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SURVEY PAPER

UNIQUENESS OF POSITIVE SOLUTIONS OF FRACTIONAL BOUNDARY VALUE PROBLEMS WITH NON-HOMOGENEOUS INTEGRAL BOUNDARY CONDITIONS

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Abstract

The authors study a type of nonlinear fractional boundary value problem with non-homogeneous integral boundary conditions. The existence and uniqueness of positive solutions are discussed. An example is given as the application of the results.

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1. Introduction

In this paper, we study the boundary value problem (BVP) consisting of the nonlinear fractional differential equation

$$-D_{0+}^{\alpha}u = F(t,u),\ 0 < t < 1, \tag{1.1}$$

and the integral boundary condition (BC)

$$u(0) = 0,\ u(1) - aI_{0+}^{\alpha}u(1) = b, \tag{1.2}$$

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where $1 < \alpha \le 2$, $0 \le a < \Gamma(\alpha+1)$, $b \in \mathbb{R}_+$ with $\mathbb{R}_+ = [0, \infty)$, $b = 0$ if $a = 0$, and Γ is the Gamma function. Here, $D^\alpha_{0+}h$ is the α-th Riemann-Liouville fractional derivative of the function $h : [0, 1] \to \mathbb{R}$ as defined by

$$D^\alpha_{0+}h(t) = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-s)^{l-\alpha-1} h(s)ds,\ l \in \mathbb{N} \text{ and } \alpha \le l < \alpha+1, \tag{1.3}$$

provided the right-hand side exists. Also, $I^\alpha_{0+}h$ is the α-th Riemann-Liouville integral of h defined by

$$I^\alpha_{0+}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} h(s)ds. \tag{1.4}$$

Fractional differential equations have attracted extensive attention as they can be applied in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc., can be modeled as fractional differential equations. We refer the reader to the monographs [21, 24, 34, 36], the papers [9, 37], and the references therein for specific applications. In addition to BVPs for ordinary differential equations, fractional differential inclusions [6, 7, 8], fractional integro-differential equations [2], fractional partial differential equations [3], and the numerical solution of such problems [12] are gaining increasing attention in the literature. For general results and background on the fractional calculus, we refer the reader to [30, 31, 32].

Fixed point theory on cones is an important tool in the study of nonlinear BVPs. The existence of positive solutions of a BVP can be proved by constructing an associated operator and finding the fixed point of the operator. This idea has been widely applied to BVPs of integer order (for example, [4, 11, 13, 15, 18, 22, 25, 27, 28, 29, 33, 35]), as well as to the study of fractional BVPs ([1, 5, 14, 17, 20, 23, 26, 39, 41, 43]). The construction of the associated operators is a key step in this approach. Generally, the operator is constructed based on the Green's function associated with the BVP, and the properties of such an operator rely heavily on those of this Green's function.

Bai and Lü [5] proved that $G_0 : [0, 1] \times [0, 1] \to \mathbb{R}_+$ defined by

$$G_0(t, s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{1.5}$$

is the Green's function for the BVP

$$\begin{cases} -D^\alpha_{0+}u = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{1.6}$$

Based on this, they obtained results on the existence of one or three positive solutions of the BVP

$$\begin{cases} -D_{0+}^{\alpha} u = f(t,u), \ 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.7}$$

using the operator T defined by

$$(Tu)(t) = \int_0^1 G_0(t,s) f(s,u(s)), \ u \in C[0,1], \tag{1.8}$$

the Krasnosel'skii fixed point theorem, and the Leggett-Williams fixed point theorem. Jiang and Yuan [23] obtained different conditions from those in [5] for BVP (1.7) to have one positive solution, and they derived conditions for it to have two positive solutions by using the same operator but different properties of the Green's function $G_0(t,s)$.

The present authors [17] considered the fractional BVP with integral BC

$$\begin{cases} -D_{0+}^{\alpha} u + au = w(t) f(t,u), \ 0 < t < 1, \\ u(0) = 0, \ u(1) = a I_{0+}^{\alpha} u(1), \end{cases} \tag{1.9}$$

and proved the following result; see [17, Theorem 2.1] for the details. For convenience, we let $0^0 = 1$.

Lemma **1.1**. *Let G_n, $n = 1, 2, \ldots$, and G be defined as follows:*

$$G_n(t,s) = \int_0^1 \ldots \int_0^1 t^{n\alpha} \Pi_{i=1}^n [\tau_i^{(i-1)\alpha} (1-\tau_i)^{\alpha-1}] G_0(t \Pi_{i=1}^n \tau_i, s) d\tau_1 \ldots d\tau_n, \tag{1.10}$$

$n = 1, 2 \ldots$, and

$$G(t,s) = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma^n(\alpha)} G_n(t,s). \tag{1.11}$$

Then G is a uniformly convergent series for $(t,s) \in [0,1] \times [0,1]$. Moreover, G is the Green's function for the BVP

$$\begin{cases} -D_{0+}^{\alpha} u + au = 0, \\ u(0) = 0, \ u(1) = a I_{0+}^{\alpha} u(1). \end{cases} \tag{1.12}$$

Replacing G_0 and $f(s,u(s))$ in (1.8) by G and $w(s)f(s,u(s))$, we obtained an operator associated with BVP (1.9). Based on it, we established the existence of any odd number or an infinite number of positive solutions of BVP (1.9).

