

THE L^p PRIMITIVE INTEGRAL

ERIK TALVILA

(Communicated by Gregor Dolinar)

ABSTRACT. For each $1 \leq p < \infty$, a space of integrable Schwartz distributions L'^p , is defined by taking the distributional derivative of all functions in L^p . Here, L^p is with respect to Lebesgue measure on the real line. If $f \in L'^p$ such that f is the distributional derivative of $F \in L^p$, then the integral is defined as $\int_{-\infty}^{\infty} fG = - \int_{-\infty}^{\infty} F(x)g(x) dx$, where $g \in L^q$, $G(x) = \int_0^x g(t) dt$ and $1/p + 1/q = 1$. A norm is $\|f\|_p' = \|F\|_p$. The spaces L'^p and L^p are isometrically isomorphic. Distributions in L'^p share many properties with functions in L^p . Hence, L'^p is reflexive, its dual space is identified with L^q , there is a type of Hölder inequality, continuity in norm, convergence theorems, Gateaux derivative. It is a Banach lattice and abstract L -space. Convolutions and Fourier transforms are defined. Convolution with the Poisson kernel is well defined and provides a solution to the half plane Dirichlet problem, boundary values being taken on in the new norm. A product is defined that makes L'^1 into a Banach algebra isometrically isomorphic to the convolution algebra on L^1 . Spaces of higher order derivatives of L^p functions are defined. These are also Banach spaces isometrically isomorphic to L^p .

©2014
Mathematical Institute
Slovak Academy of Sciences

1. Introduction

One way of defining an integral is through properties of its primitive. This is a function whose derivative is in some sense equal to the integrand. For example, if $f: [a, b] \rightarrow \mathbb{R}$ then $f \in L^1$ if and only if there is an absolutely continuous function F such that $F'(x) = f(x)$ for almost all $x \in (a, b)$. This then provides a descriptive definition of the Lebesgue integral in terms of the primitive F and

2010 Mathematics Subject Classification: Primary 46E30, 46F10, 46G12; Secondary 42A38, 42A85, 46B42, 46C05.

Key words: Lebesgue space, Banach space, Schwartz distribution, generalised function, primitive, integral, Fourier transform, convolution, Banach lattice, Hilbert space, Poisson integral.

the fundamental theorem of calculus formula $\int_a^b f(x) dx = F(b) - F(a)$. This same approach can be used to define Henstock-Kurzweil and wide Denjoy integrals. See [6] for the relevant spaces of primitives.

In this paper we define integrals of tempered distributions by taking L^p as the space of primitives for $1 \leq p < \infty$. Such functions need not have pointwise derivatives so the distributional derivative is used. The distributions integrable in this sense are the weak derivative of L^p functions but have many properties similar to L^p functions. This approach was followed in [16] with the continuous primitive integral. The primitives were functions continuous on the extended real line. The space of distributions integrable in this sense is a Banach space under the Alexiewicz norm, isometrically isomorphic to the space of primitives with the uniform norm. Primitives were taken to be regulated functions in [18]. A function on the real line is regulated if it has a left limit and a right limit at each point. This again led to a Banach space of distributions that was isometrically isomorphic to the space of primitives with the uniform norm. The space of distributions that have a continuous primitive integral is the completion of L^1 and the space of Henstock-Kurzweil integrable functions in the Alexiewicz norm. The regulated primitive integral provides the completion of the signed Radon measures in the Alexiewicz norm. In the current paper we again define a Banach space of distributions, only now it is isometrically isomorphic to an L^p space.

The outline of the paper is as follows.

Section 2 provides some notation for integrals and distributions. Then in Section 3 we define the space L'^p to be the set of distributions that are the distributional derivative of L^p functions on the real line. This is our space of integrable distributions. We define I^q to be the absolutely continuous functions that are the indefinite integral of functions in L^q . See Definition 3.1. Such functions are multipliers and the integral is defined as $\int_{-\infty}^{\infty} fG := - \int_{-\infty}^{\infty} F(x)G'(x) dx$ where $f \in L'^p$ such that $f = F'$ for $F \in L^p$ and $G(x) = \int_0^x g(t) dt$ for a function $g \in L^q$ (Definition 3.5). Here, $1 \leq p < \infty$ and $1/p + 1/q = 1$. We are therefore defining $\int_{-\infty}^{\infty} fG$ in terms of the Lebesgue integral $\int_{-\infty}^{\infty} F(x)G'(x) dx$ with respect to Lebesgue measure. Primitives are unique since there are no constant functions in L^p (Theorem 3.2). A norm on L'^p is then $\|f\|_p' = \|F\|_p$. This makes L'^p into a Banach space isometrically isomorphic to L^p . Many properties of L^p are then inherited by L'^p . The remaining theorems and propositions in this section show that L'^p is separable, the dual space is isometrically isomorphic to

