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RESEARCH PAPER

THE FBM-DRIVEN ORNSTEIN-UHLENBECK PROCESS: PROBABILITY DENSITY FUNCTION AND ANOMALOUS DIFFUSION

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Abstract

This paper deals with the Ornstein-Uhlenbeck (O-U) process driven by the fractional Brownian motion (fBm). Based on the fractional Itô formula, we present the corresponding fBm-driven Fokker-Planck equation for the nonlinear stochastic differential equations driven by an fBm. We then apply it to establish the evolution of the probability density function (PDF) of the fBm-driven O-U process. We further obtain the closed form of such PDF by combining the Fourier transform and the method of characteristics. Interestingly, the obtained PDF has an infinite variance which is significantly different from the classical O-U process. We reveal that the fBm-driven O-U process can describe the heavy-tailedness or anomalous diffusion. Moreover, the speed of the sub-diffusion is inversely proportional to the viscosity coefficient, while is proportional to the Hurst parameter. Finally, we carry out numerical simulations to verify the above findings.

MSC 2010: Primary 60G22; Secondary 26A33, 35R60, 60H15, 35Q84

Key Words and Phrases: fractional Brownian motion, Ornstein-Uhlenbeck process, anomalous diffusion, Fokker-Planck equation

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1. Introduction

Since the celebrated Ornstein-Uhlenbeck (O-U) process was originally introduced by Uhlenbeck and Ornstein [29] as a suitable model for the velocity process in the Brownian diffusion, it has been of fundamental importance for theoretical studies and practical applications in physical science and financial mathematics [26, 15, 20] and references therein. It has been found that the O-U process is a univariate continuous Markov process, has a bounded variance and admits a stationary probability distribution [16, 17]. In finance the O-U process is called the Vasiček model and is one of the earliest stochastic models used to describe interest rates, currency exchange rates, and commodity prices [30]. In physics the O-U process is a prototype of a noisy relaxation process, whose probability density function (PDF) can be described by the Fokker-Planck equation [8, 14, 11]. However, the basic feature of all stochastic differential equations (SDEs) driven by Brownian motion, including the O-U process, is that the variance of an increment is proportional to the time increment. So they are not suitable to modeling the anomalous diffusion observed in the Eulerian systems [23, 24, 28], where scaling power-law of non-integer order appears universal as an empirical description of such complex phenomena. In order to overcome these drawbacks, Mandelbrot and Van Nees proposed the concept of fractional Brownian motion (fBm) in [23] and then applied the fBm to investigate the geometry of the Eulerian turbulence field in [24]. Moreover, it is worth noting that the noise term in the SDEs is used to describe the interaction between the (small) system and its (large) environment. The non-independence over disjoint time intervals in noise term applied by the environment to the system makes the fBm more useful since it can exhibit long-range dependence. For this reason, and also because the fBm includes the fundamental classical Brownian motion as a special case when $H = 1/2$, there has been an increasing interest in the research activity related to the SDEs driven by an fBm (or, fBm-driven SDEs). The interested reader can find more properties and applications of the fBm in [3, 25] and references cited therein.

Now it is appropriate to state some basic properties of the fBm. According to [3, 25], the standard fBm $(B^H(t), t \geq 0)$ is defined as a self-similar centered Gaussian process with covariance function

$$Cov(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

where $0 < H < 1$ is called the Hurst parameter. When $H = 1/2$, the fBm becomes the usual Brownian motion. In contrast to the usual Brownian motion, the fBm is neither Markovian, nor a semi-martingale [27] for

$H \neq 1/2$. Therefore, the beautiful classical theory of stochastic analysis [10] is not applicable to the fBm-driven SDEs. But several contributions in the literature have been already devoted to stochastic integration and differentiation with respect to fBm [22, 9, 12, 1, 13, 6, 4, 19]. The theory on existence and uniqueness of solutions of SDEs driven by fBm has been discussed (see [25] for a detailed review). Some sufficient and necessary conditions for reducing the nonlinear fBm-driven SDEs to the linear ones were constructed in our previous work [33], which provides an effective approach to solve some linear and nonlinear fBm-driven SDEs.