In this paper, we consider BVP (1.1), (1.2). An associated operator is constructed based on G defined by (1.11). By further analyzing properties of G and using mixed monotone operator theory, a result on the existence and uniqueness of positive solutions is proved. By *positive solutions* we mean a function $u \in C[0,1]$ satisfying BVP (1.1), (1.2) and $u(t) > 0$ for $t \in (0,1)$. An example is also given to demonstrate our result. Note that the BCs of (1.7) and (1.9) are special cases of BC (1.2) when $a = b = 0$ and $b = 0$ respectively. So our result can be applied to BVPs (1.7) and (1.9).

This paper is organized as follows: after this introduction, the main result and example are presented in Section **2**. All the proofs are given in Section **3**.

2. Main Results

In this paper, we let $F(t,u) = f(t,u,u) + r(t,u) - au$, where $f:\ [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $r : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, and a is the same as in BC (1.2). We assume throughout this paper, and without further mention, that the following conditions hold:

(H1) $f(t,\cdot,y)$ is increasing for any fixed t and y on $\mathbb{R}_+$, and $f(t,x,\cdot)$ is decreasing for any fixed t and x on $\mathbb{R}_+$;

(H2) there exists $\theta \in (0,1)$ such that

$$f(t, \kappa x, \kappa^{-1} y) \geq \kappa^{\theta} f(t,x,y)$$

for $t \in [0,1]$, $\kappa \in (0,1)$, $x \in \mathbb{R}_+$, and $y \in \mathbb{R}_+$;

(H3) $r(t,\cdot)$ is increasing for any fixed t, and $r(t,0) \not\equiv 0$ on $[0,1]$;

(H4) $r(t,\kappa x) \geq \kappa r(t,x)$ for $t \in [0,1]$, $\kappa \in (0,1)$, and $x \in \mathbb{R}_+$;

(H5) there exists $\eta > 0$ such that

$$f(t,x,y) \geq \eta\,(r(t,x) + \delta)$$

for $t \in [0,1]$, $x \in \mathbb{R}_+$, and $y \in \mathbb{R}_+$, where

$$\delta = \begin{cases} 0, & a = 0, \\ \dfrac{b(2\underline{\phi} + 1)}{\underline{\phi} \int_0^1 [(\alpha-1)(1-s)^{\alpha-1} s/\Gamma(\alpha)] ds}, & a > 0, \end{cases} \tag{2.1}$$

with

$$\underline{\phi} = \min_{t \in [0,1]} \left[1 - t + \frac{a\Gamma(\alpha) t^{\alpha}}{\Gamma(2\alpha)} \left(1 - \frac{t}{2}\right) \right] > 0. \tag{2.2}$$

Remark **2.1**. We would like to make a few comments on the form of the nonlinear term f above. The analysis in this paper mainly relies on mixed monotone operator theory. To apply such theory, one alternative way in the literature is to write the nonlinearity as $f(t,x)$ and assume that $f(t,x)$ can be decomposed into $f(t,x) = g(t,x) + h(t,x)$, where $g : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing in the second argument, h :

$[0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nonincreasing in the second argument, and there exists $\theta \in (0,1)$ such that

$$g(t, \kappa x) \geq \kappa^\theta g(t,x) \tag{2.3}$$

and

$$h(t, \kappa^{-1} x) \geq \kappa^\theta h(t,x) \tag{2.4}$$

for $t \in [0,1]$, $\kappa \in (0,1)$, and $x \geq 0$. The reader may refer to [10] for a related discussion.

Here, the nonlinear term f is written as a function of three arguments. Then, to apply mixed monotone operator theory, we need to assume that the conditions (H1) and (H2) above are satisfied. By writing f this way, a larger class of functions can be covered. For instance, if $f(t,x,y) = \sqrt[3]{x}/\sqrt{y+1}$, then, $f(t,x,x)$ cannot be decomposed into a summation of two functions g and h satisfying (2.3) and (2.4), but $f(t,x,y)$ does satisfy (H1) and (H2) with $\theta = 5/6$.

Let $X = C[0,1]$ be the Banach space of continuous functions on $[0,1]$ with the standard maximum norm $\|\cdot\|$. Define a cone $P \subset X$ and a function $w:\ [0,1] \to \mathbb{R}$ by

$$P = \{u \in X \mid u(0) = 0,\ u(t) \geq 0 \text{ on } [0,1]\}, \tag{2.5}$$

and

$$w(t) = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma^n(\alpha)} w_n(t), \tag{2.6}$$

where

$$w_n(t) = \frac{\Gamma^{n+1}(\alpha) t^{(n+1)\alpha - 1}}{\Gamma((n+1)\alpha)} \left(1 - \frac{t}{n+1}\right),\ n \in \mathbb{N}_0, \tag{2.7}$$

and $\mathbb{N}_0 = \{0,1,2,\dots\}$ is the set of all nonnegative integers. Then we have the following lemma.