L^q , the unit ball is uniformly convex, L'^p is reflexive, it is invariant under translations, there is continuity in norm, the Schwartz space of rapidly decreasing C^∞ functions is dense, there are versions of the Hölder and Hanner inequalities. A Gateaux derivative is computed. We prove a norm convergence theorem and define an equivalent norm. Under pointwise operations L^p is a Banach lattice. If $f, g \in L'^p$ with primitives $F, G \in L^p$, define $f \preceq g$ whenever $F(x) \leq G(x)$ for almost all $x \in \mathbb{R}$. This makes L'^p into a Banach lattice that is lattice isomorphic to L^p . It is then an abstract L -space in the sense of Kakutani. Under this ordering there is a version of the dominated convergence theorem. At the end of this section we show how to approximate the integral by a sequence of derivatives of step functions.

In Section 4 various examples are given. Proposition 4.1 gives an integral condition with a power growth weight that ensures a function is in L'^p . Examples are given of functions or distributions in L'^p that are not in L^1_{loc} , not in any L^p space, are the difference of translations of the Dirac distribution, have conditionally convergent integrals, have principal value integrals, have primitives whose pointwise derivative vanishes almost everywhere or exists nowhere.

Convolutions are defined as $\ast: L'^p \times I^q \rightarrow L^\infty$ for p and q conjugate. Various properties are proved in Theorem 5.1. In this case there are many results similar to convolutions defined on $L^p \times L^q$, such as uniform continuity. Convolutions are defined in Definition 5.2 as $\ast: L'^p \times L^q \rightarrow L'^r$ and $\ast: L^p \times L'^q \rightarrow L'^r$ using a sequence in $L^q \cap I^q$ that converges to a given function in L^q . Here, $p, q, r \in [1, \infty)$ such that $1/p + 1/q = 1 + 1/r$. Properties of the convolution that mirror properties of convolutions in $L^p \times L^q$ are proved in Theorem 5.3. A different type of product is defined in Theorem 5.4 that makes L'^1 into a Banach algebra isometrically isomorphic to the convolution algebra on L^1 .

Since L'^1 is isometrically isomorphic to L^1 , Fourier transforms can be defined directly using the usual integral definition. If $f \in L'^1$, then its Fourier transform is given by the integral $\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-ist} dt = i s \hat{F}(s)$, where $F \in L^1$ is the primitive of f . It is shown that this agrees with the definition for tempered distributions. Fourier transforms of distributions in L'^1 are continuous functions and the Riemann-Lebesgue lemma takes the form $\hat{f}(s) = o(s)$ as $|s| \rightarrow \infty$. Many of the usual properties of L^1 Fourier transforms continue to hold in L'^1 . See Theorem 6.2.

In Section 7 some special properties of L'^2 are considered. The space L'^2 is isometrically isomorphic to L^2 so it is a Hilbert space. The inner product is $(f, g) = (F, G) = \int_{-\infty}^{\infty} F(x)G(x) dx$, where $f, g \in L'^2$ with respective primitives $F, G \in L^2$. The Fourier transform is defined from the L^2 Fourier transform of the primitive.

In Section 8 spaces of distributions are constructed by taking the n th distributional derivative of L^p functions. Each such space is then a separable Banach space, isometrically isomorphic to L^p . Most of the results for L'^p continue to hold in these spaces.

In Section 9 it is shown how the half plane Poisson integral can be defined for distributions that are the n th derivative of an L^p function. There are direct analogues of the usual L^p results, such as boundary values being taken on in the $\|\cdot\|'_p$ norm.

In Section 10 we sketch out how these integrals can be defined in $\mathbb{R}^n$.

2. Notation

All statements regarding measures will be with respect to Lebesgue measure, denoted λ . For $1 \leq p < \infty$, the Lebesgue space on the real line is L^p , which consists of the measurable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_{-\infty}^{\infty} |f(x)|^p dx < \infty$. To distinguish between other types of integrals introduced later, Lebesgue integrals will always explicitly show the integration variable and differential as above. The L^p spaces have norm $\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}$. And, L^∞ is the set of bounded measurable functions with norm $\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}} |f(x)|$. For $1 \leq p \leq \infty$ each L^p is a Banach space. If $1 < p < \infty$ then its conjugate exponent is $q \in \mathbb{R}$ such that $p^{-1} + q^{-1} = 1$. For $p = 1$, $q = \infty$. The locally integrable functions are L^p_{loc} and a measurable function $f \in L^p_{loc}$ if $f\chi_{[a,b]} \in L^p$ for each compact interval $[a, b]$. The set of absolutely continuous functions on the real line is denoted $AC(\mathbb{R})$ and consists of the functions $F: \mathbb{R} \rightarrow \mathbb{R}$ such that F is absolutely continuous on each compact interval in $\mathbb{R}$. And, $F \in AC(\mathbb{R})$ if and only if there is $f \in L^1_{loc}$ such that $F(x) = F(0) + \int_0^x f(t) dt$.

