Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 15, NUMBER 3 (2012)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

A FOURIER GENERALIZED CONVOLUTION TRANSFORM AND APPLICATIONS TO INTEGRAL EQUATIONS

Nguyen Xuan Thao [1], **Vu Kim Tuan** [2], **Nguyen Thanh Hong** [3]

Abstract

We introduce an integral transform related to a Fourier sine– Fourier – Fourier cosine generalized convolution and prove a Watson type theorem for the transform. As applications we obtain solutions of some integral equations in closed form.

MSC 2010: 33C10, 44A35, 45E10, 45J05, 47A30, 47B15

Key Words and Phrases: convolution, Hölder inequality, Hausdorff-Young inequality, Fourier transform, Fourier sine transform, Fourier cosine transform, integral equation, Watson theorem

1. Preliminaries

The Fourier cosine and the Fourier sine transforms are defined as follows (see [11, 12])

$$(F_c f)(y) \equiv F_c[f](y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy \, dx = \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int_0^\infty f(x) \frac{\sin xy}{x} dx, \tag{1.1}$$

$$(F_s f)(y) \equiv F_s[f](y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin xy \, dx = \sqrt{\frac{2}{\pi}} \frac{d}{dy} \int_0^\infty f(x) \frac{1 - \cos xy}{x} dx, \tag{1.2}$$

pp. 493–508 , DOI: 10.2478/s13540-012-0035-y

for $f \in L_1(\mathbb{R}_+)$. The second integrals in (1.1) and (1.2) are also well-defined for $f \in L_2(\mathbb{R}_+)$.

In 1951, I. N. Sneddon introduced a convolution of two functions f and g for the Fourier cosine transform (see [11])

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[g(|x-u|) + g(x+u)]du, \ x > 0, \tag{1.3}$$

which satisfies the following factorization equality

$$F_c[f \underset{F_c}{*} g](y) = (F_c f)(y)(F_c g)(y), \ \forall y > 0, \ f, g \in L_1(\mathbb{R}_+). \tag{1.4}$$

I. N. Sneddon was also the first author who introduced the convolution for two different integral transforms (see [11])

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u)[g(|x-u|) - g(x+u)]du, \ x > 0, \tag{1.5}$$

and showed that this convolution satisfies the following factorization identity (see [1, 11])

$$F_s[f \underset{1}{*} g](y) = (F_s f)(y)(F_c g)(y), \ y > 0, \quad f, g \in L_1(\mathbb{R}_+). \tag{1.6}$$

In this factorization equality, there are two integral transforms F_c and F_s involved. Convolutions whose factorization identities contain more than one integral transforms are called generalized convolutions (see [7]).

Another generalized convolution for the Fourier cosine and Fourier sine integral transforms has the form (see [8])

$$(f \underset{2}{*} g)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y)[g(x+y) - \operatorname{sign}(y-x) g(|y-x|)]dy, \ x > 0. \tag{1.7}$$

This generalized convolution satisfies the following factorization identity

$$F_c[f \underset{2}{*} g](y) = (F_s f)(y)(F_s g)(y), \ y > 0, \quad f, g \in L_1(\mathbb{R}_+).$$

The generalized convolutions of two functions h and f with a weight function $\gamma(y) = e^{-y} \sin y$ for three transforms - the Fourier cosine, the Fourier and the Fourier sine integral transforms were studied in [9, 14]

$$\begin{aligned}(h \overset{\gamma}{\underset{1}{*}} f)(x) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_0^\infty \Big[& k(x,u,v-1) - k(x,u,1-v) \\ & - k(x,u,v+1) + k(x,u,-v-1) \Big] h(u) f(v) dv du, \ x > 0,\end{aligned} \tag{1.8}$$

and

$$(h \overset{\gamma}{\underset{2}{*}} f)(x) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \Big[k(x,u,v-1) + k(x,u,1-v)$$
$$- k(x,u,v+1) - k(x,u,-v-1)\Big] h(u) f(v) dv du, \ x > 0. \quad (1.9)$$

Here, $k(x,u,v) = \dfrac{1+iu}{(1+iu)^2 + (x+v)^2}$.

For $h \in L_1(\mathbb{R}, \sqrt{1+x^2})$ and $f \in L_1(\mathbb{R}_+)$, the generalized convolutions (1.8), (1.9) are well-defined and the following factorization properties hold (see [9, 14])

$$F_s[h \overset{\gamma}{\underset{1}{*}} f](y) = e^{-y} \sin y (Fh)(y)(F_c f)(y), \ y > 0, \quad (1.10)$$

$$F_c[h \overset{\gamma}{\underset{2}{*}} f](y) = e^{-y} \sin y (Fh)(y)(F_s f)(y), \ y > 0. \quad (1.11)$$

Here, F denotes the Fourier transform (see [11])

$$(Ff)(y) \equiv F[f](y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x) dx = \frac{1}{\sqrt{2\pi}} \frac{d}{dy} \int_{-\infty}^{\infty} \frac{1 - e^{-ixy}}{ix} f(x) dx. \quad (1.12)$$

for $f \in L_1(\mathbb{R})$. The second integral in (1.12) is also well-defined for $f \in L_2(\mathbb{R})$.

In this paper, we are interested in an integral transform related to the generalized convolutions (1.5) and (1.8), namely, a transform of the form

$$g(x) = (K_{h_1,h_2} f)(x) = \left(1 - \frac{d^2}{dx^2}\right) \{(h_1 \overset{\gamma}{\underset{1}{*}} f)(x) + (h_2 \underset{1}{*} f)(x)\}. \quad (1.13)$$

In Section **2** we derive some properties for the generalized convolutions (1.8), (1.9) in $L_p(\mathbb{R}_+)$. In Section **3**, a Watson type theorem for the transform (1.13) is proved. In Sections **4** and **5**, as applications we obtain solutions in closed form of an integral equation and a system of integral equations.