Lemma **2.1**. *The series w is uniformly convergent on $[0,1]$ and $w \in P$. Moreover, $w(t) > 0$ for $t \in (0,1)$.*

Let G be defined by (1.11) and w be defined by (2.6). Define a series c and a set $P_w \subset P$ by

$$c(t) = \sum_{n=0}^{\infty} \frac{a^n \Gamma(\alpha)}{\Gamma((n+1)\alpha)} t^{(n+1)\alpha - 1}, \quad t \in [0,1], \tag{2.8}$$

and

$$P_w = \left\{u \in P \mid \underline{d}_u w(t) \leq u(t) \leq \overline{d}_u w(t) \text{ on } [0,1] \text{ for some } \overline{d}_u \geq \underline{d}_u > 0\right\}, \tag{2.9}$$

where $\overline{d}_u$ and $\underline{d}_u$ depend on u. By Lemma **3.6** below, c is uniformly convergent on $[0,1]$. We then have the following theorem.

THEOREM **2.1**. *Assume that (H1)–(H5) hold. Then:*
(a) BVP (1.1), (1.2) has a unique positive solution $u(t)$ with $u \in P_w$;
(b) for any $u_0, v_0 \in P_w$, the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_{n+1}(t) = \int_0^1 G(t,s)[f(s,u_n(s),v_n(s)) + r(s,u_n(s))]ds + bc(t) \tag{2.10}$$

and

$$v_{n+1}(t) = \int_0^1 G(t,s)[f(s,v_n(s),u_n(s)) + r(s,v_n(s))]ds + bc(t) \tag{2.11}$$

for $n \in \mathbb{N}_0$ satisfy $\|u_n - u\| \to 0$ and $\|v_n - u\| \to 0$ as $n \to \infty$.

REMARK **2.2**. When $a = 0$ and $b \neq 0$, BVP (1.1), (1.2) has no solution in P_w. In fact, when $a = 0$, by (2.6) and (2.7), $w(1) = 0$. Hence $\varphi(1) = 0$ for any $\varphi \in P_w$. If u is a solution of BVP (1.1), (1.2), then $u(1) = b \neq 0$. Therefore, $u \notin P_w$.

To demonstrate the applicability of our result, let us consider the following example.

EXAMPLE **2.1**. Assume α, a, and b satisfy the conditions for BVP (1.1), (1.2), $\theta_1, \theta_2 \in (0,1)$, $m > 0$, and δ is defined by (2.1). Then the BVP

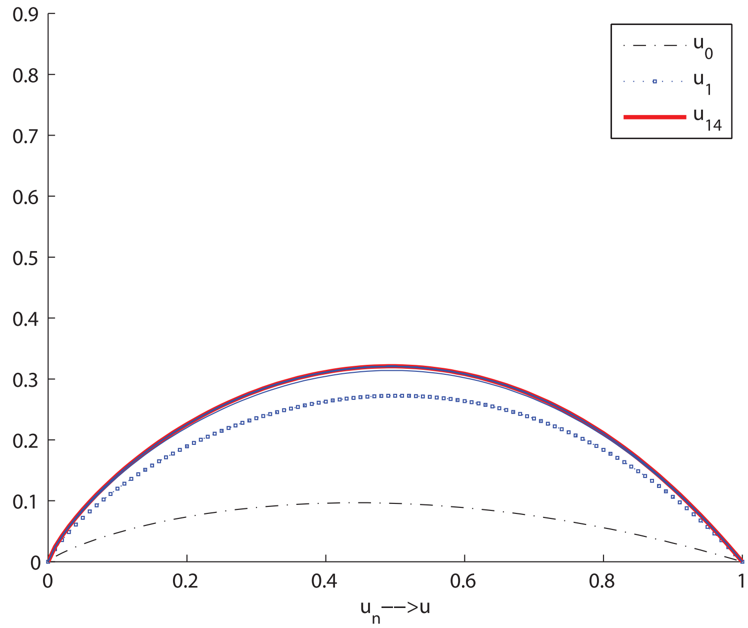
$$\begin{cases} -D_{0+}^{\alpha} u = t^2 + u^{\theta_1} + (u+m)^{-\theta_2} + \delta - au, \\ u(0) = 0, \;\; u(1) - aI_{0+}^{\alpha} u(1) = b, \end{cases} \tag{2.12}$$

has a unique positive solution. To see this, let

$$f(t,x,y) = \frac{t^2}{2} + \frac{x^{\theta_1}}{2} + (y+m)^{-\theta_2} + \delta, \text{ and } r(t,x) = \frac{t^2}{2} + \frac{x^{\theta_1}}{2}.$$

Clearly (H1)–(H5) hold when $\theta = \max\{\theta_1, \; \theta_2\}$ and $\eta = 1$. Hence, by Theorem **2.1** (a), BVP (2.12) has a unique positive solution.

