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RESEARCH PAPER

EXISTENCE RESULTS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS VIA KURATOWSKI MEASURE OF NONCOMPACTNESS

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Abstract

In this paper, we study the existence of mild solutions for a class of semilinear fractional differential equations with nonlocal conditions in Banach spaces. The results are obtained by using convex-power condensing operator and fixed point theory. An example is presented to illustrate the main result.

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Key Words and Phrases: Semilinear fractional differential equation, Kuratowski measure of noncompactness, convex-power condensing operator, nonlocal condition, mild solution

1. Introduction

In this paper, we are concerned with the following semilinear fractional differential equation with nonlocal conditions

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + f(t, u(t)), \ t \in (0, b], \\ u(0) = g(u), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, ${}^C D_t^\alpha$ is the α-order Caputo derivative operator, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (C_0-semigroup)$\{T(t)\}_{t\geq 0}$, f, g are given functions to be specified later.

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In the last decades, fractional calculus and fractional differential equations have attracted much attention, we refer to [1, 2, 9, 15, 20, 21, 24, 25], and references therein. It is found that many phenomena can be modeled with the help of fractional derivatives or integrals, such as fractional Brownian motion [5], anomalous diffusion [23, 30], etc.

Semilinear differential equations with nonlocal conditions was initiated by Byszewski [12]. The nonlocal condition shows better effects in some applications than the classical initial condition, see, for example, Deng [14]. Nonlocal semilinear differential equations has been studied extensively. Lin and Liu [26] studied semilinear integro-differential equations under Lipschitz type conditions. Fan [17] obtained the existence of impulsive semilinear differential equations with nonlocal conditions. Benchohra and Ntouyas [10] discussed second-order differential equations with nonlocal conditions under compactness conditions. Liu and Chang [27] studied the existence of a class of partial differential equations with nonlocal conditions under compactness conditions.

The measure of noncompactness has been used to study differential equations in Banach spaces. Garcia-Falset [18] studied the existence and asymptotic bahavior of solutions for Cauchy problems with nonlocal initial datum generated by accretive operators in Banach spaces. Xue [34] studied the existence of integral solutions for nonlinear differential equations with nonlocal initial conditions under non-compactness conditions. Cui and Yan [13] discussed the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces. Zhu et al. [36] used a fixed point theorem to prove the existence results of mild solutions of first order semilinear differential equations with Hausdorff measure of non-compactness. It should be pointed that in these papers the authors used the Sadovskii fixed point theorem which is concerned with condensing operators. Sun and Zhang [33] introduced a new operator, namely, convex-power condensing operator, which is a generalization of condensing operators. They proved a fixed point theorem of convex-power condensing operator, which generalizes the Schauder fixed point theorem and Sadovskii fixed point theorem. More recently, Zhu and Li [35] combined fixed point theorems with convex-power condensing operators to obtain the existence results for first order semilinear differential systems with nonlocal initial conditions in Banach spaces.

The purpose of this paper is to use convex-power condensing operator and fixed point theorem to establish the existence results for the nonlocal semilinear fractional differential equation (1.1) when the coefficient operator A generates a C_0-semigroup which is continuous in the uniform operator topology for $t > 0$.

The rest of this paper is organized as follows: In Section **2**, we give the basic definitions and notations. In Section **3**, we investigate the existence of mild solutions of problem (1.1). Finally, in Section **4**, we present an example to illustrate the main theorem.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $\mathbb{B}(X)$ be the space of all bounded linear operators on X. By $C([0,b];X)$, we denote the space of X-valued continuous functions $u:[0,b]\to X$ with the norm $\|u\|_{C([0,b];X)} = \sup_{0\le t\le b}\|u(t)\|$. By $L^1([0,b];X)$, we denote the space of X-valued Bochner integrable functions $u:[0,b]\to X$ with the norm $\|u\|_{L_1([0,b];X)} = \int_0^b \|u(t)\|dt$. Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ be the set of natural numbers, real numbers, complex numbers, respectively. Let $R_+ = [0,\infty)$. For every $U\subset X$, by $\overline{co}\ U$ we denote the convex closure of U. If A is a linear operator in X, then $\rho(A)$ means the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A, where I is the identity operator on X. By $*$ we denote the convolution of functions

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau,\ t\ge 0.$$

For $\beta\ge 0$, let

$$g_\beta(t) = \begin{cases} \dfrac{t^{\beta-1}}{\Gamma(\beta)}, & t>0,\\ 0, & t\le 0, \end{cases} \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. Let $g_0(t) = \delta(t)$, the delta distribution. The set of operators $\{g_\alpha\}_{t\ge 0}$ is a semigroup, i.e. $g_\alpha * g_\beta = g_{\alpha+\beta}$ for all $\alpha,\ \beta\ge 0$.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha>0$ of $u:[0,b]\to X$ is defined by

$$J_t^\alpha u(t) = (g_\alpha * u)(t). \tag{2.2}$$

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha\in(0,1)$ of $u:[0,b]\to X$ is defined by

$$D_t^\alpha u(t) = \frac{d}{dt}J_t^{1-\alpha}u(t). \tag{2.3}$$

Definition 2.3. The Caputo fractional derivative of order $\alpha\in(0,1)$ of $u:[0,b]\to X$ is defined by

$${}^C D_t^\alpha u(t) = D_t^\alpha (u(t) - u(0)). \tag{2.4}$$

The Laplace transform formula for the Riemann-Liouville fractional integral is given by

$$L\{J_t^\alpha u(t)\} = \frac{1}{\lambda^\alpha}\hat{u}(\lambda), \tag{2.5}$$

where $\hat{u}(\lambda)$ is the Laplace transform of u given by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)dt,\ \Re\lambda > \omega, \tag{2.6}$$

where ω is some real number, $\Re\lambda$ stands for the real part of the complex number λ.
The Laplace transform formula for the Caputo fractional derivative is defined by

$$L\{{}^C D_t^\alpha u(t)\} = \lambda^\alpha \hat{u}(\lambda) - \lambda^{\alpha-1}u(0). \tag{2.7}$$

Definition 2.4. The Mittag-Leffler function (see [16], Vol.3, Ch. 18) is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)},\ \alpha, \beta > 0,\ z \in \mathbb{C}. \tag{2.8}$$

When $\beta = 1$, set $E_\alpha(z) = E_{\alpha,1}(z)$.