To describe the property of long memory and anomalous diffusion, it is natural to adopt the simple stochastic model, the analogue of the O-U process driven by the fBm. This fBm-driven O-U process [21] was obtained by replacing the Brownian motion in the classical O-U process by the fBm. More recently the fBm-driven O-U process has attracted considerable attention from many researchers because of its wide potential applications, such as parameter estimation [21, 7, 18, 5, 31], the local time [32], the sharp large deviation [2]. However, the comprehensive research of the fBm-driven O-U process is just at the beginning. More interesting properties of the fBm-driven O-U process are to be discovered. In the present paper, we will focus on the probability density function (PDF) of the fBm-driven O-U process by constructing the Fokker-Planck equation associated with the fBm.

The main contributions of this paper are: (i) to present the fBm-driven Fokker-Planck equation for the nonlinear fBm-driven SDEs; (ii) to reveal that the fBm-driven O-U process can exhibit anomalous diffusion.

The outline is given as follows. In Section **2** the corresponding fBm-driven Fokker-Planck equation for the nonlinear fBm-driven SDEs is constructed. In Section **3** the evolution of the PDF of the fBm-driven O-U process is obtained by applying the established fBm-driven Fokker-Planck equation. The analytic solution and asymptotic properties of such PDF are further analyzed. Numerical simulations are performed to verify the above theoretical analysis. Finally, some concluding remarks are given in Section **4**.

2. Derivation of the fBm-driven Fokker-Planck equation

Consider a nonlinear fBm-driven SDE of the form

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB^H(t), \tag{2.1}$$

where $f(t, x(t))$ and $g(t, x(t))$ are real-valued functions and $B^H(t)$ is a standard fBm with Hurst parameter H. Additionally, we assume that $f(t, x(t))$ and $g(t, x(t)) \neq 0$ possess the indicated derivatives. Then, applying the fractional Itô formula [12, Theorem 4.3] to the process $h(x)$ gives

$$\begin{aligned} dh(x) =& \Big(\frac{dh(x)}{dx} f(t,x(t)) + \frac{d^2h(x)}{dx^2} g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds\Big)dt \\ &+ \frac{dh(x)}{dx} g(t,x(t))dB^H(t), \end{aligned} \tag{2.2}$$

where the function $\phi(s,t)$ is defined by

$$\phi(s,t) = H(2H-1)|s-t|^{2H-2}.$$

Taking the expectation of equation (2.2), one has

$$\begin{aligned} E\Big[dh(x)\Big] =& E\Big[\frac{dh(x)}{dx} f(t,x(t)) + \frac{d^2h(x)}{dx^2} g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds\Big]dt \\ &+ E\Big[\frac{dh(x)}{dx} g(t,x(t))dB^H(t)\Big]. \end{aligned}$$

Therefore, we get

$$E\Big[\frac{dh(x)}{dt}\Big] = E\Big[\frac{dh(x)}{dx} f(t,x(t)) + \frac{d^2h(x)}{dx^2} g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds\Big].$$

Denote by $p(t,x)$ the PDF of a particle at time t and at point x, we have

$$E\Big[h(x)\Big] = \int_{-\infty}^{\infty} h(x)p(t,x)dx,$$

which implies that

$$E\Big[\frac{dh(x)}{dt}\Big] = \int_{-\infty}^{\infty} h(x)\frac{\partial p(t,x)}{\partial t}dx.$$

Thus we further get

$$\begin{aligned} \int_{-\infty}^{\infty} h(x)\frac{\partial p(t,x)}{\partial t}dx =& \int_{-\infty}^{\infty} \Big(\frac{dh(x)}{dx} f(t,x(t)) + \frac{d^2h(x)}{dx^2} g(t,x(t)) \\ &\times \int_0^t g(s,x(s))\phi(s,t)ds\Big)p(t,x)dx. \end{aligned} \tag{2.3}$$