As a special case of (2.12), let $a = b = 0$, $\alpha = 1.8$, $\theta_1 = 1/3$, $\theta_2 = 1/4$, $m = 1$, $u_0 = w/3$, and $v_0 = 3w$ with w defined by (2.6). The first 14 iterations of $\{u_n\}$ and $\{v_n\}$ given in (2.10) and (2.11) have been computed. Figure 2.1 shows that the distinction between u_n and v_n is relatively small after 3 iterations. This is consistent with Theorem **2.1** (b).



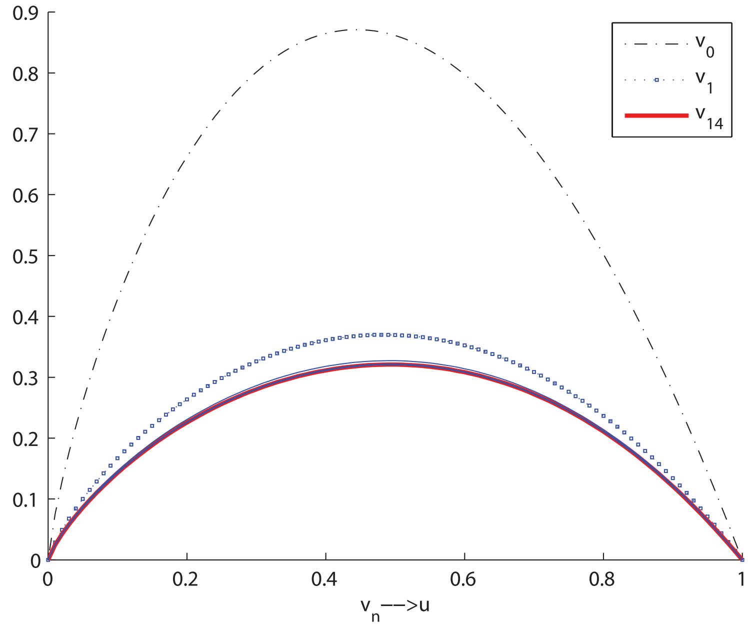


Figure 2.1. $u_n \to u$ and $v_n \to u$

3. Proofs

We first prove Lemma **2.1**.

Proof of Lemma **2.1**. From (2.7) and the monotonicity of the Gamma function, for $n \geq 1$ and $t \in [0,1]$, we have

$$\begin{aligned}\frac{a^n}{\Gamma^n(\alpha)} w_n(t) &\leq \frac{a^n \Gamma(\alpha) t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} < \frac{\Gamma^n(\alpha+1)\Gamma(\alpha) t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \\ &\leq \frac{\Gamma(\alpha)\Gamma^n(3)}{\Gamma(n+1)} = \frac{\Gamma(\alpha)\Gamma^n(3)}{n!}.\end{aligned}$$

Clearly, $\sum_{n=1}^{\infty}[\Gamma(\alpha)\Gamma^n(3)/n!]$ is convergent. By (2.6) and the Weierstrass M-test, w is uniformly convergent on $[0,1]$. It follows that $w \in C[0,1]$ and $w(0) = 0$. Moreover, for $n \geq 0$ and $t \in (0,1)$,

$$w_n(t) = \frac{\Gamma^{n+1}(\alpha) t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \left(1 - \frac{t}{n+1}\right) > \frac{\Gamma^{n+1}(\alpha) t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \left(1 - \frac{1}{n+1}\right).$$

Hence, $w(t) \geq 0$ on $[0,1]$ and $w(t) > 0$ on $(0,1)$. Clearly, $w \in P$. □

The next two lemmas provide lower and upper bounds for the functions G_0 and G given in (1.5) and (1.11).

LEMMA **3.1**. *For* $(t,s) \in [0,1] \times [0,1)$,

$$\frac{(\alpha-1)t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} \le G_0(t,s) \le \frac{t^{\alpha-1}(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha)}. \quad (3.1)$$

For $(t,s) \in (0,1)\times(0,1)$, (3.1) was obtained in [23, Theorem 1.1]. Using the fact that $G_0(0,\cdot) = G_0(1,\cdot) = G_0(\cdot,0) \equiv 0$, we see that (3.1) holds on $[0,1] \times [0,1)$.

LEMMA **3.2**. *Let* w *be defined by* (2.6). *For* $(t,s) \in [0,1] \times [0,1)$,

$$\frac{(\alpha-1)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} w(t) \le G(t,s) \le \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} w(t). \quad (3.2)$$

P r o o f. First, we show that for $n \in \mathbb{N}_0$ and $(t,s) \in [0,1] \times [0,1)$,

$$\frac{(\alpha-1)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} w_n(t) \le G_n(t,s) \le \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} w_n(t). \quad (3.3)$$

