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RESEARCH PAPER

ANTI-PERIODIC FRACTIONAL BOUNDARY VALUE PROBLEMS WITH NONLINEAR TERM DEPENDING ON LOWER ORDER DERIVATIVE

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Abstract

This paper studies a new class of anti-periodic boundary value problems of fractional differential equations with nonlinear term depending on lower order fractional derivative. Some existence and uniqueness results are obtained by applying some standard fixed point principles. Several examples are given to illustrate the results.

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Key Words and Phrases: fractional differential equations, anti-periodic boundary conditions, fixed point theorems

1. Introduction

In recent years, a variety of problems involving differential equations and inclusions of fractional order have been investigated by several researchers. Fractional differential equations appear naturally in a number of fields such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. For theory and applications of fractional calculus, see e.g. ([17], [19], [20], [21]). Some recent works on fractional differential equations and inclusions can be found in a series of papers ([1], [3], [4], [5], [6], [8], [12], [13], [15], [16], [18], [24]) and the references therein.

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Anti-periodic boundary value problems occur in the mathematical modelling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic boundary conditions, see ([2], [7], [9], [11], [14], [23]) and the references therein.

In this paper, we investigate the existence and uniqueness of solutions for an anti-periodic fractional boundary value problem given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), {}^cD^r x(t)), \ t \in [0,T], \ T > 0, \ 1 < q \leq 2, \ 0 < r \leq 1, \\ x(0) = -x(T), \qquad {}^cD^p x(0) = -{}^cD^p x(T), \ 0 < p < 1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function.

In the case $p = 1$, we have the usual antiperiodic conditions $x(0) = -x(T)$, $x'(0) = -x'(T)$. In the case $q = 2, r = 1$, the equation in (1.1) is precisely $x''(t) = f(t, x(t), x'(t))$. We point out that our results do not reduce to the case p=1 when $p \to 1^-$. Hence we consider a new class of anti-periodic boundary value conditions involving fractional derivatives.

2. Preliminaries

Let us recall some basic definitions ([17], [19], [21]).

Definition **2.1**. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \ q > 0,$$

provided the integral exists.

Definition **2.2**. For at least $n-$times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \ n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q.

To obtain the fixed point problem equivalent to (1.1), we consider the following lemma. The proof of this lemma is given in [10]. However, for the reader's convenience, we will provide the outline of the proof.

LEMMA **2.1**. *For any $y \in C[0,T]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t),\ 0 < t < T,\ 1 < q \le 2, \\ x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T),\ 0 < p < 1, \end{cases} \tag{2.1}$$

is

$$x(t) = \int_0^T G(t,s)y(s)ds, \tag{2.2}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{q-1} - \frac{1}{2}(T-s)^{q-1}}{\Gamma(q)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)}, \\ \qquad s \le t, \\ -\dfrac{(T-s)^{q-1}}{2\Gamma(q)} + \dfrac{\Gamma(2-p)(T-2t)(T-s)^{q-p-1}}{2T^{1-p}\Gamma(q-p)},\ t < s. \end{cases} \tag{2.3}$$

(Note that this Green's function depends on q and p.)

P r o o f. For $1 < q \le 2$, it is well known [17] that the general solution of the equation ${}^cD^q x(t) = y(t)$ can be written as

$$x(t) = x(t) = I^q y(t) - b_1 - b_2 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds - b_1 - b_2 t, \tag{2.4}$$

where $b_1, b_2 \in R$ are arbitrary constants. Using the boundary conditions of (2.1) in (2.4) together with the fact that ${}^cD^p b = 0$ (b is a constant), ${}^cD^p t = \frac{t^{1-p}}{\Gamma(2-p)}$, ${}^cD^p I^q y(t) = I^{q-p}y(t)$, the values of b_1, b_2 are found to be

$$b_1 = \frac{1}{2\Gamma(q)} \int_0^T (T-s)^{q-1} y(s)ds - \frac{T^p\Gamma(2-p)}{2} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} y(s)ds,$$
$$b_2 = \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} y(s)ds.$$

Inserting these values in (2.4), we get (2.2). This completes the proof. □

3. Existence results

For $0 < r \le 1$, let $\mathcal{C} = \{x : x,\ {}^cD^r x \in C([0,T],\mathbb{R})\}$ denote the Banach space of all continuous functions defined on $[0,T]$ into $\mathbb{R}$ endowed with a topology of uniform convergence with the norm $\|x\| = \sup\{|x(t)| + |{}^cD^r x(t)|, t \in [0,T]\}$.