2. Generalized convolution properties

In this section, we prove the existence of the generalized convolutions (1.8) and (1.9) for weaker conditions, namely, for $h \in L_1(\mathbb{R})$ and $f \in L_1(\mathbb{R}_+)$. Besides, we obtain some operator properties of the generalized convolutions (1.8) and (1.9) in $L_p(\mathbb{R}_+)$.

THEOREM **2.1**. *For $h \in L_1(\mathbb{R})$ and $f \in L_1(\mathbb{R}_+)$, the generalized convolutions* (1.8) *and* (1.9) *are well-defined, belong to $L_1(\mathbb{R}_+)$, and satisfy the factorization identities* (1.10) *and* (1.11). *Moreover, the generalized convolutions* (1.8) *and* (1.9) *also belong to $L_p(\mathbb{R}_+)$ for $p \geq 2$, and the following Parseval type identities hold*

$$(h \overset{\gamma}{\underset{1}{*}} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y} \sin y (Fh)(y)(F_c f)(y) \sin xy \, dy, \ x > 0, \tag{2.1}$$

$$(h \overset{\gamma}{\underset{2}{*}} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y} \sin y (Fh)(y)(F_s f)(y) \cos xy \, dy, \ x > 0. \tag{2.2}$$

P r o o f. Using formula 1.4.1 from [2], we have

$$\begin{aligned} &k(x,u,v-1) - k(x,u,1-v) - k(x,u,v+1) + k(x,u,-v-1) \qquad (2.3)\\ =&\frac{1}{4}\int_0^\infty [\cos t(x+v-1) - \cos t(x+v+1) + \cos t(x-v-1) \\ &- \cos t(x-v+1)] e^{-(1+iu)t} dt \\ =&\int_0^\infty \sin(xt) \sin t \cos(tv) e^{-(1+iu)t} dt. \end{aligned}$$

Consequently,

$$\begin{aligned} &|k(x,u,v-1) - k(x,u,1-v) - k(x,u,v+1) + k(x,u,-v-1)| \qquad (2.4)\\ \leq &\int_0^\infty |\sin(xt) \sin t \cos(tv) e^{-(1+iu)t}| dt \leq \int_0^\infty e^{-t} dt = 1. \end{aligned}$$

Similarly,

$$|k(x,u,v-1) + k(x,u,1-v) - k(x,u,v+1) - k(x,u,-v-1)| \leq 1. \tag{2.5}$$

Formulas (2.4) and (2.5) yield

$$\left|(h \overset{\gamma}{\underset{1}{*}} f)\right| \leq \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^\infty \int_0^\infty |h(u)||f(v)| dv du = \frac{1}{(2\pi)^{3/2}} \|h\|_{L_1(\mathbb{R})} \|f\|_{L_1(\mathbb{R}_+)} < \infty, \tag{2.6}$$

$$\big|(h\overset{\gamma}{\underset{2}{*}}f)\big| \leq \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{0}^{\infty}|h(u)||f(v)|dvdu = \frac{1}{(2\pi)^{3/2}}\|h\|_{L_1(\mathbb{R})}\|f\|_{L_1(\mathbb{R}_+)} < \infty. \tag{2.7}$$

It shows the existence of the generalized convolutions (1.8) and (1.9).

Besides, formulas (2.4) and (2.5), and conditions $h \in L_1(\mathbb{R}), f \in L_1(\mathbb{R}_+)$ yield the absolute convergence of the integrals (1.8) and (1.9). Putting (2.3) into (1.8) and using Fubini's theorem we obtain the Parseval type equality (2.1). Similarly we get (2.2).

Now we prove that the generalized convolutions (1.8) and (1.9) belong to the space $L_1(\mathbb{R}_+)$. First, note that $h(x) = g_1(x) + g_2(x)$, where $g_1, g_2 \in L_1(\mathbb{R})$ are defined by

$$g_1(x) = \frac{h(x)+h(-x)}{2}, \quad g_2(x) = \frac{h(x)-h(-x)}{2}.$$

Since g_1 is an even function, and g_2 is an odd function, we have $(Fg_1)(y) = (F_cg_1)(y)$, $(Fg_2)(y) = -i(F_sg_2)(y)$. Moreover, using formulas (1.6.6, p.28) and (2.9.20, p.79) in [2] we obtain

$$e^{-y}\sin y = \sqrt{\frac{2}{\pi}}F_c\Big[\frac{2-\tau^2}{\tau^4+4}\Big](y) = \sqrt{\frac{2}{\pi}}F_s\Big[\frac{2\tau}{\tau^4+4}\Big](y).$$

Therefore, the Parseval identity (2.1) can be rewritten in the following form

$$\begin{aligned}(h\overset{\gamma}{\underset{1}{*}}f)(x) &= F_s\left[e^{-y}\sin y\big((F_cg_1)(y) - i(F_sg_2)(y)\big)(F_cf)(y)\right](x) \qquad (2.8)\\ &=\sqrt{\frac{2}{\pi}}F_s\left[F_s\Big[\frac{2\tau}{\tau^4+4}\Big](y)F_c[g_1 \underset{F_c}{*} f](y) - iF_c\Big[\frac{2-\tau^2}{\tau^4+4}\Big](y)F_s[g_2 \underset{1}{*} f](y)\right](x)\\ &=\sqrt{\frac{2}{\pi}}\Big(\frac{2\tau}{\tau^4+4}\underset{1}{*}(g_1 \underset{F_c}{*} f)(\tau)\Big)(x) - i\sqrt{\frac{2}{\pi}}\Big((g_2 \underset{1}{*} f)(\tau)\underset{1}{*}\frac{2-\tau}{\tau^4+4}\Big)(x),\end{aligned}$$