In fact, by Lemma **3.1** and (2.7), (3.3) is true for $n = 0$. For $n \ge 1$, set $\Pi = \Pi_{i=1}^n$, and then by (1.10) and Lemma **3.1**,

$$\begin{aligned}
&G_n(t,s)\\
&\le \int_0^1 \cdots \int_0^1 \frac{t^{n\alpha}\Pi[\tau_i^{(i-1)\alpha}(1-\tau_i)^{\alpha-1}](t\Pi\tau_i)^{\alpha-1}(1-t\Pi\tau_i)(1-s)^{\alpha-2}}{\Gamma(\alpha)} d\tau_1 \dots d\tau_n\\
&= \int_0^1 \cdots \int_0^1 \frac{t^{(n+1)\alpha-1}\Pi[\tau_i^{i\alpha-1}(1-\tau_i)^{\alpha-1}](1-t\Pi\tau_i)(1-s)^{\alpha-2}}{\Gamma(\alpha)} d\tau_1 \dots d\tau_n\\
&= \frac{t^{(n+1)\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \int_0^1 \cdots \int_0^1 \Pi[\tau_i^{i\alpha-1}(1-\tau_i)^{\alpha-1}](1-t\Pi\tau_i) d\tau_1 \dots d\tau_n\\
&= \frac{t^{(n+1)\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \left[\Pi B(i\alpha,\alpha) - t\Pi B(i\alpha+1,\alpha)\right], \quad (t,s) \in [0,1] \times [0,1),
\end{aligned}$$

where B is the beta function. Then by (2.7),

$$\begin{aligned}
G_n(t,s) \le& \frac{t^{(n+1)\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \left[\Pi \frac{\Gamma(i\alpha)\Gamma(\alpha)}{\Gamma((i+1)\alpha)} - t\Pi \frac{\Gamma(i\alpha+1)\Gamma(\alpha)}{\Gamma((i+1)\alpha+1)}\right]\\
=& \frac{t^{(n+1)\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \left[\frac{\Gamma^{n+1}(\alpha)}{\Gamma((n+1)\alpha)} - \frac{t\Gamma(\alpha+1)\Gamma^n(\alpha)}{\Gamma((n+1)\alpha+1)}\right]\\
=& \frac{t^{(n+1)\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \left[\frac{\Gamma^{n+1}(\alpha)}{\Gamma((n+1)\alpha)} - \frac{t\Gamma^{n+1}(\alpha)}{(n+1)\Gamma((n+1)\alpha)}\right]\\
=& \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} w_n(t).
\end{aligned}$$

Similarly, we can prove

$$G_n(t,s) \geq \frac{(\alpha-1)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} w_n(t).$$

Hence, (3.3) holds for $n \in \mathbb{N}_0$. By (1.11), (3.3), and (2.6), we see that (3.2) holds. □

The following lemmas on fractional integrals and inverse operators are needed to construct an operator associated with BVP (1.1), (1.2). We refer the reader to [5, Lemma 2.2], [21, Property 2.1 and Lemma 2.3], and [17, Lemma 3.2] for the details.

Lemma **3.3**. *Assume that $u \in C(0,1) \cap L(0,1)$ with an α-th fractional derivative that belongs to $C(0,1) \cap L(0,1)$. Then $I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$ for some C_1, $C_2 \in \mathbb{R}$.*

Lemma **3.4**. *If $\alpha > 0$, $\beta > 0$, and $n \in \mathbb{N}$, then*

$$(I^{\alpha}_{0})^n (t^{\beta-1}) = I^{n\alpha}_{0}(t^{\beta-1}) = \frac{\Gamma(\beta)}{\Gamma(\beta+n\alpha)} t^{\beta+n\alpha-1}.$$

Lemma **3.5**. *Let I^{α}_{0+} be defined by (1.4) and $\mathcal{I}$ be the identity operator. Assume $0 \leq a < \Gamma(\alpha+1)$. Then $\mathcal{I} - aI^{\alpha}_{0+}$ is invertable and*

$$(\mathcal{I} - aI^{\alpha}_{0+})^{-1} = \sum_{n=0}^{\infty} (aI^{\alpha}_{0+})^n,$$

where $(aI^{\alpha}_{0+})^0 = \mathcal{I}$ and $(aI^{\alpha}_{0+})^n u = aI^{\alpha}_{0+}((aI^{\alpha}_{0+})^{n-1} u)$ for $u \in X$ and $n \in \mathbb{N}_0$.

Using these lemmas, we can prove the following one.

Lemma **3.6**. *The function $u(t)$ is a solution of the BVP consisting of the equation*

$$-D^{\alpha}_{0+} u + au = h(t), \ 0 < t < 1, \tag{3.4}$$

and BC (1.2) if and only if

$$u(t) = \int_0^1 G(t,s)h(s)ds + bc(t), \tag{3.5}$$

where G is defined by (1.11) and the series c in (2.8) is uniformly convergent on $[0,1]$. Moreover, $c(t) \geq 0$ on $[0,1]$ and $c(0) = 0$.

P r o o f. For any $h \in C[0,1]$, let u be a solution of BVP (3.4), (1.2). Taking the α-integral of both sides of (3.4) and applying Lemma **3.3**, we have

$$((\mathcal{I} - aI_{0+}^{\alpha})u)(t) = -(I_{0+}^{\alpha}h)(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \tag{3.6}$$

for some constants C_1 and C_2. The first part of BC (1.2) implies $C_2 = 0$ and the second part implies

$$C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + b. \tag{3.7}$$

Then, by (1.4), (3.6), and (3.7),

$$\begin{aligned} ((\mathcal{I} - aI_{0+}^{\alpha})u)(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} t^{\alpha-1} h(s)ds + bt^{\alpha-1} \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + bt^{\alpha-1} \\ &= \int_0^1 G_0(t,s)h(s)ds + bt^{\alpha-1}, \end{aligned}$$

where G_0 is defined by (1.5).