In the sequel, we need the following known fixed point theorems.

THEOREM **3.1**. ([22]) *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X \mid u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

THEOREM **3.2**. ([22]) *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \ \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

In relation to (1.1), we define an operator $\mathcal{G} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}(\mathcal{G}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s)) ds \\ &- \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^c D^r x(s)) ds \\ &+ \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \\ &\times \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s, x(s), {}^c D^r x(s)) ds, \ t \in [0, T].\end{aligned} \tag{3.1}$$

Observe that the problem (3.1) has a solution if and only if the operator $\mathcal{G}$ has a fixed point.

THEOREM **3.3**. *Assume that there exists $\nu \in C([0, T], \mathbb{R}^+)$ such that $|f(t, x(t), {}^c D^r x(t))| \leq \nu(t)$ for $t \in [0, T]$, $x \in \mathcal{C}$, with $\max_{t\in[0,T]} |\nu(t)| = \|\nu\|$. Then the problem (1.1) has at least one solution.*

P r o o f. For $x \in \mathcal{C}$, let $(Fx)(t) = f(t, x(t), {}^c D^r x(t)), \ t \in [0, T]$. The operator $\mathcal{G}$ maps $\mathcal{C}$ into $\mathcal{C}$ since f is continuous and

$$({}^c D^r \mathcal{G}x)(t) = (I^{q-r} Fx)(t) - \frac{b_2 t^{1-r}}{\Gamma(2-r)},$$

where b_2 is a constant (see the proof of Lemma **2.1**) given by

$$b_2 = \frac{\Gamma(2-p)}{T^{1-p}} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} (Fx)(s) ds.$$

We now show that the operator $\mathcal{G}$ is completely continuous. Let $\Omega \subset \mathcal{C}$ be bounded. Then, $\forall x \in \Omega$ together with the assumption $|f(t, x(t), {}^c D^r x(t))| \leq \nu(t)$, we get

$$
\begin{aligned}
|(\mathcal{G}x)(t)| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s))| ds \\
& + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x(s), {}^c D^r x(s))| ds \\
& + \frac{\Gamma(2-p)(T-2t)}{2T^{1-p}} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x(s), {}^c D^r x(s))| ds \\
\leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \nu(s) ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} \nu(s) ds \\
& + \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} \nu(s) ds \\
\leq & \frac{T^q [3\Gamma(q-p+1) + \Gamma(q+1)\Gamma(2-p)] \, \|\nu\|}{2\Gamma(q+1)\Gamma(q-p+1)} = M_1,
\end{aligned}
\tag{3.2}
$$

Furthermore,

$$
\begin{aligned}
|({}^c D^r \mathcal{G}x)(t)| \leq & \int_0^t \frac{(t-s)^{q-r-1}}{\Gamma(q-r)} |f(s, x(s), {}^c D^r x(s))| ds \\
& + \frac{\Gamma(2-p)t^{1-r}}{T^{1-p}\Gamma(2-r)} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x(s), {}^c D^r x(s))| ds \\
\leq & \frac{T^{q-r} \left[\Gamma(q-p+1)\Gamma(2-r) + \Gamma(q-r+1)\Gamma(2-p)\right] \|\nu\|}{\Gamma(q-r+1)\Gamma(2-r)\Gamma(q-p+1)} = M_2.
\end{aligned}
\tag{3.3}
$$

Now, for $t_1, t_2 \in [0, T]$, we have

$$
\begin{aligned}
& |(\mathcal{G}x)(t_2) - (\mathcal{G}x)(t_1)| \\
\leq & \Big| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s), {}^c D^r x(s)) ds \\
& - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s), {}^c D^r x(s)) ds \\
& - \frac{\Gamma(2-p)(t_2 - t_1)}{T^{1-p}} \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} f(s, x(s), {}^c D^r x(s)) ds \Big| \\
\leq & \left| \frac{t_2^q - t_1^q}{\Gamma(q+1)} - \frac{\Gamma(2-p)(t_2 - t_1)T^{q-1}}{\Gamma(q-p+1)} \right| \|\nu\|,
\end{aligned}
$$