Definition 2.5. ([32]). The Mainardi function is defined by

$$M_\alpha(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)},\ 0 < \alpha < 1,\ z \in \mathbb{C}. \tag{2.9}$$

The Laplace transform of the Mainardi's function $M_\alpha(r)$ is (see [28],(F.24)):

$$\int_0^\infty e^{-r\lambda} M_\alpha(r)dr = E_\alpha(-\lambda). \tag{2.10}$$

By (2.8) and (2.10), it is clear that

$$\int_0^\infty M_\alpha(r)dr = 1,\ 0 < \alpha < 1. \tag{2.11}$$

On the other hand, $M_\alpha(z)$ satisfies the following equality (see [28], P.249)

$$\int_0^\infty \frac{\alpha}{r^{\alpha+1}} M_\alpha(1/r^\alpha)e^{-\lambda r}dr = e^{-\lambda^\alpha}. \tag{2.12}$$

and the equality (see [28], (F.33))

$$\int_0^\infty r^\delta M_\alpha(r)dr = \frac{\Gamma(\delta+1)}{\Gamma(\alpha\delta+1)},\ \delta > -1,\ 0 < \alpha < 1. \tag{2.13}$$

DEFINITION 2.6. ([3]). Let X be a Banach space and Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\beta : \Omega_X \to [0, \infty)$ defined by (here $B \in \Omega_X$)

$$\beta(B) = \inf \left\{ \varepsilon > 0 : \ B = \bigcup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) \leq \varepsilon \text{ for } i = 1, \ldots, n \right\}, \tag{2.14}$$

here $\operatorname{diam} B_i = \sup\{|x - y| : x, y \in B_i\}$.

Let us recall the basic properties of Kuratowski measure of noncompactness.

LEMMA 2.1. [19] *Let S and T be bounded sets of X and a be a real number. Then the noncompactness measure has the following properties*:

(1) $\beta(S) = 0$ *if and only if S is a relatively compact set.*
(2) $S \subset T$ *implies that* $\beta(S) \leq \beta(T)$.
(3) $\beta(\overline{S}) = \beta(S)$.
(4) $\beta(S \bigcup T) = \max\{\beta(S), \beta(T)\}$.
(5) $\beta(aS) = |a|\beta(S)$.
(6) $\beta(S + T) \leq \beta(S) + \beta(T)$.
(7) $\beta(\overline{co}S) = \beta(S)$, *where $\overline{co}S$ is the convex closure of S.*
(8) $|\beta(S) - \beta(T)| \leq 2d_h(S, T)$, *where $d_h(S, T)$ denotes the Hausdorff distance between the sets S and T, that is,*

$$d_h(S, T) = \max\{\sup_{x \in S} d(x, T), \sup_{x \in T} d(x, S)\},$$

where $d(\cdot, \cdot)$ denotes the distance from an element of X to a set of X.

LEMMA 2.2. ([4]) *If $B \subset C([0, b]; X)$ is bounded, then $\beta(B(t)) \leq \beta(B)$ for all $t \in [0, b]$, where $B(t) = \{x(t); x \in B\} \subset X$. Furthermore if B is equicontinuous on $[0, b]$, then $\beta(B(t))$ is continuous on $[0, b]$, and $\beta(B) = \sup\{\beta(B(t)); t \in [0, b]\}$.*

LEMMA 2.3. ([33]) *Let $T \subset C([0, b];\ X)$ be equicontinuous, $u_0 \in C([0, b]; X)$, then $\overline{co}\{T, u_0\} \subset C([0, b]; X)$ is equicontinuous.*

DEFINITION 2.7. ([7]). A locally compact space is called *countable at infinity* (also sometimes *σ-compact*) when it can be covered by a sequence of compact subsets.

LEMMA 2.4. ([22], Corollary 3.1.(b)) *Let X be a Banach space and let T be a locally compact Hausdorff space which is countable at infinity. Here β denotes the Kuratowskii measure of noncompactness, μ denotes a regular Borel measure on T, $H = \{f_1, f_2, \ldots.\}$ denotes a countable set of strongly*

μ-measurable functions $f_n : T \to X$ $(n \geq 1)$. Set $h(t) := \sup_{n\geq 1} \|f_n(t)\|$ $(t \in T)$, $S = \{\int f_n d\mu\}_{n=1}^{\infty}$. If T is metrizable and H satisfies the "uniformly integrability" condition $\int h d\mu < \infty$, then we have

$$\beta(S) \leq 2 \int \beta(H(t)) d\mu(t), \quad t \in T. \tag{2.15}$$

LEMMA 2.5. ([11]) *Let X be a Banach space and let $B \subset X$ be bounded. β denotes the Kuratowskii measure of noncompactness. Then for every $\varepsilon > 0$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset B$ such that*

$$\beta(B) \leq 2\beta(\{x_n\}_{n=1}^{\infty}) + \varepsilon. \tag{2.16}$$

Let X be a real Banach space and let $D \subset X$ is a closed convex set. $\mathcal{A} : D \to D$, $x_0 \in D$. For every $S \subset D$, set

$$\mathcal{A}^{(1,x_0)}(S) = \mathcal{A}(S), \quad \mathcal{A}^{(n,x_0)}(S) = \mathcal{A}(\overline{co}\{\mathcal{A}^{(n-1,x_0)}(S), x_0\}), \; n = 2, 3, \ldots . \tag{2.17}$$