Define $T_0 : X \to X$ by

$$(T_0 h)(t) = \int_0^1 G_0(t,s)h(s)ds + bt^{\alpha-1}. \tag{3.8}$$

Then, $(\mathcal{I} - aI_{0+}^{\alpha})u = T_0 h$. By Lemma **3.5**,

$$\begin{aligned} u(t) &= \left((\mathcal{I} - aI_{0+}^{\alpha})^{-1} T_0 h\right)(t) = \left(\sum_{n=0}^{\infty} (aI_{0+}^{\alpha})^n T_0 h\right)(t) \\ &= \sum_{n=0}^{\infty} (aI_{0+}^{\alpha})^n \left(\int_0^1 G_0(t,s)h(s)ds\right) + \sum_{n=0}^{\infty} (aI_{0+}^{\alpha})^n \left(bt^{\alpha-1}\right). \end{aligned} \tag{3.9}$$

By the proof of [17, Theorem 2.1],

$$\sum_{n=0}^{\infty} (aI_{0+}^{\alpha})^n \left(\int_0^1 G_0(t,s)h(s)ds\right) = \int_0^1 G(t,s)h(s)ds.$$

By Lemma **3.4**,

$$(aI_{0+}^{\alpha})^n \left(bt^{\alpha-1}\right) = \frac{a^n b\Gamma(\alpha)}{\Gamma((n+1)\alpha)} t^{(n+1)\alpha-1}, \ n = 0, 1, \ldots.$$

Note that for $n > 0$,

$$\frac{a^n b\Gamma(\alpha)}{\Gamma((n+1)\alpha)} t^{(n+1)\alpha-1} < \frac{b\Gamma^n(\alpha+1)\Gamma(\alpha)}{\Gamma((n+1)\alpha)} \le \frac{b\Gamma^n(3)\Gamma(\alpha)}{\Gamma(n+1)}.$$

Hence,

$$bc(t) = b\sum_{n=0}^{\infty} \frac{a^n\Gamma(\alpha)}{\Gamma((n+1)\alpha)} t^{(n+1)\alpha-1}$$

is uniformly convergent on $[0,1]$. Clearly, $c \in C[0,1]$, $c(t) \ge 0$ on $[0,1]$, and $c(0) = 0$. Therefore by (3.9), (3.5) holds.

The converse can be verified by reversing the argument; we omit the details. □

Define $T : P \to P$ by

$$(Tu)(t) = \int_0^1 G(t,s)[f(s,u(s),u(s)) + r(s,u(s))]ds + bc(t). \tag{3.10}$$

Then we immediately have the following lemma.

Lemma **3.7**. *The function $u(t)$ is a solution of BVP* (1.1), (1.2) *if and only if u is a fixed point of T.*

P r o o f. Clearly, Eq. (1.1) is equivalent to

$$-D_{0+}^{\alpha} u + au = f(t,u,u) + r(t,u).$$

The conclusion then follows from Lemma **3.6**. □

We will use mixed monotone operator theory to prove Theorem **2.1**. The following definitions and lemma are needed. The reader is referred to [19, Lemma 2.1], [16, 38, 40], [41, Corollary 2.3], and [42, Theorems 2.1 and 2.3] for the details.

Definition **3.1**. Let $(X, \|\cdot\|)$ be a Banach space and $\mathbf{0}$ be the zero element of X.

(a) A nonempty closed convex set $P \subset X$ is said to be a cone if it satisfies (i) $u \in P$ and $\lambda > 0 \Longrightarrow \lambda u \in P$; (ii) $u \in P$ and $-u \in P \Longrightarrow u = \mathbf{0}$.

(b) A cone P is said to be normal if there exists a constant $D > 0$ such that, for all $u, v \in X$, $\mathbf{0} \le u \le v \Longrightarrow \|u\| \le D\|v\|$. The constant D is called the normality constant of P.

(c) The Banach space $(X, \|\cdot\|)$ is partially ordered by a normal cone $P \subset E$, i.e., $u \le v$ if $v - u \in P$. If $u \le v$ and $u \ne v$, then we write $u < v$ or $v > u$.

(d) For any $u, v \in X$, we use the notation $u \sim v$ to mean that there exist $\underline{d} > 0$ and $\overline{d} > 0$ such that $\underline{d}v \le u \le \overline{d}v$.

Clearly, P defined by (2.5) is a normal cone with normality constant $D = 1$. Let w and P_w be defined by (2.6) and (2.9). Then $u \sim w$ for any $u \in P_w$.

Definition **3.2**. An operator $\mathcal{A} : P_w \times P_w \to X$ is said to be mixed monotone if $\mathcal{A}(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., for $x_1, x_2, y_1, y_2 \in P_w$, we have

$$x_1 \le x_2,\ y_1 \ge y_2 \Longrightarrow \mathcal{A}(x_1, y_1) \le \mathcal{A}(x_2, y_2).$$

Moreover, an element $u \in P_w$ is said to be a fixed point of $\mathcal{A}$ if $\mathcal{A}(u, u) = u$.