After the integration by parts we find that

$$\int_{-\infty}^{\infty} \frac{dh(x)}{dx} f(t,x(t))p(t,x)dx = -\int_{-\infty}^{\infty} h(x)\frac{\partial\big(f(t,x(t))p(t,x)\big)}{\partial x}dx, \tag{2.4}$$

since $p(t, \pm\infty) = 0$. In a similar way, we further have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d^2 h(x)}{dx^2} g(t, x(t)) \int_0^t g(s, x(s))\phi(s,t)ds \; p(t,x)dx \\ = \int_{-\infty}^{\infty} h(x) \frac{\partial^2 \big(g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds \; p(t,x)\big)}{\partial x^2} dx.\end{aligned} \tag{2.5}$$

From equations (2.3), (2.4) and (2.5), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} h(x) \Big(\frac{\partial p(t,x)}{\partial t} + \frac{\partial \big(f(t,x(t))p(t,x)\big)}{\partial x} \\ - \frac{\partial^2 \big(g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds \; p(t,x)\big)}{\partial x^2} \Big) dx = 0,\end{aligned}$$

which leads to

$$\begin{aligned}\frac{\partial p(t,x)}{\partial t} + \frac{\partial \big(f(t,x(t))p(t,x)\big)}{\partial x} \\ - \frac{\partial^2 \big(g(t,x(t)) \int_0^t g(s,x(s))\phi(s,t)ds \; p(t,x)\big)}{\partial x^2} = 0.\end{aligned} \tag{2.6}$$

This is called the fBm-driven Fokker-Planck equation for stochastic differential equation (2.1) because the diffusion term in (2.1) is the fBm. It is of interest that when $H = 1/2$ equation (2.6) will reduce to the classical Fokker-Planck equation for the SDE driven by Brownian motion.

3. Probability density function for fBm-driven O-U process

We consider the Ornstein-Uhlenbeck (O-U) process driven by an fBm

$$dx(t) = -\theta x(t)dt + \sigma dB^H(t), \; x(0) = x_0, \tag{3.1}$$

where $B^H(t)$ is an fBm with Hurst index $0 < H < 1$ and $\theta > 0$ is an unknown parameter. By the method presented in our previous paper [33], the solution of equation (3.1) can be expressed by

$$x(t) = x_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dB^H(t). \tag{3.2}$$

For $\theta = 1$, $\sigma = 0.3$, $x(0) = 1$, and $t \in [0, 10]$, three sample trajectories of the fBm-driven O-U process (3.1) are shown in Fig. 3.1 for different Hurst parameters $H = 0.25$, $H = 0.5$ and $H = 0.75$, respectively.

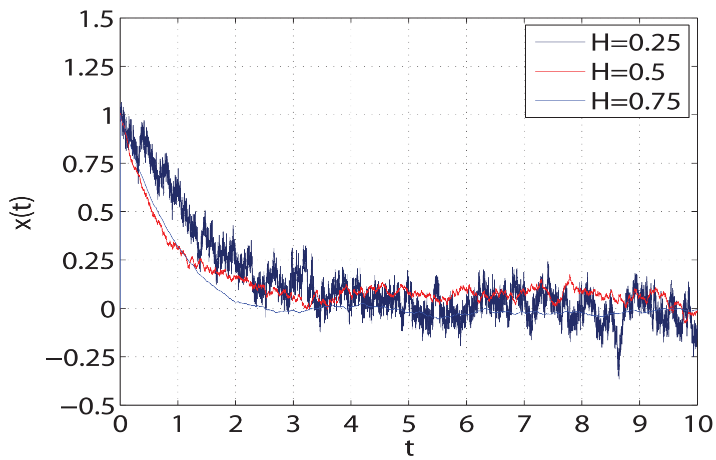

Fig. 3.1: Three sample trajectories of system (3.1)