where $(\cdot \underset{F_c}{*} \cdot)$ is defined by (1.3), and $(\cdot \underset{1}{*} \cdot)$ is defined by (1.5). Similarly, the Parseval identity (2.2) can be rewritten in the following form

$$\begin{aligned}(h\overset{\gamma}{\underset{2}{*}}f)(x) &= F_c\left[e^{-y}\sin y\big((F_cg_1)(y) - i(F_sg_2)(y)\big)(F_sf)(y)\right](x) \qquad (2.9)\\ &=\sqrt{\frac{2}{\pi}}F_c\left[F_s\Big[\frac{2\tau}{\tau^4+4}\Big](y)F_s[f \underset{1}{*} g_1](y) - iF_c\Big[\frac{2-\tau^2}{\tau^4+4}\Big](y)F_c[g_2 \underset{2}{*} f](y)\right](x)\\ &=\sqrt{\frac{2}{\pi}}\Big(\frac{2\tau}{\tau^4+4}\underset{2}{*}(f \underset{1}{*} g_2)(\tau)\Big)(x) - i\sqrt{\frac{2}{\pi}}\Big(\frac{2-\tau}{\tau^4+4}\underset{F_c}{*}(g_2 \underset{2}{*} f)(\tau)\Big)(x),\end{aligned}$$

where the convolution $(\cdot \underset{2}{*} \cdot)$ is defined by (1.7).

From formulas (2.8), (2.9) and the properties of convolutions (1.3), (1.5), (1.7), it is easy to see that generalized convolutions (1.8) and (1.9) belong to $L_1(\mathbb{R}_+)$.

Finally, from Parseval identities (2.1), (2.2), and the Lebesgue-Riemann lemma, since $h \in L_1(\mathbb{R}), f \in L_1(\mathbb{R}_+)$, we see that $(Fh)(y)$, $(F_s f)(y)$ and $(F_c f)(y)$ are continuous functions, vanish at infinity, and hence bounded. Therefore $e^{-y}\sin y\,(Fh)(y)(F_c f)(y)$ and $e^{-y}\sin y(Fh)(y)(F_s f)(y)$ are functions in space $L_q(\mathbb{R}_+)$, for $1 \le q \le 2$. Consequently, generalized convolutions (1.8) and (1.9) belong to $L_p(\mathbb{R}_+)$ for all $p \ge 2$, and the factorization equalities (1.10) and (1.11) hold. □

Recall the Hausdorff-Young inequality for the Fourier transform (see [4])

$$\|Ff\|_{L_{p'}(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}}\|f\|_{L_p(\mathbb{R})}, \tag{2.10}$$

here, $f \in L_p(\mathbb{R})$, $1 \le p \le 2$, and p' is its conjugate exponential $\frac{1}{p} + \frac{1}{p'} = 1$. In case f is an odd or an even function, the Hausdorff-Young inequality has the form

$$\|F_s f\|_{L_{p'}(\mathbb{R}_+)} \le \sqrt{\frac{2}{\pi}}\|f\|_{L_p(\mathbb{R}_+)}, \quad \|F_c f\|_{L_{p'}(\mathbb{R}_+)} \le \sqrt{\frac{2}{\pi}}\|f\|_{L_p(\mathbb{R}_+)}. \tag{2.11}$$

Theorem **2.2**. *Let $h \in L_p(\mathbb{R})$, $f \in L_q(\mathbb{R}_+)$, $1 \le p, q \le 2$ with respective conjugate exponentials p', q'. Then the generalized convolutions $(h \overset{\gamma}{\underset{1}{*}} f)$, $(h \overset{\gamma}{\underset{2}{*}} f)$ belong to $L_r(\mathbb{R}_+)$ for all $r \ge 2$, and the factorization equalities (1.10), (1.11) hold. Moreover, for $1 < r \le 2$ satisfying $\dfrac{1}{p'} + \dfrac{1}{q'} < \dfrac{1}{r}$ and r' be the conjugate exponential of r, the following estimates hold:*

$$\begin{aligned} \|h \overset{\gamma}{\underset{1}{*}} f\|_{L_{r'}(\mathbb{R}_+)} &\le \frac{\sqrt{2}}{(rs)^{\frac{1}{rs}}\pi^{3/2}}\|h\|_{L_p(\mathbb{R})}\|f\|_{L_q(\mathbb{R}_+)}, \\ \|h \overset{\gamma}{\underset{2}{*}} f\|_{L_{r'}(\mathbb{R}_+)} &\le \frac{\sqrt{2}}{(rs)^{\frac{1}{rs}}\pi^{3/2}}\|h\|_{L_p(\mathbb{R})}\|f\|_{L_q(\mathbb{R}_+)}. \end{aligned} \tag{2.12}$$

Here $s > 1$ is defined by $\dfrac{r}{p'} + \dfrac{r}{q'} + \dfrac{1}{s} = 1$, and the integrals are understood in mean value principle, if necessary.

P r o o f. We will prove for the generalized convolution $(h \overset{\gamma}{\underset{1}{*}} f)$. A proof for the generalized convolution $(h \overset{\gamma}{\underset{2}{*}} f)$ is similar.