Definition **3.3**. An operator $\mathcal{B} : P_w \to X$ is said to be sub-homogeneous if it satisfies

$$\mathcal{B}(\kappa u) \ge \kappa \mathcal{B}(u) \quad \text{for all } u \in P_w \text{ and } \kappa \in (0, 1).$$

Lemma **3.8**. *Let $\theta \in (0, 1)$ and $\mathcal{A} : P_w \times P_w \to X$ be a mixed monotone operator satisfying*

$$\mathcal{A}(\kappa u, \kappa^{-1} v) \ge \kappa^{\theta} \mathcal{A}(u, v) \quad \text{for all } u, v \in P_w \text{ and } \kappa \in (0, 1). \tag{3.11}$$

Assume that $\mathcal{B} : P_w \to X$ is an increasing sub-homogeneous operator and the following conditions hold:

(i) $\mathcal{A}(w, w) \in P_w$ and $\mathcal{B}(w) \in P_w$;

(ii) there exists a constant $\eta > 0$ such that $\mathcal{A}(u, v) \ge \eta \mathcal{B}(u)$ for all $u, v \in P_w$.

Then, we have:

(a) the equation $\mathcal{A}(u, u) + \mathcal{B}(u) = u$ has a unique solution u in P_w;

(b) for any initial values $u_0, v_0 \in P_w$, the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_{n+1} = \mathcal{A}(u_n, v_n) + \mathcal{B}(u_n),$$
$$v_{n+1} = \mathcal{A}(v_n, u_n) + \mathcal{B}(v_n),$$

for $n \in \mathbb{N}_0$ satisfy $\|u_n - u\| \to 0$ and $\|v_n - u\| \to 0$ as $n \to \infty$.

In the sequel, we define $\mathcal{A} :\ P \times P \to X$ and $\mathcal{B} :\ P \to X$ by

$$\mathcal{A}(u, v)(t) = \int_0^1 G(t, s) f(s, u(s), v(s)) ds \tag{3.12}$$

and

$$\mathcal{B}(u)(t) = \int_0^1 G(t, s) r(s, u(s)) ds + bc(t). \tag{3.13}$$

Remark **3.1**. Clearly, $Tu = \mathcal{A}(u, u) + \mathcal{B}(u)$.

Proof of Theorem **2.1**. Let $\mathcal{A}$, $\mathcal{B}$, and w be defined by (3.12), (3.13), and (2.6). Then, by Lemma **3.7** and Remark **3.1**, we see that $u(t)$ is a solution of BVP (1.1), (1.2) if and only if $u = \mathcal{A}(u,u)+\mathcal{B}(u)$. Moreover, from the monotonicity of f and r assumed in (H1) and (H3), $\mathcal{A}$ is mixed monotone and $\mathcal{B}$ is increasing. For $u, v \in P_w$ and $\kappa \in (0,1)$, from (H2), we have

$$\begin{aligned}\mathcal{A}(\kappa u, \kappa^{-1}v)(t) =& \int_0^1 G(t,s) f(s, \kappa u(s), \kappa^{-1}v(s)) ds\\ \geq& \kappa^\theta \int_0^1 G(t,s) f(s, u(s), v(s)) ds = \kappa^\theta \mathcal{A}(u,v)(t),\end{aligned}$$

for $t \in [0,1]$, i.e., (3.11) of Lemma **3.8** holds. Similarly, by (H4),

$$\begin{aligned}\mathcal{B}(\kappa u)(t) =& \int_0^1 G(t,s) r(s, \kappa u(s)) ds + bc(t)\\ \geq& \kappa \int_0^1 G(t,s) r(s, u(s)) ds + \kappa bc(t) = \kappa \mathcal{B}(u)(t),\end{aligned}$$

for $t \in [0,1]$, i.e., $\mathcal{B}$ is sub-homogeneous.

By (3.2),

$$\begin{aligned}\mathcal{A}(w,w)(t) =& \int_0^1 G(t,s) f(s, w(s), w(s)) ds\\ \leq& w(t) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s, w(s), w(s)) ds \leq d_1 w(t)\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}(w,w)(t) =& \int_0^1 G(t,s) f(s, w(s), w(s)) ds\\ \geq& w(t) \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} f(s, w(s), w(s)) ds \geq d_2 w(t),\end{aligned}$$

for $t \in [0,1]$, where

$$d_1 = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s, \|w\|, 0) ds \tag{3.14}$$

and

$$d_2 = \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-1}s}{\Gamma(\alpha)} f(s, 0, \|w\|) ds. \tag{3.15}$$

Clearly, $d_2 \leq d_1$. By (H5),

$$f(s, 0, \|w\|) \geq \eta r(s,0) \geq 0, \quad s \in [0,1]. \tag{3.16}$$

Note that $r(s,0) \not\equiv 0$ on $[0,1]$ by (H3). Then, from (3.14)–(3.16), we have $d_1 \geq d_2 > 0$, so $\mathcal{A}(w,w) \in P_w$.