Next we focus on the evolution of the PDF of system (3.1). According to the fBm-driven Fokker-Planck equation (2.6), one has

$$\frac{\partial p(t,x)}{\partial t} = \theta \frac{\partial\big(x(t)p(t,x)\big)}{\partial x} + H\sigma^2 t^{2H-1} \frac{\partial^2 p(t,x)}{\partial x^2}, \tag{3.3}$$

with initial condition $p(0,x) = \delta(x - x_0)$. Denote the Fourier transform with respect to x, for each fixed t, of $p(t,x)$ by

$$\widehat{p}(t,\xi) = \mathcal{F}\{p(t,x)\} = \int_{-\infty}^{\infty} u(t,\omega) e^{-i\xi\omega} d\omega.$$

So applying the Fourier transform to both sides of equation (3.3) gives

$$\frac{\partial \widehat{p}(t,\xi)}{\partial t} + \theta\xi\widehat{p}(t,\xi) = -H\sigma^2 t^{2H-1}\xi^2\widehat{p}(t,\xi). \tag{3.4}$$

By the method of characteristics, we suppose that

$$\frac{d\widehat{p}(t,\xi)}{dt} = \frac{\partial\widehat{p}(t,\xi)}{\partial t} + \frac{\partial\widehat{p}(t,\xi)}{\partial \xi}\frac{d\xi}{dt} = \frac{\partial\widehat{p}(t,\xi)}{\partial t} + \theta\xi\frac{\partial\widehat{p}(t,\xi)}{\partial \xi},$$

which implies that

$$\frac{d\xi}{dt} = \theta\xi,$$

and its solution is expressed by $\xi = \xi_0 e^{\theta t}$. It means that the left hand side of equation (3.4) is the total time derivative along each of the curve $\xi(t,\xi_0)$. On this curve, we then get

$$\frac{d\widehat{p}(t,\xi)}{dt} = -H\sigma^2 t^{2H-1}\xi^2\widehat{p}(t,\xi) = -H\sigma^2\xi_0^2 t^{2H-1} e^{2\theta t}\widehat{p}(t,\xi).$$

Since $\widehat{p}(0,\xi)=\mathcal{F}\{\delta(x-x_0)\}=e^{-i\xi x_0}$, we further obtain

$$\begin{aligned}\widehat{p}(t,\xi) &= \widehat{p}(0,\xi_0)\exp\Big(\int_0^t -H\sigma^2\xi_0^2 s^{2H-1}e^{2\theta s}ds\Big)\\ &= \exp\Big(-i\xi_0 x_0 - H\sigma^2\xi_0^2\int_0^t s^{2H-1}e^{2\theta s}ds\Big)\\ &= \exp\Big[-i\xi\Big(x_0e^{-\theta t}\Big)-\frac{\xi^2}{2}\Big(2H\sigma^2e^{-2\theta t}\int_0^t s^{2H-1}e^{2\theta s}ds\Big)\Big].\end{aligned}$$

Let $\mu(t)=x_0e^{-\theta t}$ and

$$\gamma^2(t)=2H\sigma^2e^{-2\theta t}\int_0^t s^{2H-1}e^{2\theta s}ds, \tag{3.5}$$

and it follows that

$$p(t,x)=\frac{1}{\sqrt{2\pi\gamma^2(t)}}\exp\Big(-\frac{(x-\mu(t))^2}{2\gamma^2(t)}\Big). \tag{3.6}$$

This means that the fBm-driven O-U process $x(t)$ is a Gaussian process with the mean $\mu(t)$ and a standard deviation $\gamma(t)$. For $\theta=1$, $\sigma=0.3$, and $x(0)=1$, we carry out the numerical simulations for evolution of the PDF (3.6) for different Hurst parameters $H=0.25$, $H=0.5$ and $H=0.75$, respectively. From Fig. 3.2, we can see that both of the mean and variance tend to zero as t from 2 to 350 when $0<H<1/2$. Moreover, comparing with the Fig. 3.2 and Fig. 3.3, we can observe that the PDF exhibits heavy-tailed behavior for the large time when $1/2<H<1$. This means the PDF (3.6) may have an infinite variance, which will be verified in Remark **3.2**.