The extended real line is $\overline{\mathbb{R}} = [-\infty, \infty]$ and $C(\overline{\mathbb{R}})$ denotes the real-valued functions that are continuous at each point of $\mathbb{R}$ and have real limits at $-\infty$ and at ∞ . Define $AC(\overline{\mathbb{R}}) = AC(\mathbb{R}) \cap \mathcal{BV}$ where $\mathcal{BV}$ are the functions of bounded variation on the real line. Then $f \in L^1$ if and only if there is $F \in AC(\overline{\mathbb{R}})$ such that $f(x) = F'(x)$ for almost all $x \in \mathbb{R}$.

The Schwartz space, $\mathcal{S}$, of rapidly decreasing smooth functions, consists of the functions $\phi \in C^\infty(\mathbb{R})$ such that for all integers $m, n \geq 0$ we have $x^m \phi^{(n)}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Elements of $\mathcal{S}$ will be termed test functions. Sequence $(\phi_j) \subset \mathcal{S}$ is said to converge to $\phi \in \mathcal{S}$ if for all integers $m, n \geq 0$, $\sup_{x \in \mathbb{R}} |x|^m |\phi_j^{(n)}(x) - \phi^{(n)}(x)| \rightarrow 0$ as $j \rightarrow \infty$. The (tempered) distributions are then the continuous

linear functionals on $\mathcal{S}$. This dual space is denoted $\mathcal{S}'$. If $T \in \mathcal{S}'$ then $T: \mathcal{S}' \rightarrow \mathbb{R}$ and we write $\langle T, \phi \rangle \in \mathbb{R}$ for $\phi \in \mathcal{S}$. If $\phi_j \rightarrow \phi$ in $\mathcal{S}$ then $\langle T, \phi_j \rangle \rightarrow \langle T, \phi \rangle$ in $\mathbb{R}$. And, for all $a_1, a_2 \in \mathbb{R}$ and all $\phi, \psi \in \mathcal{S}$, $\langle T, a_1\phi + a_2\psi \rangle = a_1\langle T, \phi \rangle + a_2\langle T, \psi \rangle$. If $f \in L^p_{loc}$ for some $1 \leq p \leq \infty$ then $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x) dx$ defines a distribution. In such case, T_f is called regular and we will often ignore the distinction between f and T_f . The differentiation formula $\langle D^n T, \phi \rangle = \langle T^{(n)}, \phi \rangle = (-1)^n \langle T, \phi^{(n)} \rangle$ ensures that all distributions have derivatives of all orders which are themselves distributions. This is known as the distributional or weak derivative. We will usually denote distributional derivatives by $D^n F$, $F^{(n)}$ or F' and pointwise derivatives by $F^{(n)}(t)$ or $F'(t)$. For $T \in \mathcal{S}'$ and $t \in \mathbb{R}$ the translation τ_t is defined by $\langle \tau_t T, \phi \rangle = \langle T, \tau_{-t} \phi \rangle$ where $\tau_t \phi(x) = \phi(x - t)$ for $\phi \in \mathcal{S}$. The Heaviside step function is $H = \chi_{(0, \infty)}$. The Dirac distribution is $\delta = H'$. The action of δ on test function ϕ is $\langle \delta, \phi \rangle = \phi(0)$. See [9, 10, 21] for more on distributions.

Laurent Schwartz introduced the notion of integrable distribution, see [15: p. 199–203] and [4]. Define the test function spaces as $\mathcal{D}_{L^p}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) \mid \phi^{(m)} \in L^p(\mathbb{R}) \text{ for each } m \geq 0\}$ for $1 \leq p < \infty$. For $p = \infty$, define $\mathcal{B} = \mathcal{D}_{L^\infty}(\mathbb{R}) = \{\phi \in C^\infty(\mathbb{R}) \mid \|\phi\|_\infty < \infty\}$ and $\dot{\mathcal{B}} = \dot{\mathcal{D}}_{L^\infty}(\mathbb{R}) = \{\phi \in \mathcal{D}_{L^\infty}(\mathbb{R}) \mid \lim_{|x| \rightarrow \infty} \phi(x) = 0\}$. For $1 \leq p < \infty$, a sequence $(\phi_n) \subset \mathcal{D}_{L^p}(\mathbb{R})$ (or in $\dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$) converges to $\phi \in \mathcal{D}_{L^p}(\mathbb{R})$ (or $\phi \in \dot{\mathcal{D}}_{L^\infty}(\mathbb{R})$), if $\lim_{n \rightarrow \infty} \|\phi_n^{(m)} - \phi^{(m)}\|_p = 0$ for each $m \geq 0$. The integrable distributions are then $\mathcal{D}'_{L^p}(\mathbb{R})$ which is the dual of $\mathcal{D}_{L^q}(\mathbb{R})$ ($1 < p < \infty$, $1/p + 1/q = 1$) and $\mathcal{D}'_{L^1}(\mathbb{R})$ which is the dual of $\dot{\mathcal{B}}$. Schwartz's main structure theorem [15: p. 201] is that if $1 \leq p \leq \infty$ and T is a distribution then $T \in \mathcal{D}'_{L^p}(\mathbb{R})$ if and only if $T = \sum_{n=0}^m F_n^{(n)}$ for some $F_n \in L^p(\mathbb{R})$ and some $m \geq 0$. If this expansion holds then the functions F_n can be taken to be bounded and continuous.