Since the integrals are understood in mean, using Fubini's theorem and formula (2.3), we see that the generalized convolution (1.8) is well-defined and the Parseval equality (2.1) holds. Furthermore, for $h \in L_p(\mathbb{R}), f \in L_q(\mathbb{R}_+)$, the Riemann-Lebesgue lemma shows that $(Fh)(y) \in C_0(\mathbb{R})$, $(F_c f)(y) \in C_0(\mathbb{R}_+)$, and hence bounded. Therefore,

$$e^{-y}\sin y (Fh)(y)(F_c f)(y) \in L_r(\mathbb{R}_+) \quad \text{for all} \quad r > 1,$$

hence $h \underset{1}{\overset{\gamma}{*}} f \in L_{r'}(\mathbb{R}_+)$ for all $r' \geq 2$, and the factorization equality (1.10) holds.

We now prove the first inequality in (2.12). Note that $r < p', r < q'$ for $1 < r \leq 2$, thanks to the Hausdorff-Young inequality and the Hölder inequality, we have

$$\|h \underset{1}{\overset{\gamma}{*}} f\|_{L_{r'}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi}} \|e^{-y}\sin y (Fh)(y)(F_c f)(y)\|_{L_r(\mathbb{R}_+)}$$

$$= \sqrt{\frac{2}{\pi}} \Big(\int\limits_0^\infty |(Fh)(y)(F_c f)(y) e^{-y}\sin y|^r dy \Big)^{\frac{1}{r}}$$

$$\leq \sqrt{\frac{2}{\pi}} \Big\{ \Big(\int\limits_{-\infty}^\infty |(Fh)(y)|^{p'} dy \Big)^{\frac{r}{p'}} \Big(\int\limits_0^\infty |(F_c f)(y)|^{q'} dy \Big)^{\frac{r}{q'}} \Big(\int\limits_0^\infty |e^{-rsy}\sin^{rs} y| dy \Big)^{\frac{1}{s}} \Big\}^{\frac{1}{r}}.$$

Therefore, applying again the Hausdorff-Young inequality we obtain

$$\|h \underset{1}{\overset{\gamma}{*}} f\|_{L_{r'}(\mathbb{R}_+)} \leq \sqrt{\frac{2}{\pi}} \|Fh\|_{L_{p'}(\mathbb{R})} \|F_c f\|_{L_{q'}(\mathbb{R}_+)} \Big(\int\limits_0^\infty e^{-rsy} dy \Big)^{\frac{1}{rs}}$$

$$\leq \frac{\sqrt{2\pi}}{(rs)^{\frac{1}{rs}}.\pi^2} \|h\|_{L_p(\mathbb{R})} \|f\|_{L_q(\mathbb{R}_+)}.$$

The proof is completed. □

Let $L_p^{\alpha,\beta}(\mathbb{R}_+)$, $\alpha > -1, \beta > 0$, be the space of all functions defined on $\mathbb{R}_+$ such that

$$\|f\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \Big(\int\limits_0^\infty x^\alpha e^{-\beta x} |f(x)|^p dx \Big)^{1/p} < \infty.$$

The existence and norm estimate of the convolutions (1.8), (1.9) for $h \in L_p(\mathbb{R}_+)$, $f \in L_q(\mathbb{R}_+)$, where $p > 1$ and q is its conjugate exponential, are stated in the following

Proposition **2.1**. *For all $h \in L_1(\mathbb{R})$ and $f \in L_1(\mathbb{R}_+)$, the generalized convolutions* (1.8), (1.9) *exist as continuous operators in $L_r^{\alpha,\beta}(\mathbb{R}_+)$,* *moreover,* $h \overset{\gamma}{\underset{1}{*}} f$, $h \overset{\gamma}{\underset{2}{*}} f$ *belong to* $L_r^{\alpha,\beta}(\mathbb{R}_+)$, *and the following estimates hold true*

$$\begin{aligned} \|h \overset{\gamma}{\underset{1}{*}} f\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} &\leq C\|h\|_{L_1(\mathbb{R}_+)}\|f\|_{L_1(\mathbb{R}_+)}, \\ \|h \overset{\gamma}{\underset{2}{*}} f\|_{L_r^{\alpha,\beta}(\mathbb{R}_+)} &\leq C\|h\|_{L_1(\mathbb{R}_+)}\|f\|_{L_1(\mathbb{R}_+)}, \end{aligned} \tag{2.13}$$

where $C = \beta^{-\frac{\alpha-1}{r}}\Gamma^{\frac{1}{r}}(\alpha+1)$.

P r o o f. With the help of formula 3.381.4 in [5]

$$\int_0^\infty x^\alpha e^{-\beta x} dx = \beta^{-\alpha+1}\Gamma(\alpha+1), \quad \beta > 0,\ \alpha > -1,$$

and estimates (2.6), (2.7), we obtain (2.13). □

3. A Watson type theorem

In this section, we consider the following transform

$$f \mapsto K_{h_1,h_2}[f](x) = g(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\{(h_1 \overset{\gamma}{\underset{1}{*}} f)(x) + (h_2 \underset{1}{*} f)(x)\}, \tag{3.1}$$

where, $(h_2 \underset{1}{*} f)(x)$ is the generalized convolution for the Fourier sine and Fourier cosine transforms (1.5).