$$\begin{aligned}
&|({}^cD^r\mathcal{G}x)(t_2) - ({}^cD^r\mathcal{G}x)(t_1)|\\
&\leq \Big|\frac{1}{\Gamma(q-r)}\int_0^{t_2}(t_2-s)^{q-r-1}f(s,x(s),{}^c D^r x(s))ds\\
&-\frac{1}{\Gamma(q-r)}\int_0^{t_1}(t_1-s)^{q-r-1}f(s,x(s),{}^c D^r x(s))ds\\
&-\frac{\Gamma(2-p)(t_2^{1-r}-t_1^{1-r})}{T^{1-p}\Gamma(2-r)}\int_0^T\frac{(T-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s),{}^c D^r x(s))ds\Big|\\
&\leq \Big|\frac{t_2^{q-r}-t_1^{q-r}}{\Gamma(q-r+1)}-\frac{\Gamma(2-p)(t_2^{1-r}-t_1^{1-r})T^{q-1}}{\Gamma(q-p+1)\Gamma(2-r)}\Big|\,\|\nu\|.
\end{aligned}$$

The functions $t,\ t^q,\ t^{q-r}$, and t^{1-r} are uniformly continuous on $[0,T]$ since $1<q\leq 2,\ q-r>0$ and $1-r\geq 0$. Hence, by the Arzela-Ascoli theorem, the sets $\{\mathcal{G}x : x\in\Omega\}$ and $\{{}^cD^r\mathcal{G}x : x\in\Omega\}$ are relatively compact in $C([0,T])$. Therefore, $\mathcal{G}(\Omega)$ is a relatively compact subset of $\mathcal{C}$.

Next, we consider the set

$$V=\{x\in\mathcal{C}\mid x=\mu\mathcal{G}x,\ 0<\mu<1\},$$

and show that the set V is bounded. Let $x\in V$, then $x=\mu\mathcal{G}x,\ 0<\mu<1$. For any $t\in[0,T]$, we have

$$\begin{aligned}
x(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^c D^r x(s))ds\\
&-\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^c D^r x(s))ds\\
&+\frac{\Gamma(2-p)(T-2t)}{2T^{1-p}}\int_0^T\frac{(T-s)^{q-p-1}}{\Gamma(q-p)}f(s,x(s),{}^c D^r x(s))ds,
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
|x(t)| = \ &\mu|(\mathcal{G}x)(t)|\leq\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s),{}^c D^r x(s))|ds\\
&+\frac{1}{2}\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s),{}^c D^r x(s))|ds\\
&+\frac{\Gamma(2-p)|T-2t|}{2T^{1-p}}\int_0^T\frac{(T-s)^{q-p-1}}{\Gamma(q-p)}|f(s,x(s),{}^c D^r x(s))|ds\\
&\leq\max_{t\in[0,T]}\left\{\frac{2t^q+T^q}{2\Gamma(q+1)}+\frac{\Gamma(2-p)|T-2t|T^{q-1}}{2\Gamma(q-p+1)}\right\}\|\nu\|=M_3.
\end{aligned}$$

Thus, $\|x\| \leq M_3$ for any $t \in [0,T]$. So, the set V is bounded. Thus, by the conclusion of Theorem **3.1**, the operator $\mathcal{G}$ has at least one fixed point, which implies that (1.1) has at least one solution. □

THEOREM **3.4**. *Let there exist positive constants τ_1, τ_2 such that $|f(t,x,y)| \leq \delta(|x|+|y|)$ with $0<|x|<\tau_1$, $0<|y|<\tau_2$, where δ is a positive constant. Then the problem (1.1) has at least one solution for δ small.*

P r o o f. In view of our assumption, we fix

$$\max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)} + \frac{\Gamma(2-p)|T-2t|T^{q-1}}{2\Gamma(q-p+1)},\right.$$
$$\left.\frac{t^{q-r}}{\Gamma(q-r+1)}\frac{\Gamma(2-p)t^{1-r}T^{q-1}}{\Gamma(2-r)\Gamma(q-p+1)}\right\}\delta \leq 1. \tag{3.5}$$