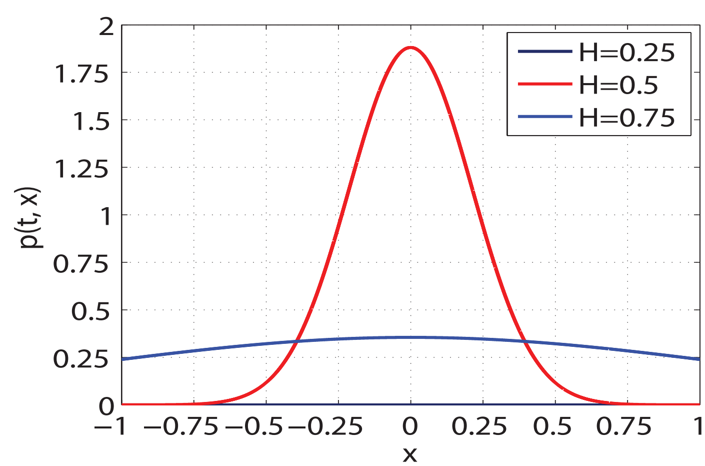

Fig. 3.2: The evolution of the PDF (3.6) for $t\in[0,350]$

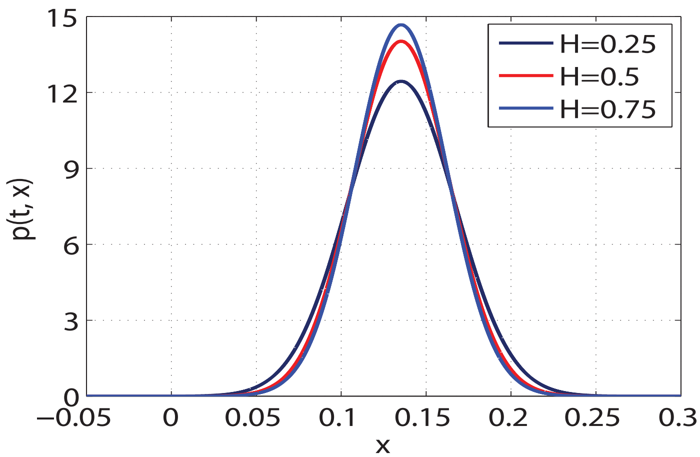

Fig. 3.3: The evolution of the PDF (3.6) for $t \in [0, 2]$

REMARK **3.1**. The limit $\theta \to 0$ gives

$$\mu(t) = x_0, \text{ and } \gamma^2(t) = \sigma^2 t^{2H}.$$

In this case, the fBm-driven O-U process (3.1) reduces to a fractional Brownian motion (fBm) with mean x_0 and variance $\sigma^2 t^{2H}$. The process with $0 < H < 1/2$ describes anti-persistent or subdiffusion, while the case $1/2 < H < 1$ relates to persistent or superdiffusion. Indeed, (3.6) becomes

$$p(t, x) = \frac{1}{\sqrt{2\pi\sigma^2 t^{2H}}} \exp\Big(- \frac{(x - x_0)^2}{2\sigma^2 t^{2H}} \Big).$$

REMARK **3.2**. It is worth noting that there is significant difference between the fBm-driven O-U process and the fBm. Formally, for the fBm the drift term is constant, whereas for the fBm-driven O-U process it is dependent on the current value of the process. In fact, for large time the asymptotic behavior of the variance is given by

$$\gamma^2(t) \sim \begin{cases} \frac{H\sigma^2}{\theta} t^{-(1-2H)}, & 0 < H < \frac{1}{2}, \\ \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}), & H = \frac{1}{2}, \\ \frac{H\sigma^2}{\theta} t^{2H-1}, & \frac{1}{2} < H < 1. \end{cases} \tag{3.7}$$

On one hand, when $0 < H \leq 1/2$ the fBm-driven O-U process has a bounded variance. Particularly, the variance decays with power-law of exponent $1 - 2H$ to zero for $0 < H < 1/2$, while the variance decays with exponential law to a constant $\sigma^2/2\theta$.