Our theory differs in that we take $T = F^{(m)}$ for some $F \in L^p(\mathbb{R})$ and some $m \geq 1$. This is a restricted form of Schwartz's definition but it has the advantage that the resulting space of distributions is a Banach space isometrically isomorphic to $L^p(\mathbb{R})$. This provides a class of distributions that behave in many ways like L^p functions.

3. The L^p primitive integral

In this section we define Banach spaces L'^p and I^q that are isometrically isomorphic to L^p and L^q , respectively. The first serves as a space of integrable distributions and the second as a space of multipliers. The distributional derivative provides a linear isometry between L^p and L'^p and many properties of

L^p are inherited by L'^p . Hence, L'^p is a separable Banach space, reflexive with dual space isometrically isomorphic to L^q for $1/p + 1/q = 1$. There is a Hölder inequality and we prove a convergence theorem. The pointwise ordering on L^p is inherited by L'^p so that it is a Banach lattice and abstract L -space. A version of the dominated convergence theorem based on this ordering is given. At the end of this section, the integral is also defined in terms of the limit of a sequence of derivatives of step functions.

DEFINITION 3.1. Let $1 \leq p \leq \infty$.

- (a) Define $L'^p = \{f \in \mathcal{S}' \mid f = F' \text{ for } F \in L^p\}$.
- (b) Define $I^p = \left\{G: \mathbb{R} \rightarrow \mathbb{R} \mid G(x) = \int_0^x g(t) dt \text{ for some } g \in L^p\right\}$.

THEOREM 3.2.

- (a) Let $1 \leq p < \infty$. If $f \in L'^p$ there is a unique function $F \in L^p$ such that $f = F'$.
- (b) Let $1 \leq p \leq \infty$. If $G \in I^p$ there is a unique function $g \in L^p$ such that $G(x) = \int_0^x g(t) dt$.

Proof.

(a) If $f \in L'^p$ and $f = F'_1 = F'_2$ then let $F = F_1 - F_2$. Hence, $F \in L^p$ and $F' = 0$ in $\mathcal{S}'$. It follows that F is a constant distribution [10: §2.4]. The only constant distribution in L^p is 0.

(b) If there are $g_1, g_2 \in L^p$ such that $G(x) = \int_0^x g_1(t) dt = \int_0^x g_2(t) dt$ for all $x \in \mathbb{R}$ then $g_1 = g_2$ almost everywhere. □

This uniqueness is within the equivalence class structure on L^p . Two functions in L^p are equivalent if they are equal almost everywhere. We always consider L^p as a disjoint union of these equivalence classes. The unique function F in Theorem 3.2 is called the primitive of f . If $f \in L'^p$ and F is its primitive in L^p then $\langle f, \phi \rangle = \langle F', \phi \rangle = -\langle F, \phi' \rangle = -\int_{-\infty}^{\infty} F(x)\phi'(x) dx$ for all $\phi \in \mathcal{S}$. Since ϕ' is in each L^q for all $1 \leq q \leq \infty$, this last integral exists by the Hölder inequality.

Now we can show that L'^p is a Banach space isometrically isomorphic to L^p .

THEOREM 3.3. Let $1 \leq p < \infty$. Let $f, f_1, f_2 \in L'^p$. Let their respective primitives in L^p be F, F_1, F_2 . Let $a_1, a_2 \in \mathbb{R}$. Define $a_1 f_1 + a_2 f_2 = (a_1 F_1 + a_2 F_2)'$. Define $\|f\|'_p = \|F\|_p$. Then L'^p has the following properties.

- (a) It is a Banach space with norm $\|\cdot\|'_p$, isometrically isomorphic to L^p .
- (b) It is separable.
- (c) Its dual space is isometrically isomorphic to I^q where q is conjugate to p .