Theorem **3.1**. *Let* $h_1 \in L_2(\mathbb{R})$, $h_2 \in L_2(\mathbb{R}_+)$, *then the condition*

$$|e^{-y}\sin y(Fh_1)(y) + (F_s h_2)(y)| = \frac{1}{1+y^2}, \tag{3.2}$$

is necessary and sufficient to ensure that the transform (3.1) *is unitary on* $L_2(\mathbb{R}_+)$. *Moreover, the inverse transform has the form*

$$f(x) = \Big(1 - \frac{d^2}{dx^2}\Big)\{(\overline{h_1} \overset{\gamma}{\underset{2}{*}} g)(x) + (\overline{h_2} \underset{2}{*} g)(x)\}. \tag{3.3}$$

P r o o f. *Sufficiency.* Suppose that h_1 and h_2 satisfy condition (3.2). It is well-known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R})$ if and only if $(Fh)(x)$, $\frac{d}{dx}(Fh)(x)$, $\frac{d^2}{dx^2}(Fh)(x) \in L_2(\mathbb{R})$ (Theorem 68, [12]). Moreover,

$$\frac{d^2}{dx^2}(Fh)(x) = F\big[(-iy)^2 h(y)\big](x).$$

In particular, in case h is an even or an odd function such that $(1+y^2)h(y) \in L_2(\mathbb{R}_+)$, the following equalities hold

$$\begin{aligned}\Big(1-\frac{d^2}{dx^2}\Big)(F_ch)(x) &= F_c\big[(1+y^2)h(y)\big](x),\\ \Big(1-\frac{d^2}{dx^2}\Big)(F_sh)(x) &= F_s\big[(1+y^2)h(y)\big](x).\end{aligned} \tag{3.4}$$

Using Theorem **2.2**, in case $p=q=r=2$, the convolution $h_1 \overset{\gamma}{\underset{1}{*}} f$ satisfies the Parseval identity (2.1). Applying the Parseval identity (1.6) for convolution $h_2 \underset{1}{*} f$, we have

$$\begin{aligned}g(x) =&\Big(1-\frac{d^2}{dx^2}\Big)F_s\big[e^{-y}\sin y(Fh_1)(y)(F_cf)(y)+(F_sh_2)(y)(F_cf)(y)\big](x)\\ =&F_s\big[(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)(F_cf)(y)\big](x).\end{aligned}$$

By virtue of the Parseval identity for the Fourier cosine and the Fourier sine transforms $\|f\|_{L_2(\mathbb{R}_+)} = \|F_cf\|_{L_2(\mathbb{R}_+)} = \|F_sf\|_{L_2(\mathbb{R}_+)}$ and from condition (3.2) we get

$$\begin{aligned}\|g\|_{L_2(\mathbb{R}_+)} =&\big\|(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)(F_cf)(y)\big\|_{L_2(\mathbb{R}_+)}\\ =&\|F_cf\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

It shows that the transformation (3.1) is unitary.

On the other hand, formula (3.2) implies that $(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)$ is bounded on $\mathbb{R}_+$, hence $(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)(F_cf)(y) \in L_2(\mathbb{R}_+)$. We have

$$(F_sg)(y) = (1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)(F_cf)(y).$$

Using condition (3.2) we obtain

$$(F_cf)(y) = (1+y^2)\big(e^{-y}\sin y(F\overline{h_1})(y)+(F_s\overline{h_2})(y)\big)(F_sg)(y).$$

Again, condition (3.2) shows that $(1+y^2)\big(e^{-y}\sin y(F\overline{h_1})(y)+(F_s\overline{h_2})(y)\big)(F_sg)(y) \in L_2(\mathbb{R}_+)$, then formula (3.4) yields

$$\begin{aligned}f(x) =& F_c\big[(1+y^2)\big(e^{-y}\sin y(F\overline{h_1})(y)+(F_s\overline{h_2})(y)\big)(F_sg)(y)\big](x)\\ =&\Big(1-\frac{d^2}{dx^2}\Big)F_c\big[(e^{-y}\sin y(F\overline{h_1})(y)+(F_s\overline{h_2})(y))(F_sg)(y)\big](x).\end{aligned}$$

Using the Parseval identity for the generalized convolution (1.9) and the Parseval identity for the generalized convolution (1.7) we have

$$f(x) = \Big(1-\frac{d^2}{dx^2}\Big)\{(\overline{h_1} \overset{\gamma}{\underset{2}{*}} g)(x) + (\overline{h_2} \underset{2}{*} g)(x)\}.$$

Therefore the inverse transformation of (3.1) has the form (3.3).

Necessity. Suppose that the transform (3.1) is unitary on $\mathbb{R}_+$, then the Parseval identities for the Fourier sine and cosine transforms yield

$$\begin{aligned}&\big\|(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)(F_cf)(y)\big\|_{L_2(\mathbb{R}_+)}\\=&\|g\|_{L_2(\mathbb{R}_+)}=\|f\|_{L_2(\mathbb{R}_+)}=\|F_cf\|_{L_2(\mathbb{R}_+)}.\end{aligned}$$

Therefore the multiplication operator $M_\theta[\cdot]$ with $\theta(y)=(1+y^2)(e^{-y}\sin y$ $(Fh_1)(y)+(F_sh_2)(y))$ is unitary on $L_2(\mathbb{R}_+)$. This equivalent to $|\theta(y)|\equiv 1$ on $\mathbb{R}_+$, namely

$$|(1+y^2)\big(e^{-y}\sin y(Fh_1)(y)+(F_sh_2)(y)\big)|=1,\ \forall y>0.$$

It shows that h_1 and h_2 satisfy condition (3.2). The proof of Theorem **3.1** is completed. □

4. A class of integral equations

Inspire of having many useful applications (see [6]), not so many integral equations can be solved in a closed form. In this section, we consider the following integral equation related to the transform (3.1)

$$\begin{aligned}&f(x)+\frac{d}{dx}(K_{\varphi,\psi}f)(x)=g(x),\\&f'(0)=0,\\&\lim_{x\to\infty}f(x)=\lim_{x\to\infty}f'(x)=0.\end{aligned}\tag{4.1}$$

Here, $\varphi(x)=(\varphi_1(\tau)\underset{F}{*}\operatorname{sech}(\tau))(x),\psi(x)=(\psi_1(\tau)\underset{1}{*}\operatorname{sech}(\tau))(x)$; φ_1,ψ_1,g are given functions in $L_1(\mathbb{R}_+)$, and f is a unknown function.