DEFINITION 2.8. ([33]) Let X be a real Banach space. $D \subset X$ is a closed convex set. $\mathcal{A} : D \to D$. If $\mathcal{A}$ is continuous and bounded, and there exist $x_0 \in D$ and positive integer n_0 such that for every non-relatively compact and bounded set $S \subset D$, we have

$$\beta(\mathcal{A}^{(n_0,x_0)}(S)) < \beta(S), \tag{2.18}$$

then $\mathcal{A}$ is called a convex-power condensing operator.

LEMMA 2.6. ([33]) *Let X be a real Banach space and let $D \subset X$ be a closed convex set. If $\mathcal{A} : D \to D$ is a convex-power condensing operator, then $\mathcal{A}$ has at least one fixed point in D.*

DEFINITION 2.9. ([31]) An n-tuple of nonnegative integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is called a multi-index and we define

$$|\gamma| = \sum_{i=1}^{n} \gamma_i$$

and

$$x^{\gamma} = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n} \quad \text{for} \quad x = (x_1, x_2, \ldots, x_n).$$

Denoting $D_k = \partial/\partial x_k$ and $D = (D_1, D_2, \ldots, D_n)$, we have

$$D^{\gamma} = D_1^{\gamma_1} D_2^{\gamma_2} \ldots D_n^{\gamma_n} = \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_2}} \frac{\partial^{\gamma_2}}{\partial x_2^{\gamma_2}} \cdots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}} .$$

3. Main results

In this section, we consider the existence results of the nonlocal problem (1.1) by virtue of convex-power condensing operator and fixed point theorem. We first consider the following fractional differential equation

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t) + \varphi(t), \ t \in (0, b], \\ u(0) = x \in X, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$, ${}^C D_t^\alpha$ is the α-order Caputo derivative operator, $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(t)(t \geq 0)$, X is a Banach space with the norm $\|\cdot\|$, $\varphi \in L^1([0, b]; X)$.

Since the coefficient operator A is the infinitesimal generator of a C_0-semigroup $T(t)$, there exists constants $N \geq 1$, $\omega \geq 0$, such that $\|T(t)\| \leq Ne^{\omega t}$, $t \geq 0$. From the subordination principle (see Theorem 3.1 in [8]), it follows that A generates an exponentially bounded solution operator $T_\alpha(t)(0 < \alpha < 1)$ (see [8], Definition 2.3), satisfying $T_\alpha(0) = I$ (the identity operator on X),

$$\|T_\alpha(t)\| \leq Ne^{\omega^{1/\alpha} t}, \ t \geq 0, \tag{3.2}$$

and

$$T_\alpha(t) = \int_0^\infty t^{-\alpha} M_\alpha(st^{-\alpha}) T(s) ds = \int_0^\infty M_\alpha(r) T(t^\alpha r) dr, \ t > 0, \tag{3.3}$$

where $M_\alpha(r)$ is the Mainardi function.

Since $\|T_\alpha(t)\| \leq N_1 e^{\omega^{1/\alpha} t}$, from the formulas (2.5) and (2.6) in [8], it follows that

$$\{\lambda^\alpha : \lambda > \omega^{1/\alpha}\} \subset \rho(A), \tag{3.4}$$

and

$$\lambda^{\alpha-1} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} T_\alpha(t) dt, \ \lambda > \omega^{1/\alpha}, \ x \in X. \tag{3.5}$$

By $\widehat{\varphi}(\lambda) = \int_0^\infty e^{-\lambda t} \varphi(t) dt$, resp. $\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t) dt$, we denote the Laplace transforms of the functions $\varphi \in L^1([0, b]; X)$ and $u \in C([0, b]; X)$, respectively. Taking the Laplace transform to both sides of ${}^C D_t^\alpha u(t) = Au(t) + \varphi(t)$ and using the initial condition $u(0) = x$, we obtain

$$\widehat{u}(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha, A)x + R(\lambda^\alpha, A)\widehat{\varphi}(\lambda). \tag{3.6}$$

Since A is the infinitesimal generator of C_0-semigroup $T(t)$,

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t) x dt, \ \lambda > \omega, \ x \in X. \tag{3.7}$$

Hence, by (3.7) and (2.12),

$$
\begin{aligned}
&R(\lambda^\alpha, A)\widehat{\varphi}(\lambda) &(3.8)\\
&= \int_0^\infty e^{-\lambda^\alpha t} T(t)\widehat{\varphi}(\lambda)dt\\
&= \int_0^\infty \int_0^\infty \alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} T(t^\alpha)\varphi(s)e^{-s\lambda} dsdt\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \alpha t^{\alpha-1} \frac{\alpha}{r^{\alpha+1}} M_\alpha(1/r^\alpha) e^{-\lambda t r} T(t^\alpha)\varphi(s)e^{-s\lambda} drdsdt\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \frac{\alpha^2}{r^{2\alpha+1}} t^{\alpha-1} M_\alpha(1/r^\alpha) e^{-\lambda t} \varphi(s)e^{-s\lambda} drdsdt\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \alpha r t^{\alpha-1} M_\alpha(r) T(t^\alpha r)\varphi(s) e^{-\lambda(t+s)} drdsdt\\
&= \int_0^\infty e^{-\lambda t}\left(\int_0^t \int_0^\infty \alpha r M_\alpha(r)(t-s)^{\alpha-1} T((t-s)^\alpha r)\varphi(s) drds\right) dt. &(3.9)
\end{aligned}
$$