PROOF. Define $D: L^p \rightarrow L'^p$ by $D(F) = F'$. By Theorem 3.2, D is injective. And D is surjective by the definition of L'^p . It then follows that L^p and L'^p are isometrically isomorphic. The other properties then follow from the corresponding properties in L^p . $\square$

The inverse $D^{-1}: L'^p \rightarrow L^p$ can be formally computed as follows. Let $x \in \mathbb{R}$ and define $G_x \in L^\infty$ by $G_x = \chi_{(-\infty, x)}$. Let δ_x be Dirac measure supported at x . Let $f = F' \in L'^p$. Then $\int_{-\infty}^{\infty} f G_x = - \int_{-\infty}^{\infty} F(t) G'_x(t) dt = \int_{-\infty}^{\infty} F(t) \tau_x \delta(t) dt = \int_{-\infty}^{\infty} F(t) d\delta_x(t) = \langle \tau_x \delta, F \rangle = F(x)$ for almost all $x \in \mathbb{R}$. This is a purely formal calculation since $\int_{-\infty}^{\infty} f G_x$ has not been defined and G_x is not in L^p for any $1 \leq p < \infty$. However, let $n \in \mathbb{N}$ such that $n > -x$ and define

$$G_{n,x}(t) = \begin{cases} 0, & t \leq -2n \\ (t + 2n)/n, & -2n \leq t \leq -n \\ 1, & -n \leq t \leq x \\ -n(t - x - 1/n), & x \leq t \leq x + 1/n \\ 0, & t \geq x + 1/n. \end{cases}$$

Then $G_{n,x} \in I^q$ for each $1 \leq q \leq \infty$. Define $F_n(x) = \int_{-\infty}^{\infty} f G_{n,x}$. Then

$$F_n(x) = - \int_{-\infty}^{\infty} F(t) G'_{n,x}(t) dt = A_n + B_n(x) \text{ where } A_n = -n^{-1} \int_{-2n}^{-n} F(t) dt \text{ and}$$

$B_n(x) = n \int_x^{x+1/n} F(t) dt$. By the Hölder inequality $A_n \rightarrow 0$ as $n \rightarrow \infty$. By the Lebesgue differentiation theorem $B_n(x) \rightarrow F(x)$ for almost every $x \in \mathbb{R}$. Hence, $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ for almost every $x \in \mathbb{R}$ as $n \rightarrow \infty$. This then gives a means of computing D^{-1} .

The case $p = \infty$ is rather different as the dual space of L^∞ is not one of the L^p spaces. See [20: IV.9 Example 5]. We postpone this case for consideration elsewhere.

The space I^q is isometrically isomorphic to the dual space of L'^p .

THEOREM 3.4. *Let $1 \leq q \leq \infty$.*

- (a) $I^q \subset AC(\mathbb{R})$.
- (b) *For $G \in I^q$ define $\|G\|_{I,q} = \|G'\|_q$ then under usual pointwise operations I^q is a Banach space isometrically isomorphic to L^q .*

This is an immediate consequence of the fundamental theorem of calculus and Theorem 3.2(b). The pointwise derivative operator $D: I^q \rightarrow L^q$ defines a linear

isometry. Note that if $G \in I^q$ for some $1 < q \leq \infty$ then G need not be bounded. The space I^∞ is the same as the Lipschitz functions that vanish at 0.

The Sobolev space $W^{1,p}(\mathbb{R})$ consists of the absolutely continuous L^p functions whose distributional derivative is also in L^p . It is a Banach space with norm $\|f\|_{1,p} = \|f\|_p + \|f'\|_p$. See [20]. Clearly, $W^{1,p}(\mathbb{R})$ is a subspace of I^p . It is not a closed subspace. For example, let $g_n = \beta_n \chi_{(0,\alpha_n)} - \beta_n \chi_{(\alpha_n,2\alpha_n)}$, where $\alpha_n, \beta_n > 0$ for each $n \in \mathbb{N}$. Suppose $1 \leq p < \infty$. Then $g_n \in L^p$ and $\|g_n\|_p = 2^{1/p} \alpha_n^{1/p} \beta_n$. Let $G_n(x) = \int_0^x g_n(t) dt$. Then $G_n \in AC(\mathbb{R})$ and $G_n(x) = 0$ for $|x| \geq 2\alpha_n$ so $G_n \in L^p$. And, $\|G_n\|_p = 2^{1/p} (p+1)^{-1/p} \alpha_n^{1+1/p} \beta_n$. For $p = \infty$ we have $\|g_n\|_\infty = \beta_n$ and $\|G_n\|_\infty = \alpha_n \beta_n$. Now let $\alpha_n = n^2$ and $\beta_n = n^{-(1+2/p)}$. Then $\|g_n\|_p \rightarrow 0$ as $n \rightarrow \infty$ so $G_n \rightarrow 0$ in I^p , while $\|G_n\|_p \rightarrow \infty$ so (G_n) does not converge in $W^{1,p}(\mathbb{R})$. For each $1 \leq p \leq \infty$ then, $W^{1,p}(\mathbb{R})$ is not closed in I^p .