Define $\Omega = \{x \in \mathcal{C} \mid \|x\| < r\}$ and take $x \in$ C such that $\|x\| = r$, that is, $x \in \partial\Omega$. As before, it can be shown that $\mathcal{G}$ is completely continuous and

$$|\mathcal{G}x(t)| \leq \max_{t\in[0,T]}\left\{\frac{2|t^q|+T^q}{2\Gamma(q+1)} + \frac{\Gamma(2-p)|T-2t|T^{q-1}}{2\Gamma(q-p+1)}\right\}\delta\|x\|, \tag{3.6}$$

$$|({}^cD^r\mathcal{G}x)(t)| \leq \max_{t\in[0,T]}\left\{\frac{t^{q-r}}{\Gamma(q-r+1)} + \frac{\Gamma(2-p)t^{1-r}T^{q-1}}{\Gamma(2-r)\Gamma(q-p+1)}\right\}\delta\|x\|,$$

where we have used $\|x\| = \sup\{|x(t)| + |{}^cD^r x(t)|, t \in [0,T]\}$. Thus, in view of (3.5), it follows that $\|\mathcal{G}x\| \leq \|x\|$, $x \in \partial\Omega$. Therefore, by Theorem **3.2**, the operator $\mathcal{G}$ has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution. □

Before proceeding further, let us introduce some notations:

$$\mathrm{N} = \max\{\mathrm{N}_1, \mathrm{N}_2\}, \tag{3.7}$$

where

$$\mathrm{N}_1 = \frac{T^q[3\Gamma(q-p+1) + \Gamma(q+1)\Gamma(2-p)]}{2\Gamma(q+1)\Gamma(q-p+1)},$$

$$\mathrm{N}_2 = \frac{T^{q-r}[\Gamma(q-p+1)\Gamma(2-r) + \Gamma(q-r+1)\Gamma(2-p)]}{\Gamma(q-r+1)\Gamma(2-r)\Gamma(q-p+1)}.$$

THEOREM **3.5**. *Assume that $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the condition*

$$|f(t,x,y) - f(t,y,\bar{y})| \leq L(|x-y| + |\bar{x}-\bar{y}|),\ \forall t \in [0,T],\ x,\ y,\ \bar{x},\ \bar{y} \in \mathbb{R}$$

with $L < \frac{1}{\mathrm{N}}$, where N *is given by (3.7). Then the anti-periodic boundary value problem (1.1) has a unique solution.*

P r o o f. Defining $\sup_{t\in[0,T]} |f(t,0,0)| = M < \infty$ and selecting

$$R \geq \frac{M\mathrm{N}}{1-L\mathrm{N}},$$

we show that $\mathcal{G}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$. For $x \in B_R$, we have

$$\begin{aligned}
& |(\mathcal{G}x)(t)| \\
\leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s),{}^c D^r x(s))| ds \\
+\ & \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s,x(s),{}^c D^r x(s))| ds \\
+\ & \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s,x(s),{}^c D^r x(s))| ds \\
\leq\ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (|f(s,x(s),{}^c D^r x(s)) - f(s,0,0)| + |f(s,0,0)|) ds \\
+\ & \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} (|f(s,x(s),{}^c D^r x(s)) - f(s,0,0)| + |f(s,0,0)|) ds \\
+\ & \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} (|f(s,x(s),{}^c D^r x(s)) - f(s,0,0)| \\
+\ & |f(s,0,0)|) ds \\
\leq\ & (LR+M)\Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2}\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
+\ & \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds\Big] \\
\leq\ & (LR+M)\Big[\frac{T^q\Big(3\Gamma(q-p+1)+\Gamma(q+1)\Gamma(2-p)\Big)}{2\Gamma(q+1)\Gamma(q-p+1)}\Big] \\
\leq\ & (LR+M)\mathrm{N} \leq R.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|({}^c D^r \mathcal{G}x)(t)| &\leq \frac{T^{q-r}[\Gamma(q-p+1)\Gamma(2-r)+\Gamma(q-r+1)\Gamma(2-p)](LR+M)}{\Gamma(q-r+1)\Gamma(2-r)\Gamma(q-p+1)} \\
&\leq (LR+M)\mathrm{N} \leq R.
\end{aligned}$$

