMA **4.1**. *Let $c \in \widetilde{\mathbb{T}}_\kappa$. For $0 < \beta \leq 1$ we have*

$$({}_c\nabla_x^{-\beta}\, {}_c\nabla_x^{\beta} f)(x) = f(x) - ({}_c\nabla_x^{-(1-\beta)} f)(c)\, \hat{h}_{\beta-1}(x,c), \tag{4.14}$$

for μ_∇-almost all $x \in (c,b] \cap \mathbb{T}$.

P r o o f. We let $\psi(x) = {}_c\nabla_x^{-(1-\beta)} f(x)$. The ∇-integral is ld-absolutely continuous, so $\int_c^x \psi^\nabla(t)\,\nabla t = \psi(x) - \psi(c)$. Then $\psi^\nabla(x) = [{}_c\nabla_x^{-(1-\beta)} f]^\nabla(x) = ({}_c\nabla_x^{\beta} f)(x)$ and $\psi(x) - \psi(c) = ({}_c\nabla_x^{-(1-\beta)} f)(x) - ({}_c\nabla_x^{-(1-\beta)} f)(c)$, meanwhile

$$\int_c^x \psi^\nabla(t)\,\nabla t = ({}_c\nabla_x^{-1}\, {}_c\nabla_x^{\beta} f)(x) = ({}_c\nabla_x^{-(1-\beta)}\, {}_c\nabla_x^{-\beta}\, {}_c\nabla_x^{\beta} f)(x).$$

Thus, differentiating by order $1-\beta$, and using ${}_c\nabla_x^{1-\beta}\, {}_c\nabla_x^{-(1-\beta)} f = f$ we have

$${}_c\nabla_x^{-\beta}\, {}_c\nabla_x^{\beta} f(x) = f(x) - {}_c\nabla_x^{-(1-\beta)} f(c)\, \hat{h}_{\beta-1}(x,c),$$

μ_∇-almost everywhere. □

THEOREM **4.4**. *Let $c \in \widetilde{\mathbb{T}}_{\kappa^{n-1}}$. Let $0 = \mu_0 < \mu_1 < \ldots < \mu_n$ be a real sequence satisfying $0 < \mu_{k+1} - \mu_k \leq 1$ for $k = 0, \ldots, n-1$. Let $\alpha_k := 1 - (\mu_{k+1} - \mu_k)$. Define the operator $L^{(\mu_k)}$ by $L^{(\mu_0)} f := f$ and $L^{(\mu_{k+1})} f := {}_c\nabla_x^{\mu_{k+1}-\mu_k} L^{(\mu_k)} f$. Let $f : \widetilde{\mathbb{T}} \to \mathbb{R}$ be a function such that for each k, $L^{(\mu_k)} f$ exists and is integrable of order α_k. Then*

$$f(x) = \sum_{k=0}^{n-1} ({}_c\nabla_x^{-\alpha_k} L^{(\mu_k)} f)(c)\, \hat{h}_{\mu_{k+1}-1}(x,c) + D_f(x), \tag{4.15}$$

where the remainder term $D_f(x)$ has integral representation

$$D_f(x) = \int_c^x \hat{h}_{\mu_n-1}(x, \rho(\tau))\, (L^{(\mu_n)} f)(\tau)\, \nabla\tau. \tag{4.16}$$

P r o o f. The proof is analogous as before. Let $n = 1$. Then

$$D_f(x) = ({}_c\nabla_x^{-\mu_1}\, {}_c\nabla_x^{\mu_1} f)(x) = f(x) - ({}_c\nabla_x^{-\alpha_0} f)(c)\, \hat{h}_{\mu_1-1}(x,c),$$

as $\mu_0 = 0$. Assume the theorem is true for n. Then evaluating $n+1$ we have

$$\begin{aligned} D_f(x) &= ({}_c\nabla_x^{-\mu_n}\, {}_c\nabla_x^{-(\mu_{n+1}-\mu_n)}\, {}_c\nabla_x^{\mu_{n+1}-mu_n} L^{(\mu_n)} f)(x) \\ &= ({}_c\nabla_x^{-\mu_n} L^{(\mu_n)} f)(x) - ({}_c\nabla_x^{-\alpha_n} L^{(\mu_n)} f)(c)\, \hat{h}_{\mu_{n+1}-1}(x,c) \\ &= f(x) - \sum_{k=0}^{n} ({}_c\nabla_x^{-\alpha_k} L^{(\mu_k)} f)(c)\, \hat{h}_{\mu_{k+1}-1}(x,c), \end{aligned}$$

