

A NOTE ON THE CHARACTERISTIC RANK AND RELATED NUMBERS

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Dedicated to Professor Masaharu Morimoto on the occasion of his 60th birthday

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ABSTRACT. This note quantifies, via a sharp inequality, an interplay between

- (a) the characteristic rank of a vector bundle over a topological space X ,
- (b) the $\mathbb{Z}_2$ -Betti numbers of X , and
- (c) sums of the numbers of certain partitions of integers.

In a particular context, (c) is transformed into a sum of the readily calculable Betti numbers of the real Grassmann manifolds.

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1. Introduction

The *characteristic rank* of a smooth closed connected d -dimensional manifold M was introduced, and in some cases also calculated, in [3] as the largest integer k , $0 \leq k \leq d$, such that each cohomology class in $H^j(M; \mathbb{Z}_2)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of M ; further results can be found in [2]. Recently, A. Naolekar and A. Thakur ([6]) have adapted this homotopy invariant of smooth closed connected manifolds to vector bundles. Given a path-connected topological space X and a real vector bundle α over X , by the *characteristic rank of α* , denoted $\text{charrank}(\alpha)$, we understand the largest k , $0 \leq k \leq \dim_{\mathbb{Z}_2}(X)$, such that every cohomology class in $H^j(X; \mathbb{Z}_2)$, $0 \leq j \leq k$, can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(\alpha) \in H^i(X; \mathbb{Z}_2)$; $\dim_{\mathbb{Z}_2}(X)$ is the supremum of all q such that the cohomology groups $H^q(X; \mathbb{Z}_2)$ do not vanish. We shall always use $\mathbb{Z}_2$ as the coefficient group for cohomology, and so we shall write $H^i(X)$ instead of $H^i(X; \mathbb{Z}_2)$ and $\dim(X)$ instead of $\dim_{\mathbb{Z}_2}(X)$. Of course, if TM denotes the tangent bundle of M , then we have $\text{charrank}(TM) = \text{charrank}(M)$. Results on the characteristic rank of vector bundles over the Stiefel manifolds can be found in [4]. In addition to being

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an interesting question in its own right, there are other reasons for investigating the characteristic rank; one of them is its close relation to the cup-length of a given space ([3], [6]).

This note presents a result on the characteristic rank that may also prove useful in some non-topological questions (for instance, in the theory of partitions; see [1]). More precisely, we quantify (Theorem 2.1), via a sharp inequality, an interplay between

- (a) the characteristic rank of a vector bundle,
- (b) the $\mathbb{Z}_2$ -Betti numbers of base space of this vector bundle, and
- (c) sums of the numbers of certain partitions of integers.

In a particular context, (c) is transformed into a sum of the readily calculable Betti numbers of the real Grassmann manifolds $G_{n,k}$ of all k -dimensional vector subspaces in $\mathbb{R}^n$.

2. The result and its proof

Let $p(a, b, c)$ be the number of partitions of c into *at most* b parts, each $\leq a$. At the same time, given a subset A of the positive integers, let $p(A, b, c)$ denote the number of partitions of c into b parts each taken from the set A . We recall that ([5: 6.7]) $p(a, b, c)$ is the same as the number of cells of dimension c in the Schubert cell decomposition of the Grassmann manifold $G_{a+b,b}$, or the same as the $\mathbb{Z}_2$ -Betti number $b_c(G_{a+b,b}) = b_c(G_{a+b,a})$. Of course, for dimensional reasons, $b_c(G_{a+b,b}) = p(a, b, c) = 0$ for $c > ab = \dim(G_{a+b,b})$. In addition, we denote by $p(c)$ the total number of partitions of c . Even if not explicitly stated, X will always mean a path-connected topological space.

For obvious reasons, we confine our considerations to those vector bundles having total Stiefel-Whitney class non-trivial. For a given space X , an ordered subset in $\{1, 2, \dots, \dim(X)\}$, with the least element (denoted by) ν and the greatest element (denoted by) κ , will be denoted by S_ν^κ . If, in addition, α is a real vector bundle over X , then $S_\nu^\kappa(\alpha)$ will denote any $S_\nu^\kappa \neq \emptyset$ such that $w_i(\alpha) = 0$ for every positive $i \notin S_\nu^\kappa$. In general, there are several possible choices of $S_\nu^\kappa(\alpha)$: one can always take $S_\nu^\kappa(\alpha) = \{1 = \nu, 2, \dots, \dim(X) = \kappa\}$, but if we know, for instance, that $w_3(\alpha) = 0$, then we can also take the set $\{1 = \nu, 2, 4, \dots, \dim(X) = \kappa\}$ in the role of $S_\nu^\kappa(\alpha)$. Now we can state and prove our result.

THEOREM 2.1. *Let α be a real vector bundle over a path-connected topological space X such that $w(\alpha) \neq 1$, and let $\text{charrank}(\alpha) \geq t$ for some t . Then, for every $j \in \{1, \dots, t\}$, we have an inequality,*

$$b_j(X) \leq \sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{x \in S_\nu^\kappa(\alpha) \mid x \leq \mu\}, s, j), \quad (1)$$

where $\mu = \min\{j, \kappa\}$.