By (3.5), (3.6), (3.8), we obtain

$$
\begin{aligned}
\widehat{u}(\lambda) = &\int_0^\infty e^{-\lambda t} T_\alpha(t) dt\\
+ &\int_0^\infty e^{-\lambda t}\left(\int_0^t \int_0^\infty \alpha r M_\alpha(r)(t-s)^{\alpha-1} T((t-s)^\alpha r)\varphi(s) drds\right) dt. \quad (3.10)
\end{aligned}
$$

From the uniqueness theorem of the Laplace transform, it follows that

$$u(t) = T_\alpha(t)x + \int_0^t \int_0^\infty \alpha r M_\alpha(r)(t-s)^{\alpha-1} T((t-s)^\alpha r)\varphi(s) drds. \quad (3.11)$$

Set

$$S_\alpha(t)x = \int_0^\infty \alpha r M_\alpha(r) T(t^\alpha r)x dr,\ t \geq 0,\ x \in X. \quad (3.12)$$

By (3.11), (3.12),

$$u(t) = T_\alpha(t)x + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\varphi(s)ds. \quad (3.13)$$

Motivated by (3.13), we give the following definition of the mild solution of nonlocal problem (1.1).

Definition 3.1. A function $u \in C([0,b];X)$ is called a mild solution of the equation (1.1) if

$$u(t) = T_\alpha(t)g(u) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,u(s))ds. \quad (3.14)$$

To prove the main results, we assume the following conditions:

(H_1) The C_0 semigroup $T(t)$ generated by A is continuous in the uniform operator topology for $t > 0$, and there exists $M \geq 1$ such that

$$\sup_{t \geq 0} \|T(t)\| \leq M. \tag{3.15}$$

(H_2) $f : [0, b] \times X \to X$ satisfies the Carathéodory condition, i.e. $f(\cdot, x)$ is measurable for all $x \in X$, and $f(t, \cdot)$ is continuous for a.e. $t \in [0, b]$.

(H_3) There exist a constant $p \in (0, \alpha)$ and a function $a \in L^{\frac{1}{p}}([0, b]; \mathbb{R}_+)$ and a real number $c > 0$ such that

$$\|f(t, u)\| \leq a(t)\|u\| + c, \;\; t \in [0, b], \;\; u \in X, \tag{3.16}$$

and

$$\sup_{t \in [0,b]} J_t^{\alpha} a(t) < \frac{1}{M}. \tag{3.17}$$

(H_4) There exists a function $L \in L^{\frac{1}{p}}([0, b]; \mathbb{R}_+)$ such that

$$\beta(f(t, B(t)) \leq L(t)\beta(B(t)) \tag{3.18}$$

for a.e. $t \in [0, b]$ and every bounded $B \subset C([0, b]; X)$.

(H_5) $g : C([0, b]; X) \to X$ is continuous and compact.

THEOREM 3.1. *Assume that the conditions (H_1), (H_2), (H_3), (H_4) and (H_5) are satisfied. Then the nonlocal problem (1.1) has at least one mild solution on $[0, b]$.*

P r o o f. Define the mapping $F : C([0, b]; X) \to C([0, b]; X)$ by

$$(Fu)(t) = T_{\alpha}(t)g(u) + \int_0^t (t - s)^{\alpha-1} S_{\alpha}(t - s) f(s, u(s)) ds. \tag{3.19}$$

From (3.3), (2.11) and (3.15), it follows that

$$\|T_{\alpha}(t)\| \leq M, \;\; t \geq 0. \tag{3.20}$$

From (3.12) and (2.13), it follows that

$$\|S_{\alpha}(t)\| \leq \alpha M \int_0^{\infty} r M_{\alpha}(r) dr \leq \frac{\alpha M}{\Gamma(1 + \alpha)} = \frac{M}{\Gamma(\alpha)}, \quad t \geq 0. \tag{3.21}$$

Let $\{u_n\}_{n=1}^{\infty} \subset C([0, b]; X)$ be a sequence in $C([0, b]; X)$ with $\lim_{n \to \infty} u_n = u$ in $C([0, b]; X)$. By (3.19), (3.20) and (3.21), we have

$$\begin{aligned} \|Fu_n - Fu\| &\leq \|T_{\alpha}(t)\| g(u_n) - g(u)\| \\ &\quad + \int_0^t (t - s)^{\alpha-1} \|S_{\alpha}(t - s)\| f(s, u_n(s) - f(s, u(s))\| ds \qquad (3.22) \\ &\leq M\|g(u_n) - g(u)\| + \frac{\alpha M}{\Gamma(1 + \alpha)} \int_0^t (t - s)^{\alpha-1} \|f(s, u_n(s) - f(s, u(s))\| ds. \end{aligned}$$

Then by the continuity of g, the continuity of f with respect to the second argument and dominated convergence theorem, $\lim_{n\to\infty} Fu_n = Fu$, that is, F is continuous on $C([0,b];X)$.