as required. □

EXAMPLE **4.4**. Let $0 < \alpha < 1$, and choose $\mu_0 = 0$, $\mu_k = \alpha + k - 1$ for $k = 1, \dots, n$. Then $\alpha_0 = 1 - (\mu_1 - \mu_0) = 1 - \alpha$ and $\hat{h}_{\mu_1 - 1}(x, c) = \hat{h}_{\alpha-1}(x, c)$. When $k > 0$, we have $\alpha_k = 0$ and $\hat{h}_{\mu_{k+1}-1}(x, c) = \hat{h}_{\alpha+k-1}(x, c)$. Also $L^{(\mu_0)}f = f$, $L^{(\mu_1)}f = {}_c\nabla_x^{\alpha} f$ and in general $L^{(\mu_k)}f = (L^{(\mu_{k-1})}f)^{\nabla} = {}_c\nabla_x^{\alpha+k-1} f$. Then (4.15) is expressed as

$$f(x) = ({}_c\nabla_x^{-(1-\alpha)} f)(c)\, \hat{h}_{\alpha-1}(x, c) + \sum_{j=0}^{n-2} ({}_c\nabla_x^{\alpha+j} f)(c)\, \hat{h}_{\alpha+j}(x, c) + R(x),$$

$$R(x) = \int_c^x \hat{h}_{\alpha+n-2}(x, \rho(\tau))\, ({}_c\nabla_\tau^{\alpha+n-1} f)(\tau)\, \nabla\tau.$$

Moreover, when $\mathbb{T} = \mathbb{R}$, we obtain the Taylor theorem from [40],

$$f(x) = \frac{(I^{1-\alpha} f)(c)\,(x-c)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=0}^{n-2} \frac{(D^{\alpha+j} f)(c)\,(x-c)^{\alpha+j}}{\Gamma(\alpha+j+1)} + R(x),$$

$$R(x) = \frac{1}{\Gamma(\alpha+n-1)} \int_c^x (x-\tau)^{\alpha+n-2}\,(D^{\alpha+n-1} f)(\tau)\, d\tau.$$

EXAMPLE **4.5**. Let $0 < \alpha < 1$. As before we shall denote

$$({}_c\nabla_x^{\alpha})^n f(x) := \underbrace{{}_c\nabla_x^{\alpha} \cdot \cdots \cdot {}_c\nabla_x^{\alpha}}_{n} f(x).$$

Let $\mu_k = k\,\alpha$, then $\alpha_k = 1 - \alpha$. So $L^{(\mu_{k+1})}f = {}_c\nabla_x^{\alpha} L^{(\mu_k)} f = ({}_c\nabla_x^{\alpha})^{k+1} f$. Then (4.15) is equal to

$$f(x) = \sum_{k=0}^{n-1} ({}_c\nabla_x^{-(1-\alpha)} ({}_c\nabla_x^{\alpha})^k f)(c)\, \hat{h}_{(k+1)\alpha-1}(x, c) + R_f(x),$$

$$R_f(x) = \int_c^x \hat{h}_{n\alpha-1}(x, \rho(\tau))\, ({}_c\nabla_x^{\alpha})^n f(\tau)\, \nabla\tau.$$

When $\mathbb{R} = \mathbb{T}$, we obtain an expression equivalent to that obtained in [41],

$$f(x) = \sum_{k=0}^{n-1} \frac{(I^{1-\alpha}(D^{\alpha})^k)(c)\,(x-c)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} + R(x),$$

$$R(x) = \frac{1}{\Gamma(n\alpha)} \int_c^x (x-\tau)^{n\alpha-1}\,(D^{\alpha})^n f(\tau)\, d\tau.$$

5. Conclusions

In this paper we have considered an axiomatic definition of fractional calculus on time scales. We defined a time scale analogue of power functions using properties which are common to the power functions on the reals, the falling factorial on the integers and the q-power function from quantum calculus. Using this as an integral kernel, we defined a time scale analogue of fractional integration and studied its properties such as integrability and constructed an index law. We have also defined fractional differentiation and have proved a generalization of Taylor's theorem using these operators.

However, even though we have shown that the power functions on the reals, integers and the quantum time scales exist, we do not have a construction which would be consistent over all time scales. Doing so would solve the open problem of defining fractional calculus on general time scales. This is left for further research.

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