In order to give a solution of the above problem, note that, for $h\in L_1(\mathbb{R}_+)$ such that $h(0)=0$, $\lim\limits_{x\to\infty}h'(x)=0$, there exist the Fourier sine and Fourier cosine transforms of h,h'. Furthermore,

$$(F_sh')(y)=-y(F_ch)(y),\tag{4.2}$$

and

$$(F_ch')(y)=\frac{1}{\sqrt{2\pi}}\int\limits_0^\infty h'(x)\cos xy dx=y(F_sh)(y).\tag{4.3}$$

Theorem **4.1**. *Suppose the following condition holds*

$$1-6F_c[\varphi_1\overset{\gamma}{\underset{2}{*}}\sinh(\tau)\operatorname{sech}^4(\tau)+\psi_1\underset{2}{*}\sinh(\tau)\operatorname{sech}^4(\tau)](y)\neq 0,\ \forall y>0.\tag{4.4}$$

Then problem (4.1) *has a unique solution in* $L_1(\mathbb{R}_+)$ *that has the form*

$$f(x)=g(x)+(g\underset{F_c}{*}l)(x).\tag{4.5}$$

Here, $l \in L_1(\mathbb{R}_+)$ *is defined by*

$$(F_c l)(y) = \frac{6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}{1 - 6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}.$$

P r o o f. Using the Parseval identity for the generalized convolution (1.8) (Theorem **2.1**), for $\varphi \in L_1(\mathbb{R}), f \in L_1(\mathbb{R}_+)$, we have $(F\varphi)(y) \in C_0(\mathbb{R}), (F_c f)(y) \in C_0(\mathbb{R}_+)$, and hence bounded. It shows that the integral (2.1) is absolutely convergent. Then we can interchange the order of integration and differentiation to get

$$\begin{aligned}
&\Big(\frac{d}{dx} - \frac{d^3}{dx^3}\Big)(\varphi \overset{\gamma}{\underset{1}{*}} f)(x) \\
=&\sqrt{\frac{2}{\pi}}\Big(\frac{d}{dx} - \frac{d^3}{dx^3}\Big)\int_0^\infty e^{-y}\sin y(F\varphi)(y)(F_c f)(y)\sin xy\,dy \\
=&\sqrt{\frac{2}{\pi}}\int_0^\infty (y+y^3)e^{-y}\sin y(F\varphi)(y)(F_c f)(y)\cos xy\,dy. \qquad (4.6)
\end{aligned}$$

The equation (4.1) can be rewritten in the form

$$f(x) + \sqrt{\frac{2}{\pi}}\Big(\frac{d}{dx} - \frac{d^3}{dx^3}\Big)\Big\{\int_0^\infty e^{-y}\sin y(F\varphi)(y)(F_c f)(y)\sin xy\,dy\Big\} + \Big(\frac{d}{dx} - \frac{d^3}{dx^3}\Big)\{(\psi \underset{1}{*} f)(x)\} = g(x). \quad (4.7)$$

Applying the Fourier cosine transform to both sides of (4.7), and using the factorization equality (1.6) and formulas (4.2), (4.3), and (4.6) we obtain

$$(F_c f)(y) + (y+y^3)\big(e^{-y}\sin y(F\varphi)(y) + (F_s\psi)(y)\big)(F_c f)(y) = (F_c g)(y). \quad (4.8)$$

By the assumption $\varphi(x) = (\varphi_1 \underset{F}{*} \operatorname{sech}(\tau))(x), \psi(x) = (\psi_1 \underset{1}{*} \operatorname{sech}(\tau))(x)$, we have

$$\begin{aligned}
(F\varphi)(y) =&(F\varphi_1)(y)F[\operatorname{sech}(\tau)](y) = \sqrt{\frac{\pi}{2}}\operatorname{sech}\frac{\pi y}{2}(F\varphi_1)(y), \\
(F_s\psi)(y) =&(F_s\psi_1)(y)F_c[\operatorname{sech}(\tau)](y) = \sqrt{\frac{\pi}{2}}\operatorname{sech}\frac{\pi y}{2}(F_s\psi_1)(y).
\end{aligned}$$

Therefore

$$(F_cf)(y)+\sqrt{\frac{\pi}{2}}y(y^2+1)\operatorname{sech}\frac{\pi y}{2}((F\varphi_1)(y)e^{-y}\sin y+(F_s\varphi_1)(y))(F_cf)(y) \\ =(F_cg)(y). \quad (4.9)$$

Using formula (see relation (1.9.4) in [2]) for $n=1$:

$$\frac{\sqrt{2\pi}}{4}(y^2+1)\operatorname{sech}\frac{\pi y}{2}=F_c[\operatorname{sech}^3(\tau)](y),$$

we can write equation (4.9) in the form

$$(F_cf)(y)+2yF_c[\operatorname{sech}^3(\tau)](y)\big((F\varphi_1)(y)e^{-y}\sin y \\ +(F_s\psi_1)(y)\big)(F_cf)(y)=(F_cg)(y).$$

On the other hand, by integration by parts we easily see that

$$yF_c[\operatorname{sech}^3(\tau)](y)=-3F_s[\sinh(\tau)\operatorname{sech}^4(\tau)](y).$$

Therefore,

$$(F_cf)(y)\big(1-6F_c[\varphi_1\overset{\gamma}{\underset{2}{*}}\sinh(\tau)\operatorname{sech}^4(\tau)](y)-6F_c[\psi_1\underset{2}{*}\sinh(\tau)\operatorname{sech}^4(\tau)](y)\big) \\ =(F_cg)(y).$$