If $a > 0$, by (2.6), (2.7), and (2.8),

$$\begin{aligned} c(t) =& w(t) + \sum_{n=0}^{\infty} \frac{a^n \Gamma(\alpha) t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \frac{t}{n+1} \leq w(t) + \frac{c(t)}{2} + \frac{t^\alpha}{2} \\ \leq& w(t) + \frac{c(t)}{2} + \frac{t^\gamma}{2}, \end{aligned}$$

where $\gamma = \alpha - 1 < \alpha$. Note that

$$\begin{aligned} w(t) &= t^\gamma \sum_{n=0}^{\infty} \frac{a^n \Gamma(\alpha) t^{(n+1)\alpha-1-\gamma}}{\Gamma((n+1)\alpha)} \left(1 - \frac{t}{n+1}\right) \\ &\geq t^\gamma \left(1 - t + \frac{a\Gamma(\alpha) t^\alpha}{\Gamma(2\alpha)} \left(1 - \frac{t}{2}\right)\right). \end{aligned}$$

By (2.2), $w(t) \geq t^\gamma \underline{\phi}$. Then,

$$c(t) \leq w(t) + \frac{c(t)}{2} + \frac{t^\gamma}{2} \leq w(t) + \frac{c(t)}{2} + \frac{w(t)}{2\underline{\phi}},$$

i.e.,

$$c(t) \leq (2 + \frac{1}{\underline{\phi}}) w(t). \tag{3.17}$$

By (3.2), (3.13), and (3.17)

$$\begin{aligned} \mathcal{B}(w)(t) =& \int_0^1 G(t,s) r(s, w(s)) ds + bc(t) \\ \leq& w(t) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} r(s, w(s)) ds + b(2 + \frac{1}{\underline{\phi}}) w(t) = d_3 w(t) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(w)(t) =& \int_0^1 G(t,s) r(s, w(s)) ds + bc(t) \\ \geq& w(t) \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-1} s}{\Gamma(\alpha)} r(s, w(s)) ds \geq d_4 w(t), \end{aligned}$$

where

$$\begin{aligned} d_3 &= \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} r(s, w(s)) ds + b(2 + \frac{1}{\underline{\phi}}), \\ d_4 &= \int_0^1 \frac{(\alpha-1)(1-s)^{\alpha-1} s}{\Gamma(\alpha)} r(s, 0) ds. \end{aligned}$$

It is easy to see that $d_3 \geq d_4 > 0$. Hence $\mathcal{B}(w) \in P_w$. Therefore, condition (i) of Lemma **3.8** is satisfied.

For $u, v \in P_w$, by (H5), (3.2), and (3.17), for $t \in (0,1]$,

$$\begin{aligned}
\mathcal{A}(u,v)(t) &= \int_0^1 G(t,s) f(s,u(s),v(s))ds \\
&\geq \eta \int_0^1 G(t,s) \left[r(s,u(s)) + \frac{b(2\underline{\phi}+1)w(t)}{\underline{\phi} w(t) int_0^1 [(\alpha-1)(1-s)^{\alpha-1} s/\Gamma(\alpha)]ds} \right] ds \\
&\geq \eta \int_0^1 G(t,s) \left[r(s,u(s)) + \frac{b(2\underline{\phi}+1)w(t)}{\underline{\phi} \int_0^1 G(t,s)ds} \right] ds \\
&\geq \eta \left[\int_0^1 G(t,s) r(s,u(s))ds + bc(t) \right] \\
&= \eta \mathcal{B}(u)(t).
\end{aligned}$$

Clearly, $\mathcal{A}(u,v)(0) = \mathcal{B}(u)(0) = 0$. Therefore, condition (ii) of Lemma **3.8** holds.

Note that if $a = b = 0$, then $G = G_0$ defined by (1.5) and $w(t) = t^{\alpha-1}(1-t)$. By similar but simpler arguments, we can prove conditions (i) and (ii) of Lemma **3.8** hold for $a = b = 0$.

Therefore, by Lemma **3.8** (a), BVP (1.1), (1.2) has a unique solution $u(t)$ in P_w, which is obviously positive, and by Lemma **3.8** (b), conclusion (b) of Theorem **2.1** holds.

Finally, we will show that if $\hat{u}(t)$ is a positive solution of BVP (1.1), (1.2), then $\hat{u} \in P_w$. In fact, if $\hat{u}(t)$ is a positive solution of BVP (1.1), (1.2), then by Lemma **3.7** and Remark **3.1**, we have

$$\hat{u}(t) = \int_0^1 G(t,s)[f(s,\hat{u}(s),\hat{u}(s)) + r(s,\hat{u}(s))]ds + bc(t).$$

Similar to what we did for the operators $\mathcal{A}$ and B, using (3.2) we can show that there exist $d_5 \geq d_6 > 0$ such that

$$d_6 w(t) \leq \hat{u}(t) \leq d_5 w(t).$$

Hence, $\hat{u} \in P_w$.

Therefore, by the claim, we see that BVP (1.1), (1.2) has a unique positive solution in X. □

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