It is clear that the fixed point of F is the mild solution of (1.1). Let $r > (M\|g(u)\| + \frac{Mcb^\alpha}{\Gamma(1+\alpha)})(1 - M\sup_{t\in[0,b]} J_t^\alpha a(t))^{-1}$. Set $B_r = \{x \in X : \|x\| \le r\}$, $W_r = \{u \in C([0,b];X) : u(t) \in B_r, t \in [0,b]\}$. We will prove that $FW_r \subset W_r$. For any $u \in W_r$, by (H_1), (3.16) and (3.17),

$$\begin{aligned}
\|(Fu)(t)\| &\le \|T_\alpha(t)g(u)\| + \|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s,u(s))ds\| \\
&\le M\|g(u)\| + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[a(s)\|u\| + c]ds \\
&\le M\|g(u)\| + rM \sup_{t\in[0,b]} J_t^\alpha a(t) + \frac{cb^\alpha M}{\alpha\Gamma(\alpha)} \le r,
\end{aligned}$$

then $FW_r \subset W_r$. Next, we show that FW_r is equicontinuous on $[0,b]$. For $u \in W_r$, $0 \le t_1 \le t_2 \le b$, by (3.19), we have

$$\begin{aligned}
&\|(Fu)(t_2) - (Fu)(t_1)\| \\
&= \|T_\alpha(t_2)g(u) + \int_0^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s) f(s,u(s))ds \\
&\quad - T_\alpha(t_1)g(u) + \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_1-s) f(s,u(s))ds\| \\
&\le \|(T_\alpha(t_2) - T_\alpha(t_1))g(u)\| \\
&\quad + \|\int_0^{t_1} (t_2-s)^{\alpha-1}(S_\alpha(t_2-s) - S_\alpha(t_1-s)) f(s,u(s))ds\| \\
&\quad + \|\int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})(S_\alpha(t_1-s) f(s,u(s))ds\| \\
&\quad + \|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s) f(s,u(s)ds\| \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \|(T_\alpha(t_2) - T_\alpha(t_1))g(u)\|, \\
I_2 &= \|\int_0^{t_1} (t_2-s)^{\alpha-1}(S_\alpha(t_2-s) - S_\alpha(t_1-s)) f(s,u(s))ds\|,
\end{aligned}$$

$$I_3 = \| \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})(S_\alpha(t_1 - s)f(s, u(s))ds\|,$$
$$I_4 = \| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha(t_2 - s)f(s, u(s)ds\|.$$

By (3.3),(2.11) and dominated convergence theorem, it is easy to show that $T_\alpha(t)$ is strongly continuous for $t \geq 0$. Since $T_\alpha(t)g(u)$ is continuous in t on $[0, b]$, then $T_\alpha(t)g(u)$ is uniformly continuous in t on $[0, b]$, hence

$$I_1 \to 0 \quad \text{as} \quad t_2 \to t_1. \tag{3.23}$$

From (H_1), (3.12), (2.13) and dominated convergence theorem, it follows that $S_\alpha(t)$ is continuous in the uniform operator topology for $t > 0$. This together with (H_2) imply that

$$I_2 \to 0 \quad \text{as} \quad t_2 \to t_1. \tag{3.24}$$

By (3.21) and (3.16), we obtain

$$\begin{aligned} I_3 &\leq \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})\|S_\alpha(t_1 - s)f(s, u(s)\|ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^{t_1} (a(s)\|u\| + c)((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})ds. \end{aligned} \tag{3.25}$$

From (3.25), we have

$$I_3 \to 0 \quad \text{as} \quad t_2 \to t_1. \tag{3.26}$$

By (3.21), we see that

$$\begin{aligned} I_4 &\leq \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|S_\alpha(t_2 - s)f(s, u(s)\|ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_{t_1}^{t_2} (a(s)\|u\| + c)(t_2 - s)^{\alpha-1} ds \\ &\leq \frac{M\|u\|}{\Gamma(\alpha)} \int_{t_1}^{t_2} a(s)(t_2 - s)^{\alpha-1} ds + \frac{Mc(t_2 - t_1)^\alpha}{\Gamma(\alpha)}. \end{aligned} \tag{3.27}$$

For $0 < p < \alpha < 1$, by the Hölder inequality,

$$\begin{aligned} &\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} a(s) ds \\ &\leq \left(\int_{t_1}^{t_2} (t_2 - s)^{\frac{\alpha-1}{1-p}} ds \right)^{1-p} \cdot \left(\int_{t_1}^{t_2} a(s)^{\frac{1}{p}} ds \right)^p \\ &\leq (t_2 - t_1)^{\alpha-p} \left(\int_{t_1}^{t_2} a(s)^{\frac{1}{p}} ds \right)^p. \end{aligned} \tag{3.28}$$

From (3.27) and (3.28), we have

$$I_4 \to 0 \quad \text{as} \quad t_2 \to t_1. \tag{3.29}$$

By (3.23), (3.24), (3.26) and (3.29), FW_r is equicontinuous on $[0, b]$.

Let $Q = \overline{co}FW_r$. It is clear that F maps Q into itself and Q is equicontinuous on $[0, b]$. In the following, we prove that $F : Q \to Q$ is a convex-power condensing operator. It is easy to see that F is continuous. Taking $u_0 \in Q$, we will show that there exists a positive integer n_0 such that for any non-relatively compact set $B \subset Q$, $\beta(F^{(n,u_0)}(B)) < \beta(B)$. For every $B \subset Q$, it is easy to show that $F^{(n,u_0)}(B) \subset W_r$ is equicontinuous. Then, $\beta(F^{(n,u_0)}(B)) = \max_{t\in[0,b]} \beta((F^{(n,u_0)}(B))(t))$, $n \in \mathbb{N}$.