From condition (4.4) we get

$$(F_cf)(y) \\ =\Big(1+\frac{6F_c[\varphi_1\overset{\gamma}{\underset{2}{*}}\sinh(\tau)\operatorname{sech}^4(\tau)+\psi_1\underset{2}{*}\sinh(\tau)\operatorname{sech}^4(\tau)](y)}{1-6F_c[\varphi_1\overset{\gamma}{\underset{2}{*}}\sinh(\tau)\operatorname{sech}^4(\tau)+\psi_1\underset{2}{*}\sinh(\tau)\operatorname{sech}^4(\tau)](y)}\Big)(F_cg)(y). \quad (4.10)$$

The Wiener-Levy theorem [10] states that if f is the Fourier transform of an $L_1(\mathbb{R})$ function, and φ is analytic in a neighborhood of the origin with $\varphi(0)=0$ that contains the domain $\{f(y),\forall y\in\mathbb{R}\}$, then $\varphi(f)$ is also the Fourier transform of an $L_1(\mathbb{R})$ function. For the Fourier cosine transform it means that if f is the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function, and φ is analytic in a neighborhood of the origin that contains the domain $\{f(y),\forall y\in\mathbb{R}_+\}$, and $\varphi(0)=0$, then $\varphi(f)$ is also the Fourier cosine transform of an $L_1(\mathbb{R}_+)$ function.

By the given condition (4.4) the function $\varphi(z)=\frac{6z}{1-6z}$ satisfies conditions of the Wiener-Levy theorem, and therefore, there exists a unique function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) = \frac{6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}{1 - 6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}.$$

Therefore, the equation (4.10) becomes

$$(F_c f)(y) = \big(1 + (F_c l)(y)\big)(F_c g) = F_c\big[g + g \underset{F_c}{*} l\big](y),$$

that implies $f(x) = g(x) + (g \underset{F_c}{*} l)(x) \in L_1(\mathbb{R}_+)$. The proof is complete. $\square$

5. A system of integral equations

Finally, we consider a system of two integral equations

$$\begin{aligned} f(x) + K_{\varphi,\psi} g(x) &= p(x), \\ g(x) + K^{-1}_{\xi,\eta} f(x) &= q(x), \end{aligned} \tag{5.1}$$

with boundary conditions

$$\begin{aligned} &f'(0) = 0, \quad g'(0) = 0, \\ &\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0 = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} g'(x). \end{aligned} \tag{5.2}$$

Theorem **5.1**. *Suppose that*

$$1 - 4F_c\big[\big(\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau\big) \underset{2}{*} \big(\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau\big)\big](y) \neq 0, \tag{5.3}$$

for any $y > 0$. The system of equations (5.1) *with boundary conditions* (5.2) *has a unique solution (f, g) in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$ in the form*

$$\begin{aligned} f(x) =& p(x) - \big((\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{1}{*} q\big)(x) + (p \underset{1}{*} l)(x) \\ &- \big(((\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{1}{*} q) \underset{1}{*} l\big)(x), \\ g(x) =& q(x) - 2\big((\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{2}{*} p\big)(x) + (q \underset{F_c}{*} l)(x) \\ &- 2\big(((\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{2}{*} p) \underset{F_c}{*} l\big)(x). \end{aligned}$$

Here, $l \in L_1(\mathbb{R}_+)$ is defined by

$$(F_c l)(y) = \frac{6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}{1 + 6F_c[\varphi_1 \overset{\gamma}{\underset{2}{*}} \sinh(\tau)\operatorname{sech}^4(\tau) + \psi_1 \underset{2}{*} \sinh(\tau)\operatorname{sech}^4(\tau)](y)}.$$

P r o o f. Applying the Fourier sine transform to the first equation, and the Fourier cosine transform to the second one, we obtain

$$\begin{gathered}(F_s f)(y) + (1+y^2)\big((F\varphi)(y)e^{-y}\sin y + (F_s\psi)(y)\big)(F_c g)(y) = (F_s p)(y),\\ (F_c g)(y) + (1+y^2)\big((F\xi)(y)e^{-y}\sin y + (F_s\eta)(y)\big)(F_s f)(y) = (F_c q)(y).\end{gathered} \tag{5.4}$$

By assumption $\varphi(x) = (\varphi_1 \underset{F}{*} \operatorname{sech}(\tau))(x)$, $\xi(x) = (\xi_1 \underset{F}{*} \operatorname{sech}(\tau))(x)$, $\psi(x) = (\psi_1 \underset{1}{*} \operatorname{sech}(\tau))(x)$, $\eta(x) = (\eta_1 \underset{1}{*} \operatorname{sech}(\tau))(x)$. Using factorization identities for the Fourier convolution and the generalized convolutions (1.5), (1.10), (1.11), and formula 1.9.4 in [2] we obtain

$$\begin{aligned} &(1+y^2)\big((F\varphi)(y)e^{-y}\sin y + (F_s\psi)(y)\big)\\ =&(1+y^2)\sqrt{\frac{\pi}{2}}\operatorname{sech}\frac{\pi y}{2}\big(e^{-y}\sin y(F\varphi_1)(y) + (F_s\psi_1)(y)\big)\\ =&2F_c[\operatorname{sech}^3\tau](y)\big(e^{-y}\sin y(F\varphi_1)(y) + (F_s\psi_1)(y)\big)\\ =&2F_s[\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s[\psi_1 \underset{1}{*} \operatorname{sech}^3\tau](y), \end{aligned}$$

and

$$\begin{aligned} &(1+y^2)\big((F\xi)(y)e^{-y}\sin y + (F_s\eta)(y)\big)\\ =&(1+y^2)\sqrt{\frac{\pi}{2}}\operatorname{sech}\frac{\pi y}{2}\big(e^{-y}\sin y(F\xi_1)(y) + (F_s\eta_1)(y)\big)\\ =&2F_c[\operatorname{sech}^3\tau](y)[e^{-y}\sin y(F\xi_1)(y) + (F_s\eta_1)(y)]\\ =&2F_s[\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s[\eta_1 \underset{1}{*} \operatorname{sech}^3\tau](y). \end{aligned}$$