Now we can define an integral on L'^p .

DEFINITION 3.5. Let $1 \leq p < \infty$ and let q be its conjugate. Let $f \in L'^p$ and let $G \in I^q$. The integral of fG is $\int_{-\infty}^{\infty} fG = - \int_{-\infty}^{\infty} F(x)G'(x) dx$ where $F \in L^p$ is the primitive of f . For each $a \in \mathbb{R}$ define $\int_{-\infty}^{\infty} fa = 0$.

This defines a bilinear product $L'^p \times I^q \rightarrow L^1$ with $(f, G) \mapsto -FG'$. Some mathematicians may wish to refer to $\int_{-\infty}^{\infty} fG$ as merely a linear functional but

we like the term integral. The Lebesgue integral $\int_{-\infty}^{\infty} F(x)G'(x) dx$ exists by the Hölder inequality. This leads to a version of the Hölder inequality in L'^p and I^q .

THEOREM 3.6 (Hölder inequality). Let $1 \leq p < \infty$ and let q be its conjugate. Let $f \in L'^p$ with primitive $F \in L^p$ and let $G \in I^q$. Then

$$\left| \int_{-\infty}^{\infty} fG \right| = \left| \int_{-\infty}^{\infty} F(x)G'(x) dx \right| \leq \|F\|_p \|G'\|_q = \|f\|_p' \|G\|_{I,q}.$$

A consequence of the Hölder inequality is the following convergence theorem.

THEOREM 3.7. Let $1 \leq p < \infty$ with conjugate q . Suppose $f, f_n \in L'^p$ and $G, G_n \in I^q$ for each $n \in \mathbb{N}$. If $\|f_n - f\|_p' \rightarrow 0$ and $\|G_n - G\|_{I,q} \rightarrow 0$ then $\int_{-\infty}^{\infty} f_n G_n \rightarrow \int_{-\infty}^{\infty} fG$.

The proof follows from the equality $f_n G_n - fG = (f_n - f)G_n + f(G_n - G)$, linearity of the distributional derivative and the fact that $\|G_n\|_{I,q}$ is bounded.

The Hölder inequality can also be used to demonstrate an equivalent norm. The corresponding result for L^p appears in [9: Proposition 6.13].

PROPOSITION 3.8. *Let $f \in L'^p$ with primitive $F \in L^p$ and conjugate q . Define $\|f\|_p'' = \sup_G \int_{-\infty}^{\infty} fG$ where the supremum is taken over all $G \in I^q$ such that $\|G\|_{I,q} \leq 1$. Then $\|f\|_p' = \|f\|_p''$.*

Proof. By the Hölder inequality, $\|f\|_p'' \leq \|f\|_p'$. Without loss of generality $f \neq 0$. Let $g(x) = \text{sgn}(F(x))|F(x)|^{p-1}\|F\|_p^{1-p}$. Then $|g(x)|^q = |F(x)|^p\|F\|_p^{-p}$ so $g \in L^q$ and $\|g\|_q = 1$. Hence, $g \in L_{loc}^1$ so we can define $G(x) = \int_0^x g(t) dt$. Then

$$\|f\|_p'' \geq \left| \int_{-\infty}^{\infty} F(x)g(x) dx \right| = \frac{1}{\|F\|_p^{p-1}} \int_{-\infty}^{\infty} |F(x)|^p dx = \|F\|_p = \|f\|_p'.$$

This shows that $\|f\|_p' = \|f\|_p''$. $\square$

Step functions can be used to give an alternate definition of the integral. If $I_n = (x_n, y_n)$ are finite, disjoint intervals for $1 \leq n \leq N$ then a step function is $\sigma = \sum_1^N a_n \chi_{I_n}$, where $a_n \in \mathbb{R}$. The integral is $\int_{-\infty}^{\infty} \sigma(x) dx = \sum_1^N a_n [y_n - x_n]$. And,

if $g \in L^q$ with $G(x) = \int_0^x g(t) dt$ then $\int_{-\infty}^{\infty} \sigma(x)g(x) dx = \sum_1^N a_n [G(y_n) - G(x_n)]$.

The step functions are dense in L^p so for each $F \in L^p$ there is a sequence of step functions (σ_n) such that $\|F - \sigma_n\|_p \rightarrow 0$. Now suppose $f = F' \in L'^p$.

Define $s = \sigma' = \sum_1^N a_n [\tau_{x_n} \delta - \tau_{y_n} \delta]$ and $s_n = D(\sigma_n)$. Then $\left| \int_{-\infty}^{\infty} (fG - s_n G) \right|$

$= \left| \int_{-\infty}^{\infty} (Fg - \sigma_n g) \right| \leq \|F - \sigma_n\|_p \|g\|_q$. Using Proposition 3.8 and taking the

supremum over all $G \in I^q$ such that $\|G\|_{I,q} \leq 1$ shows $\|f - s_n\|_p' \rightarrow 0$.