On the other hand, there is obvious difference for $1/2 < H < 1$. In this case, the fBm-driven O-U process has an infinite variance, and the infinite variance behaves like a power-law function of exponent $2H-1$. This means the fBm-driven O-U process can characterize subdiffusion phenomena when $1/2 < H < 1$. From (3.7), the speed of the subdiffusion is proportional to the Hurst parameter H, while it is inversely proportional to the viscosity coefficient θ.

In the following, we carry out some numerical simulations to illustrate the above findings. Given the parameters $\theta = 1$, $\sigma = 0.3$, $x(0) = 1$, and $t \in [0, 350]$, the variance function of fBm-driven O-U process (3.1) is plotted in Fig. 3.4 for $H = 0.25$, $H = 0.5$, and 0.75 respectively. From Fig. 3.4, we can observe that the variance tends to zero when $0 < H < 1/2$, equals 0.045 when $H = 1/2$, and keeps the trend of increasing when $1/2 < H < 1$.

Next we discuss the speed of the the subdiffusion according to the numerical simulation. Fix the parameters $\theta = 1$, $\sigma = 0.3$, $x(0) = 1$, and $t \in [0, 350]$, the evolution of the variance (3.5) by choosing two different Hurst parameter $H = 0.6$ and $H = 0.9$ is shown in Fig. 3.5. Indeed, we find that the quotient $\gamma^2(t)/t^{2H-1}$ equals 0.0809 for $H = 0.9$, and 0.0540 for $H = 0.6$. This implies that $0.0809/0.540 = 1.498 \approx 1.5 = 0.9/0.6$. In other words, the speed of the subdiffusion is proportional to the Hurst parameter H. Furthermore, the evolution of the variance (3.5) is shown in Fig. 3.6 by choosing two different viscosity coefficient $\theta = 0.5$ and $\theta = 1$. From Fig. 3.6, we can find that the quotient $\gamma^2(t)/t^{2H-1}$ equals 0.1438 for $\theta = 0.5$, and 0.0719 for $\theta = 1$. Thus $0.1438/0.0719 = 2 = 1/0.5$, which implies that the speed of the subdiffusion is inversely proportional to the viscosity coefficient θ.

4. Concluding remarks

In this paper, the fBm-driven O-U process has been investigated. Firstly, we have presented the fBm-driven Fokker-Planck equation for the nonlinear fBm-SDEs by applying the fractional Itô formula. Then we have established the PDF of the fBm-driven O-U process. We have further obtained the closed form of this PDF by combining the Fourier transform and the method of characteristics. In particular, the fBm-driven O-U process has a bounded variance for $0 < H \leq 1/2$ whereas the variance is infinite with a power-law exponent $2H-1$ for $1/2 < H < 1$. It means the fBm-driven O-U process can describe the subdiffusion in the latter case. The speed of the subdiffusion is proportional to the Hurst parameter H, while is inversely proportional to the viscosity coefficient θ. Finally, we have carried out numerical simulations to illustrate the theoretical results.