Thus we get $\mathcal{G}x \in B_R$. Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, we obtain

$$
\begin{aligned}
&|(\mathcal{G}x_1)(t) - (\mathcal{G}x_2)(t)| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^c D^r x_1) - f(s, x_2, {}^c D^r x_2)| ds \\
&+ \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} |f(s, x_1, {}^c D^r x_1) - f(s, x_2, {}^c D^r x_2)| ds \\
+ \;& \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} |f(s, x_1, {}^c D^r x_1) - f(s, x_2, {}^c D^r x_2)| ds \\
&\leq L\|x_1 - x_2\| \Big[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + \frac{1}{2} \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
+ \;& \frac{\Gamma(2-p)}{2T^{1-p}} |T-2t| \int_0^T \frac{(T-s)^{q-p-1}}{\Gamma(q-p)} ds \Big] \\
&\leq \Big[\frac{LT^q \Big(3\Gamma(q-p-1) + \Gamma(q+1)\Gamma(2-p) \Big)}{2\Gamma(q+1)\Gamma(q-p-1)} \Big] \|x_1 - x_2\| \\
&= L\mathrm{N}_1 \|x_1 - x_2\|.
\end{aligned}
$$

In a similar manner, we have

$$
\begin{aligned}
&|({}^c D^r \mathcal{G}x_1)(t) - ({}^c D^r \mathcal{G}x_2)(t)| \\
&\leq \frac{LT^{q-r}[\Gamma(q-p+1)\Gamma(2-r) + \Gamma(q-r+1)\Gamma(2-p)]\delta\|x\|}{\Gamma(q-r+1)\Gamma(2-r)\Gamma(q-p+1)} \|x_1 - x_2\| \\
&= L\mathrm{N}_2 \|x_1 - x_2\|.
\end{aligned}
$$

Consequently, we obtain $|(\mathcal{G}x_1)(t) - (\mathcal{G}x_2)(t)| \leq L\mathrm{N}\|x_1 - x_2\|$. As $L < \frac{1}{\mathrm{N}}$, $(\mathrm{N} = \max\{\mathrm{N}_1, \mathrm{N}_2\})$, therefore $\mathcal{G}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

EXAMPLE **3.1**. Consider the following anti-periodic fractional boundary value problem

$$
\begin{cases}
{}^C D^q x(t) = 3t^2 e^{-x^2(t)} \ln(4 + 3\sin^2 x(t)) + \dfrac{|{}^c D^r x(t)|}{1 + |{}^c D^r x(t)|}, \; 0 < t < 1, \\
x(0) = -x(1), \qquad {}^c D^p x(0) = -{}^c D^p x(1), \; 0 < p < 1,
\end{cases}
\tag{3.8}
$$

where $1 < q \leq 2$, $0 < r \leq 1$ and $T = 1$.

Clearly, $|f(t,x(t),^c D^r x(t))| \leq (1+3(\ln 7)t^2) = \nu(t)$. So, the hypothesis of Theorem **3.3** holds. Therefore, the conclusion of Theorem **3.3** applies to antiperiodic fractional boundary value problem (3.8).

Example **3.2**. Consider the following anti-periodic fractional boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = L\Big(x(t) + tan^{-1}({}^cD^{\frac{1}{2}}x(t))\Big), \; L>0, \; t\in[0,2], \\ x(0) = -x(2), \qquad {}^cD^{\frac{3}{4}}x(0) = -{}^cD^{\frac{3}{4}}x(2), \end{cases} \tag{3.9}$$

where $q = 3/2,\ r = 1/2,\ p = 3/4,\ f(t,x,y) = L\Big(x + tan^{-1}y\Big)$, and $T = 2$. Clearly,

$$|f(t,x,y)-f(t,\bar{x},\bar{y})| \leq L(|x-\bar{x}|+|tan^{-1}y-tan^{-1}\bar{y}|) \leq L(|x-\bar{x}|+|y-\bar{y}|),$$

where we have used the fact that $|(tan^{-1}y)'| = \frac{1}{1+y^2} < 1$. Further, $L < 1/N_1$,

$$N_1 = 4\sqrt{\frac{2}{\pi}} + \frac{\sqrt{2}\Gamma(1/4)}{3\Gamma(3/4)},\ N_2 = 2 + \frac{4\Gamma(1/4)}{3\sqrt{\pi}\Gamma(3/4)},\ (N_1 > N_2).$$

Thus, all the assumptions of Theorem **3.5** are satisfied. Hence, the fractional boundary value problem (3.9) has a unique solution on $[0,2]$.

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