Hence, we can approximate distributions in L'^p by differences of Dirac distributions as with s . The integral of sG is defined $\int_{-\infty}^{\infty} sG = - \int_{-\infty}^{\infty} \sigma(x)g(x) dx = - \sum_1^N a_n [G(y_n) - G(x_n)]$. This gives an alternative definition of the integral.

If $G \in I^q$ then there is $g \in L^q$ such that $G(x) = \int_0^x g(t) dt$. This imposes the

arbitrary condition $G(0) = 0$. If instead, we take $G_a(x) = \int_a^x g(t) dt$ for some fixed $a \in \mathbb{R}$ then G and G_a differ by a constant. This does not affect the integral in Definition 3.5 since it only depends on the derivative of G .

If F and G are absolutely continuous functions then the integration by parts formula is

$$\int_{-\infty}^{\infty} F'(x)G(x) \, dx = \lim_{x \rightarrow \infty} F(x)G(x) - \lim_{x \rightarrow -\infty} F(x)G(x) - \int_{-\infty}^{\infty} F(x)G'(x) \, dx$$

provided these limits exist. When FG vanishes at $\pm\infty$ the integral in Definition 3.5 agrees with the Lebesgue integral. Similarly, it agrees with the Henstock-Kurzweil and wide Denjoy integrals. See [6] for the relevant spaces of primitives for these integrals. These limit terms are omitted in Definition 3.5, as they are in the definition of the distributional derivative. For $F \in L^p$ and $G \in I^q$ the product FG vanishes in the following weak sense as $|x| \rightarrow \infty$.

PROPOSITION 3.9. *Let $F \in L^p$ and $G \in I^q$ where $1 \leq p < \infty$ and q is conjugate to p . For $0 < M < N$ and $\varepsilon > 0$ define $E_{(M,N),\varepsilon} = \{x \in (M, N) \mid |F(x)G(x)| > \varepsilon\}$. Then*

$$\frac{\lambda(E_{(M,N),\varepsilon})}{N-M} \leq \begin{cases} \frac{\|F\chi_{(M,N)}\|_p \|G'\|_q (N^q - M^q)^{1/q}}{\varepsilon q^{1/q} (N-M)}, & 1 < p < \infty \\ \frac{\|F\chi_{(M,N)}\|_1 \|G'\|_\infty N}{\varepsilon (N-M)}, & p = 1. \end{cases} \quad (3.1)$$

Proof. The Hölder inequality gives $\|FG\chi_{(M,N)}\|_1 \leq \|F\chi_{(M,N)}\|_p \|G\chi_{(M,N)}\|_q$. For $1 < p < \infty$ use Jensen's inequality, for example, [9: p. 109], to get

$$\|G\chi_{(M,N)}\|_q^q = \int_M^N \left| \int_0^x G'(t) \frac{dt}{x} \right|^q x^q \, dx \leq \|G'\|_q^q (N^q - M^q)/q.$$

For $p = 1$ we have $\|G\chi_{(M,N)}\|_\infty = \sup_{x \in (M,N)} \left| \int_0^x G'(t) \, dt \right| \leq \|G'\|_\infty N$. As in the proof of the Chebyshev inequality [9: 6.17] we have $\|FG\chi_{(M,N)}\|_1 \geq \varepsilon \lambda(E_{(M,N),\varepsilon})$. The result now follows. $\square$

For $1 \leq p < \infty$, let $M, N \rightarrow \infty$ in (3.1) such that $M \leq \delta N$ for some $0 < \delta < 1$. Since $F \in L^p$ we have $\|F\chi_{(M,N)}\|_p \rightarrow 0$. Hence, the measure of $E_{(M,N),\varepsilon}$ relative to the interval (M, N) tends to 0 in this limit. In this weak sense, $F(x)G(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Due to the isometry, properties of L^p that depend only on the norm carry over to L'^p . Here we list a few such properties, each with a reference to the L^p result. Proofs of the L'^p result follow from the L^p result using the fact that the distributional derivative provides a linear isometry and isomorphism between L'^p and L^p .

THEOREM 3.10.

- (a) For $1 \leq p < \infty$, $\mathcal{S}$ is dense in L^p .
- (b) L^p is uniformly convex for $1 < p < \infty$. Reflexivity follows by Milman's theorem.
- (c) The unit ball of L^p is strictly convex for $1 < p < \infty$.
- (d) Hanner's inequality holds in L^p for $1 \leq p < \infty$.
- (e) Homogeneity of norm for $1 \leq p < \infty$: If $f \in \mathcal{S}'$ then $f \in L^p$ if and only if $\tau_x f \in L^p$ for each $x \in \mathbb{R}$. If $f \in L^p$ then $\|\tau_x f\|'_p = \|f\|'_p$.
- (f) Continuity in norm for $1 \leq p < \infty$: $\lim_{x \rightarrow 0} \|\tau_x f - f\|'_p = 0$ for each $f \in L^p$.
- (g) Gateaux derivative: Let $f, g \in L^p$ for $1 < p < \infty$. Define $M(t) = \|f + tg\|'_p$. Then M is differentiable and $|M'(t)| \leq \|g\|'_p$.