By (H_4), Lemma 2.1, Lemma 2.5, Lemma 2.4, Lemma 2.2 and (3.21), we have for every non-relatively compact set $B \subset Q$ and $\varepsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset B$, such that

$$\begin{aligned}
&\beta((F^{(1,u_0)}(B))(t)) = \beta((F(B))(t)) \\
&\le \beta(T_\alpha(t)g(B)) + \beta\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, B(s))ds\right) \\
&= \beta\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, B(s))ds\right) \\
&\le 2\beta\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, \{u_n\}_{n=1}^{\infty})ds\right) + \varepsilon \\
&\le 4\int_0^t (t-s)^{\alpha-1} \beta(S_\alpha(t-s) f(s, \{u_n\}_{n=1}^{\infty}))ds + \varepsilon \\
&\le \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta(f(s, \{u_n\}_{n=1}^{\infty}))ds + \varepsilon \\
&\le \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s)\beta(\{u_n\}_{n=1}^{\infty})ds + \varepsilon \\
&\le \frac{4M\beta(B)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s)ds + \varepsilon.
\end{aligned} \tag{3.30}$$

From (H4), $L \in L^{\frac{1}{p}}([0, b]; \mathbb{R}_+)$, where $0 < p < \alpha < 1$. Then for every $\eta > 0$ there exists a function $y \in C([0, b]; \mathbb{R}_+)$ such that,

$$\left(\int_0^t |L(s) - y(s)|^{\frac{1}{p}} ds\right)^p < \eta, \quad t \in [0, b].$$

Therefore, by the Hölder inequality,

$$\begin{aligned}
&\int_0^t (t-s)^{\alpha-1}|L(s)-y(s)|ds \\
&\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-p}}ds\right)^{1-p} \cdot \left(\int_0^t |L(s)-y(s)|^{\frac{1}{p}}ds\right)^p \\
&\le t^{\alpha-p}\left(\int_0^t |L(s)-y(s)|^{\frac{1}{p}}ds\right)^p \le b^{\alpha-p}\eta. \qquad (3.31)
\end{aligned}$$

Since $y \in C([0,b];\mathbb{R}_+)$, there exists a constant M_1 such that $|y(t)| \le M_1$. This together with (3.30) and (3.31) yield

$$\beta((F^{(1,u_0)}(B))(t)) \le \frac{4M\beta(B)}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}|L(s)-y(s)|ds$$

$$+\int_0^t (t-s)^{\alpha-1}|y(s)|ds\Big) + \varepsilon \le \frac{4M\beta(B)}{\Gamma(\alpha)}(b^{\alpha-p}\eta + \frac{M_1 t^\alpha}{\alpha}) + \varepsilon. \qquad (3.32)$$

Since $\varepsilon > 0$ is arbitrary, then it follows from (3.32) that

$$\beta((F^{(1,u_0)}(B))(t)) \le \frac{4M\beta(B)}{\Gamma(\alpha)}(b^{\alpha-p}\eta + \frac{M_1 t^\alpha}{\alpha})$$

$$= \left(\frac{4Mb^{\alpha-p}\eta}{\Gamma(\alpha)} + \frac{4MM_1 t^\alpha}{\Gamma(\alpha+1)}\right)\beta(B) = \left(d + \frac{ht^\alpha}{\Gamma(\alpha+1)}\right)\beta(B), \qquad (3.33)$$

where $d = 4Mb^{\alpha-p}\eta((\Gamma(\alpha))^{-1}$, $h = 4MM_1$.

From one hand, we have

$$\begin{aligned}
&\beta((F^{(2,u_0)}(B))(t)) = \beta(F((\overline{co}\{F^{(1,u_0)}B, u_0\})(t))) \\
&\quad \le \beta((T_\alpha(t)g(\overline{co}\{F^{(1,u_0)}(B))(s), u_0(s)\}) \\
&\quad + \beta\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,\overline{co}\{((F^{(1,u_0)}(B))(s)), u_0(s)\})ds\right) \\
&= \beta\left(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s,\overline{co}\{((F^{(1,u_0)}(B))(s)), u_0(s)\})ds\right). \quad (3.34)
\end{aligned}$$

On the other hand, by Lemma 2.5 and Lemma 2.4 and (3.21), there is a sequence $\{v_n\}_{n=1}^\infty \subset \overline{co}\{(F^{(1,u_0)}B, u_0\}$ such that

$$
\begin{aligned}
&\beta\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s, \overline{co}\{(F^{(1,u_0)}(B))(s), u_0(s)\}) ds\right) \\
&\leq 2\beta\left(\int_0^t (t-s)^{\alpha-1} (S_\alpha(t-s) f(s, \{v_n\}_{n=1}^\infty)) ds\right) + \varepsilon \\
&\leq 4\int_0^t (t-s)^{\alpha-1} \beta(S_\alpha(t-s) f(s, \{v_n\}_{n=1}^\infty)) ds + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta(f(s, \{v_n\}_{n=1}^\infty)) ds + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s) \beta(\{v_n\}_{n=1}^\infty) ds + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s) \beta\{\overline{co}\{((F^{(1,u_0)}(B))(s)), u_0(s)\} ds + \varepsilon \\
&\leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L(s) \beta((F^{(1,u_0)}(B))(s)) ds + \varepsilon. \qquad (3.35)
\end{aligned}
$$