Therefore, system (5.4) is equivalent to

$$\begin{gathered}(F_s f)(y) + 2\big(F_s[\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s[\psi_1 \underset{1}{*} \operatorname{sech}^3\tau](y)\big)(F_c g)(y) = (F_s p)(y),\\ (F_c g)(y) + 2\big(F_s[\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s(\eta_1 \underset{1}{*} \operatorname{sech}^3\tau)(y)\big)(F_s f)(y) = (F_c q)(y).\end{gathered} \tag{5.5}$$

Solving system (5.5) we obtain

$$\begin{aligned}\Delta =&1 - 4\big(F_s[\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s[\psi_1 \underset{1}{*} \operatorname{sech}^3\tau](y)\big)\times\\ &\times\big(F_s[\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau](y) + F_s(\eta_1 \underset{1}{*} \operatorname{sech}^3\tau)(y)\big)\\ =&1 - 4F_c\big[\big(\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau\big) \underset{2}{*} \big(\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau\big)\big](y).\end{aligned}$$

From condition (5.3), the Wiener-Levy theorem shows that there exists a function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_c l)(y) =$$
$$= \frac{4F_c\big[\big(\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau\big) \underset{2}{*} \big(\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau\big)\big](y)}{1 - 4F_c\big[\big(\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau\big) \underset{2}{*} \big(\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau\big)\big](y)}.$$

It implies $\dfrac{1}{\Delta} = 1 + (F_c l)(y)$. Therefore, solving system (5.5) we obtain

$$f(x) = p(x) - \big((\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{1}{*} q\big)(x) + (p \underset{1}{*} l)(x)$$
$$- \big(((\varphi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \psi_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{1}{*} q) \underset{1}{*} l\big)(x) \in L_1(\mathbb{R}_+),$$

and

$$g(x) = q(x) - 2\big((\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{2}{*} p\big)(x) + (q \underset{F_c}{*} l)(x)$$
$$- 2\big(((\xi_1 \overset{\gamma}{\underset{1}{*}} \operatorname{sech}^3\tau + \eta_1 \underset{1}{*} \operatorname{sech}^3\tau) \underset{2}{*} p) \underset{F_c}{*} l\big)(x) \in L_1(\mathbb{R}_+).$$

The proof is completed. $\square$

Acknowledgements

This research is supported partially by Vietnam's National Foundation for Science and Technology Development (NAFOSTED), Grant 101.01-2011.05.

References

[1] F. Al-Musallam, V.K. Tuan, Integral transforms related to a generalized convolution. *Results Math.* **38**, No 3-4 (2000), 197–208.

[2] H. Bateman, A. Erdélyi, *Tables of Integral Transforms, Vol. I.* McGraw-Hill Book Co., New York-Toronto-London (1954).

[3] L.E. Britvina, A class of integral transforms related to the Fourier cosine convolution. *Integr. Trans. Spec. Func.*, **16**, No 5-6 (2005), 379–389.

[4] J. Duoadikoetxea, *Fourier Analysis.* AMS, Providence, Rhode Island (2001).

[5] I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series, and Products, 7ed.* Academic Press (2007).

[6] Y.N. Grigoriev, N.H. Ibragimov, V.F. Kovalev, S.V. Meleshko, *Symmetries of Integro-Differential Equations with Applications in Mechanics and Plasma Physics.* Lect. Notes Phys. 806, Springer, Dordrecht (2010).

[7] V.A. Kakichev, N.X. Thao, On the design method for the generalized integral convolutions. *Izv. Vyssh. Uchebn. Zaved. Mat.*, No 1 (1998), 31–40 (In Russian).

[8] V.A. Kakichev, N.X. Thao, V.K. Tuan, On the generalized convolutions for Fourier cosine and sine transforms. *East-West J. of Math.*, **1**, No 1 (1998), 85–90.

[9] N.M. Khoa, On the generalized convolution with a weight function for Fourier cosine, Fourier and Fourier sine transforms. *Southeast Asian Bull. Math.*, **33** (2009), 285–298.

[10] R.E.A.C. Paley, N. Wiener, *Fourier Transforms in the Complex Domain.* AMS, New York (1934).

[11] I.N. Sneddon, *Fourier Transforms.* McGray-Hill, New York (1951).

[12] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals, 3ed.* Chelsea Publishing Co., New York (1986).

[13] V.K. Tuan, Integral transforms of Fourier cosine convolution type. *J. Math. Anal. Appl.* **229**, No 2 (1999), 519–529.

[14] N.X. Thao, N.M. Khoa, On the generalized convolution for Fourier sine, Fourier and Fourier cosine transforms. In: *Proc. Function Spaces in Complex and Clifford Analysis*, National University Publisher, Hanoi (2008), 223–240.

[1] *Department of Mathematics*
School of Applied Mathematics & Informatics
Hanoi University of Science and Technology
1 Dai Co Viet, Hanoi, VIETNAM

e-mail: thaonxbmai@yahoo.com

[2] *Department of Mathematics, University of West Georgia*
1601 Maple Street, Carrollton, GA 30118, USA

e-mail: vu@westga.edu *Received: April 12, 2012*

[3] *Department of Mathematics, High School for Gifted*
Hanoi National University of Education
136 Xuan Thuy, Hanoi, VIETNAM

e-mail: hongdhsp1@yahoo.com

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **15**, No 3 (2012), pp. 493–508;
DOI: 10.2478/s13540-012-0035-y