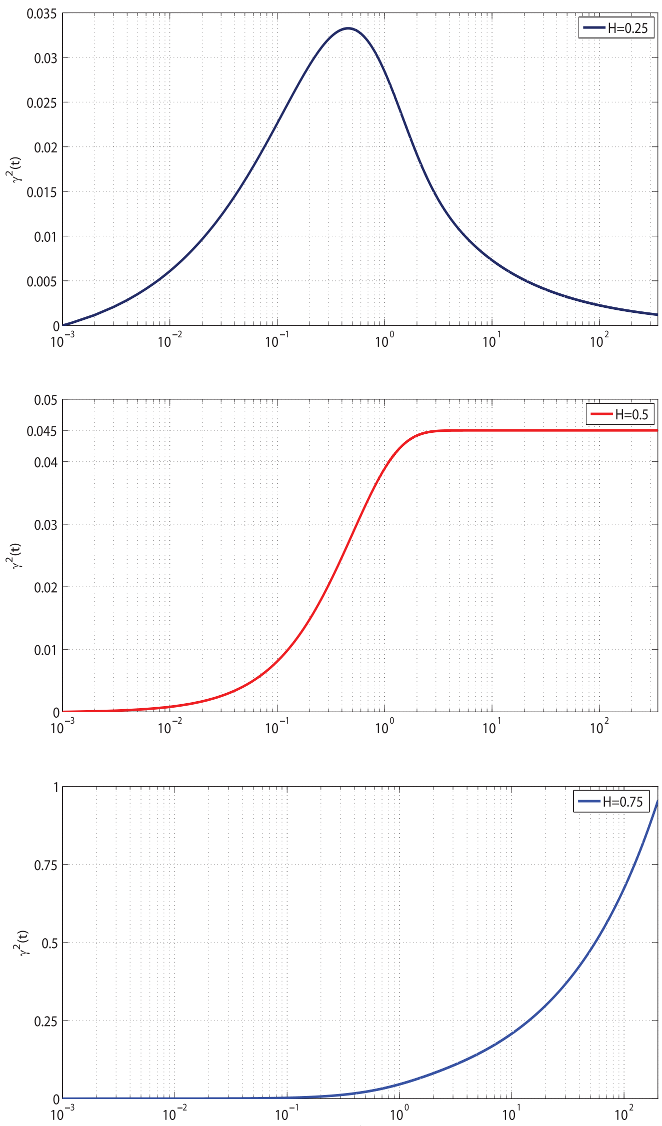

Fig. 3.4: The evolution of the variance (3.5)

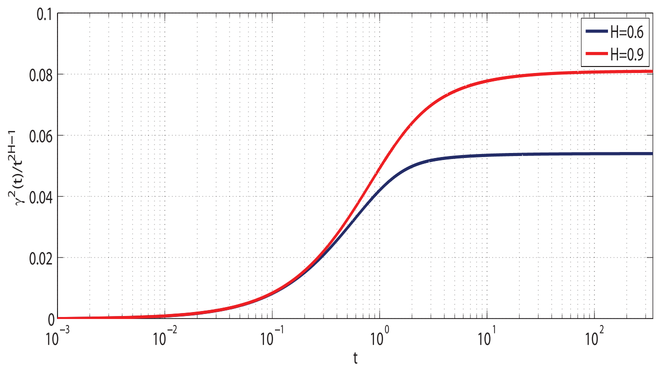

Fig. 3.5: The speed of subdiffusion of equation (3.1) for different H

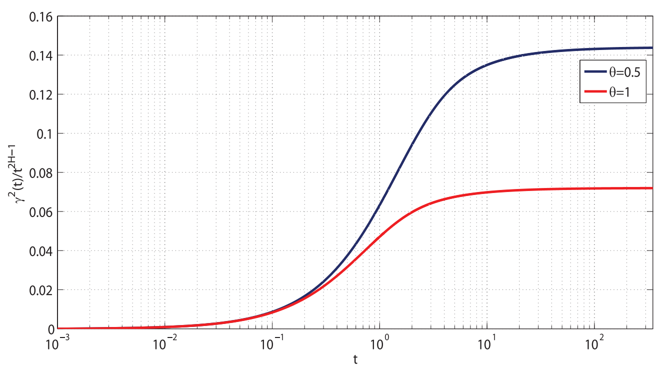

Fig. 3.6: The speed of subdiffusion of equation (3.1) for different θ

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