By (3.33), (3.34), (3.35), (3.31), it follows that

$$
\begin{aligned}
\beta((F^{(2,u_0)}(B))(t)) &\leq \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |L(s)-y(s)| \beta((F^{(1,u_0)}(B))(s)) ds \\
&\quad + \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s)| \beta((F^{(1,u_0)}(B))(s)) ds + \varepsilon \\
&\leq \frac{4M\beta(B)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |L(s)-y(s)| (d + \frac{hs^\alpha}{\Gamma(\alpha+1)}) ds \\
&\quad + \frac{4M\beta(B)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |y(s)| (d + \frac{hs^\alpha}{\Gamma(\alpha+1)}) ds + \varepsilon \\
&\leq \frac{4M\beta(B)}{\Gamma(\alpha)} (d + \frac{ht^\alpha}{\Gamma(\alpha+1)}) \int_0^t (t-s)^{\alpha-1} |L(s)-y(s)| ds \\
&\quad + \frac{4MM_1\beta(B)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (d + \frac{hs^\alpha}{\Gamma(\alpha+1)}) ds + \varepsilon \\
&\leq d(d + \frac{ht^\alpha}{\Gamma(\alpha+1)}) \beta(B) + \frac{dht^\alpha}{\Gamma(\alpha+1)} \beta(B) \\
&\quad + \frac{h^2\beta(B)}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} s^\alpha ds. \qquad (3.36)
\end{aligned}
$$

It is easy to see that

$$\int_0^t (t-s)^{\alpha-1}s^\alpha ds = \int_0^1 (t-t\tau)^{\alpha-1}(t\tau)^\alpha t d\tau = t^{2\alpha}\int_0^1 (1-\tau)^{\alpha-1}\tau^\alpha d\tau. \tag{3.37}$$

By the Beta function,

$$B(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}d\tau, \quad x>0,\ y>0, \tag{3.38}$$

and

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{3.39}$$

we have

$$\int_0^t (t-s)^{\alpha-1}s^\alpha ds = t^{2\alpha}B(\alpha,\alpha+1) = \frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}. \tag{3.40}$$

This, together with (3.36) yield

$$\begin{aligned}\beta((F^{(2,u_0)}(B))(t)) \le{}& d(d+\frac{ht^\alpha}{\Gamma(\alpha+1)})\beta(B) + \frac{dht^\alpha}{\Gamma(\alpha+1)}\beta(B)\\ &+ \frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}\beta(B)+\varepsilon. \end{aligned}\tag{3.41}$$

Since $\varepsilon > 0$ is arbitrary, then it follows from (3.41) that

$$\beta((F^{(2,u_0)}(B))(t)) \le \left(d^2 + \frac{2dht^\alpha}{\Gamma(\alpha+1)} + \frac{(ht^\alpha)^2}{\Gamma(2\alpha+1)}\right)\beta(B).$$

Proceeding by induction, we obtain

$$\begin{aligned}\beta((F^{(n,u_0)}(B))(t)) \le{}& (d^n + \frac{C_n^1 d^{n-1}ht^\alpha}{\Gamma(\alpha+1)} + \frac{C_n^2 d^{n-2}(ht^\alpha)^2}{\Gamma(2\alpha+1)}\\ &+\cdots+\frac{(ht^\alpha)^n}{\Gamma(n\alpha+1)})\beta(B). \end{aligned}\tag{3.42}$$

Then,

$$\begin{aligned}\beta((F^{(n,u_0)}(B)) &= \sup_{t\in[0,b]}\beta((F^{(n,u_0)}(B))(t))\\ \le (d^n + \frac{C_n^1 d^{n-1}hb^\alpha}{\Gamma(\alpha+1)} &+ \frac{C_n^2 d^{n-2}(hb^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{(hb^\alpha)^n}{\Gamma(n\alpha+1)})\beta(B), \quad n\in\mathbb{N}. \end{aligned}\tag{3.43}$$

By Proposition 1.1 in [6], we have

$$\Gamma(1+x)^{2/x} \ge \frac{1}{e^2}(x+1)(x+2), \quad x\ge 1. \tag{3.44}$$

Since $0<\alpha<1$, then there exists $k_1\in\mathbb{N}$, such that $k_1\alpha\ge 1$ and $0<(k_1-1)\alpha<1$. By (3.44), for $n\ge k_1$,

$$\Gamma(1+n\alpha) \geq \frac{(n\alpha+1)^{n\alpha}}{e^{n\alpha}}. \tag{3.45}$$

Since $d = 4Mb^{\alpha-p}\eta((\Gamma(\alpha))^{-1}$ and $\eta > 0$ is arbitrary, there exists a positive number m such that $0 < m + d < 1$. And there exists $k_2 \in \mathbb{N}$, such that for $n \geq k_2$,

$$0 < \frac{hb^\alpha e^\alpha}{(n\alpha+1)^\alpha} < m. \tag{3.46}$$

Set

$$S_n = d^n + \frac{C_n^1 d^{n-1} hb^\alpha}{\Gamma(\alpha+1)} + \frac{C_n^2 d^{n-2} (hb^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{(hb^\alpha)^n}{\Gamma(n\alpha+1)}. \tag{3.47}$$

Let $k = \max\{k_1, k_2\}$, then for $n \geq k$,

$$\begin{aligned} S_n = d^n &+ \frac{C_n^1 d^{n-1} hb^\alpha}{\Gamma(\alpha+1)} + \frac{C_n^2 d^{n-2} (hb^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{C_n^k d^{n-k} (hb^\alpha)^k}{\Gamma(k\alpha+1)} \\ &+ \frac{C_n^{k+1} d^{n-k-1} (hb^\alpha)^{k+1}}{\Gamma((k+1)\alpha+1)} + \frac{C_n^{k+2} d^{n-k-2} (hb^\alpha)^{k+2}}{\Gamma((k+2)\alpha+1)} \\ &+ \cdots + \frac{C_n^{n-1} d (hb^\alpha)^{n-1}}{\Gamma((n-1)\alpha+1)} + \frac{(hb^\alpha)^n}{\Gamma(n\alpha+1)}. \end{aligned} \tag{3.48}$$

Since d can be sufficiently small, it is easy to prove $\lim_{n\to\infty} C_n^i d^{n-i} = 0$, $i = 1, 2, \cdots, k$. Thus,

$$d^n + \frac{C_n^1 d^{n-1} hb^\alpha}{\Gamma(\alpha+1)} + \frac{C_n^2 d^{n-2} (hb^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{C_n^k d^{n-k} (hb^\alpha)^k}{\Gamma(k\alpha+1)} \to 0 \tag{3.49}$$

as $n \to \infty$.