In particular, if the set $\{x \in S_\nu^\kappa(\alpha) \mid x \leq \mu\}$ in (1) is gapless, then (1) turns into

$$b_j(X) \leq \sum_{\frac{j}{\mu} \leq s \leq \frac{j}{\nu}} b_{j-\nu s}(G_{\mu-\nu+s,s}). \quad (2)$$

P r o o f. Let us fix one of the sets $S_\nu^\kappa(\alpha)$. If $\text{charrank}(\alpha) \geq t$ for some t then, for every $j \in \{1, 2, \dots, t\}$, the $\mathbb{Z}_2$ -vector space $H^j(X)$ is spanned by all the products of the form

$$w_\nu^{i_\nu}(\alpha) \cup \dots \cup w_\mu^{i_\mu}(\alpha) \in H^j(X). \quad (3)$$

Since $\nu(i_\nu + \dots + i_\mu) \leq \nu i_\nu + \dots + \mu i_\mu = j$ (thus $i_\nu + \dots + i_\mu \leq \frac{j}{\nu}$), the number of generators of the form (3), that is, an upper bound for $b_j(X)$, is the number of partitions of j into at most $\lfloor \frac{j}{\nu} \rfloor$ positive parts each taken from the set $\{x \in S_\nu^\kappa(\alpha) \mid x \leq \mu\}$. In other words, this upper bound is $\sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{x \in S_\nu^\kappa(\alpha) \mid x \leq \mu\}, s, j)$ as was asserted.

In order to transform (1) into (2) when the set $\{x \in S_\nu^\kappa(\alpha) \mid x \leq \mu\}$ in (1) is gapless, it suffices to know that

$$\sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{\nu, \nu+1, \nu+2, \dots, \mu\}, s, j) = \sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\mu-\nu, s, j-\nu s); \quad (4)$$

this equality is verified by the following elementary considerations.

Let $P(j)_{\{\nu, \nu+1, \dots, \mu\}}^x$ be the set of partitions of j into x positive parts each taken from the set $\{\nu, \nu+1, \dots, \mu\}$ and let $P(l)_{\{1, 2, \dots, \mu-\nu\}}^{\leq x}$ be the set of partitions of l ($l \geq 0$) into at most x parts each taken from the set $\{1, 2, \dots, \mu-\nu\}$ (the set $P(0)_{\{1, 2, \dots, \mu-\nu\}}^{\leq x}$ for all $x > 0$ and $\mu-\nu > 0$ contains just one element, namely the empty partition). Of course, the total number of elements in $P(l)_{\{1, 2, \dots, \mu-\nu\}}^{\leq x}$ is $p(\mu-\nu, x, l)$.

Each element of $P(j)_{\{\nu, \nu+1, \dots, \mu\}}^x$ has the form

$$a_1 + \dots + a_{i(\nu)} + a_{i(\nu)+1} + \dots + a_x,$$

where $i(\nu) \geq 0$, $a_1 = \dots = a_{i(\nu)} = \nu < a_{i(\nu)+1} \leq a_{i(\nu)+2} \leq \dots \leq a_x$ and $a_1 + \dots + a_{i(\nu)} + a_{i(\nu)+1} + \dots + a_x = j$. The map

$$P(j)_{\{\nu, \nu+1, \dots, \mu\}}^x \longrightarrow P(j-\nu x)_{\{1, 2, \dots, \mu-\nu\}}^{\leq x},$$

$a_1 + \dots + a_{i(\nu)} + a_{i(\nu)+1} + \dots + a_x \mapsto (a_{i(\nu)+1} - \nu) + (a_{i(\nu)+2} - \nu) + \dots + (a_x - \nu)$ is bijective; indeed, the inverse map is readily seen to be

$$P(j-\nu x)_{\{1, 2, \dots, \mu-\nu\}}^{\leq x} \longrightarrow P(j)_{\{\nu, \nu+1, \dots, \mu\}}^x,$$

$$b_1 + b_2 + \dots + b_{x-i} \mapsto \underbrace{\nu + \dots + \nu}_{i \text{ times}} + (b_1 + \nu) + \dots + (b_{x-i} + \nu).$$

Thus (4) is verified, and the proof of Theorem 2.1 is finished. $\square$

Example 2.2. We recall ([5: Theorem 7.1]) that the cohomology algebra of the infinite Grassmannian $G_{\infty, k}$ can be identified with a polynomial algebra,

$$H^*(G_{\infty, k}) = \mathbb{Z}_2[w_1, \dots, w_k],$$

where $w_i \in H^i(G_{\infty,k})$ is the i th Stiefel-Whitney class of the universal k -plane bundle $\gamma_{\infty,k}$. Thus $\text{charrank}(\gamma_{\infty,k}) = \infty$ and we may take $S_{\nu}^{\kappa}(\gamma_{\infty,k}) = \{1 = \nu, 2, \dots, k = \kappa\}$. Since for $X = G_{\infty,k}$ there are no relations among the generators of the form (3), inequalities (1) and (2) turn into one of the following equalities for any positive integer j :

$$b_j(G_{\infty,k}) = p(j) = \sum_{s=1}^j b_{j-s}(G_{j-1+s,s}) \quad \text{if } j \leq k, \quad (5)$$

$$b_j(G_{\infty,k}) = \sum_{s=1}^j p(\{1, 2, \dots, k\}, s, j) = \sum_{s=\lceil \frac{j}{k} \rceil}^j b_{j-s}(G_{k-1+s,s}) \quad \text{if } j > k. \quad (6)$$

In a similar way, one can see that (1) and (2) are also sharp for $X = \tilde{G}_{\infty,k}$, the infinite oriented Grassmannian.

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