On the other hand, by (3.45) and (3.46), we have

$$\begin{aligned} &\frac{C_n^{k+1} d^{n-k-1} (hb^\alpha)^{k+1}}{\Gamma((k+1)\alpha+1)} + \frac{C_n^{k+2} d^{n-k-2} (hb^\alpha)^{k+2}}{\Gamma((k+2)\alpha+1)} + \cdots + \frac{C_n^{n-1} d (hb^\alpha)^{n-1}}{\Gamma((n-1)\alpha+1)} \\ &\quad + \frac{(hb^\alpha)^n}{\Gamma(n\alpha+1)} \\ &\leq C_n^{k+1} d^{n-k-1} m^{k+1} + C_n^{k+2} d^{n-k-2} m^{k+2} + \cdots + C_n^{n-1} d m^{n-1} + m^n \\ &\leq (m+d)^n \to 0 \end{aligned} \tag{3.50}$$

as $n \to \infty$. By (3.49) and (3.50), we have $\lim_{n\to\infty} S_n = 0$. This together with (3.43) imply that there exists a sufficiently large positive integer n_0 such that

$$\left(d^{n_0} + \frac{C_{n_0}^1 d^{n_0-1} hb^\alpha}{\Gamma(\alpha+1)} + \frac{C_{n_0}^2 d^{n_0-2} (hb^\alpha)^2}{\Gamma(2\alpha+1)} + \cdots + \frac{(hb^\alpha)^{n_0}}{\Gamma(n_0\alpha+1)} \right) < 1. \tag{3.51}$$

This shows that $F : Q \to Q$ is a convex-power condensing operator. From Lemma 2.6, it follows that F has at least one fixed point. This fixed point is the desired mild solution of nonlocal problem (1.1).

4. Example

As an application, we consider the following nonlinear system of the form

$$\begin{cases} {}^C D_t^{\frac{1}{3}} u(t,x) = -A(x,D)u(t,x) + \omega(t,u(t,x)),\ t \in (0,8],\ x \in \Omega, \\ D^\gamma u(t,x) = 0,\ t \in [0,b], x \in \partial\Omega \text{ for } |\gamma| \le m, \\ u(0,x) = \displaystyle\int_\Omega \int_0^b h(t,x,y,u(t,y)dtdy,\ x \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $A(x,D) = \sum_{\gamma \le 2m} a_\gamma(x) D^\gamma$ is strongly elliptic in Ω.

Let $X = L^2(\Omega)$. We define $A : D(A) \subset X \to X$ by $Au = A(x,D)u$ for every $u \in D(A)$, where

$$D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega).$$

From Theorem 7.2.7 in [31], it follows that the operator $-A$ is the infinitesimal generator of an analytic and contraction semigroup of operators on $X = L^2(\Omega)$. Hence this semigroup is continuous in the uniform operator topology for $t > 0$.

We define $f : [0,b] \times X \to X$ and $g : C([0,b]; X) \to X$ by

$$f(t,u)(x) = \omega(t,u(t,x)) = t^{-\frac{2}{3}} \arctan u(t),\ x \in \Omega.$$

$$g(u)(x) = \int_\Omega \int_0^b h(t,x,y,u(t,y)dtdy,\ x \in \Omega.$$

It is easy to see that $f : (0,8] \times X \to X$ satisfies the Carathéodory condition and for $u, v \in X$,

$$\|f(t,u) - f(t,v)\| \le t^{-\frac{2}{3}} \|u - v\|.$$

We assume that:

(i) $h : [0,8] \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition.

(ii) $|h(t,x,y,r) - h(t,\overline{x},y,r)| \le \phi_k(t,x,\overline{x},y)$ for $(t,x,y), (t,\overline{x},y) \in [0,8] \times \Omega \times \Omega$ and $|r| \le k$, where $\phi_k \in L^1([0,8] \times \Omega \times \Omega \times \Omega, \mathbb{R}_+)$ satisfies

$$\lim_{\overline{x} \to x} \int_\Omega \int_0^8 \phi_k(t,x,\overline{x},y)dtdy = 0$$

uniformly for $x \in \Omega$.

(iii) $|h(t,x,y,r)| \leq \frac{|r|}{bm(\Omega)} + \psi(t,x,y)$, where $m(\Omega)$ denotes the Lebesgue measure of Ω in $\mathbb{R}^n$, $r \in \mathbb{R}$ and $\int_0^8 \int_{\Omega\times\Omega} \phi^2(t,x,y)dxdydt < \infty$.

It is clear that f satisfies the hypotheses (H_2) and it is easy to show that f satisfies the hypotheses (H_3). According to the arguments in [29], it is easy to see that g satisfies the hypothesis (H_4). Noting that (4.1) can be rewritten as problem (1.1). By Theorem 3.1, we conclude that the problem (4.1) has at least one mild solution $u \in C([0,8]; L^2(\Omega))$.

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