Proof.

- (a) See [9: p. 245].
- (b) See [20: p. 126] and [8].
- (c) See [14: p. 112].
- (d) See [12: p. 49].
- (e) See [14: p. 182].
- (f) See [14: p. 182].
- (g) See [12: p. 51].

□

Each L^p space is a Dedekind complete Banach lattice under the partial order: $F \leq G$ if and only if $F(x) \leq G(x)$ for almost all $x \in \mathbb{R}$. This lattice structure is inherited by L^p . Here we list only a few lattice properties shared by L^p and L^p . See [11] for the lattice defined by primitives of Henstock-Kurzweil integrable functions.

THEOREM 3.11. In L^p define $f \preceq g$ if and only if $F \leq G$, where F and G are the respective primitives in L^p .

- (a) L^p is a Banach lattice, Dedekind complete and lattice isomorphic to L^p .
- (b) $|f| = |D(F)| = D|F|$ and $\|f\|'_p = \|f\|'_p$.
- (c) The space L^{1^1} is an abstract L -space.

Proof. See [20: XII.2, XII.3].

□

A version of the dominated convergence theorem is then the following [5: Theorems 7.2, 7.8].

PROPOSITION 3.12. Let $1 \leq p < \infty$. Let $(f_n) \subset L^p$ with respective primitives $(F_n) \subset L^p$. Suppose there is a measurable function F such that $F_n \rightarrow F$ almost everywhere or in measure. Suppose there is $g \in L^p$ such that $|f_n| \preceq g$ in L^p . Then $F' \in L^p$ and $\lim_{n \rightarrow \infty} \|f_n - F'\|'_p = 0$.

If $F \in L^p$ and $g \in L^q$ with $G(x) = \int_0^x g(t) dt$ then $\int_{-\infty}^{\infty} F(x)g(x) dx$ can be computed using an increasing sequence of step functions as follows. First write $F = F^+ - F^-$ and $g = g^+ - g^-$. The product Fg then is a linear combination of four products of positive functions. Hence, it suffices to consider the case $F \geq 0$ and $g \geq 0$. In the following we take a supremum over step functions $\sigma \leq F$ where $\sigma = \sum_{n=1}^{N_\sigma} a_n^{(\sigma)} \chi_{(x_n^{(\sigma)}, y_n^{(\sigma)})}$. Here, $\sigma \leq F$ means $\sigma(x) \leq F(x)$ for almost all $x \in \mathbb{R}$. Then $\int_{-\infty}^{\infty} F(x)g(x) dx = \sup_{\sigma \leq F} \sum_{n=1}^{N_\sigma} a_n^{(\sigma)} [G(y_n^{(\sigma)}) - G(x_n^{(\sigma)})]$. But this is equivalent to $\int_{-\infty}^{\infty} fG = - \sup_{\sigma' \preceq f} \int_{-\infty}^{\infty} \sigma' G$. And, for $\sigma = \sum_{n=1}^N a_n \chi_{(x_n, y_n)}$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} \sigma' G &= \int_{-\infty}^{\infty} \sum_{n=1}^N a_n [\tau_{x_n} \delta - \tau_{y_n} \delta] G = \sum_{n=1}^N a_n \left[\int_{-\infty}^{\infty} (\tau_{x_n} \delta) G - \int_{-\infty}^{\infty} (\tau_{y_n} \delta) G \right] \\ &= \sum_{n=1}^N a_n (\langle \tau_{x_n} \delta, G \rangle - \langle \tau_{y_n} \delta, G \rangle) = \sum_{n=1}^N a_n [G(x_n) - G(y_n)]. \end{aligned}$$

This then furnishes an equivalent definition of the integral.

4. Examples

In this section we give examples of functions in L'^p that are not in any L^p space. We also show that L'^p contains some functions that are not in L^1_{loc} and some functions that have conditionally convergent integrals. A simple integral condition is given in Proposition 4.1 that ensures a function is in L'^p . It is shown that differences of translated Dirac distributions can be in L'^p . Distributions in L'^p can have primitives that have no pointwise derivative at any point or a pointwise derivative that vanishes almost everywhere.

First note that L'^p contains many functions (i.e. regular distributions). For example, $\mathcal{S} \subset L^p$ for each $1 \leq p < \infty$. Let $F \in \mathcal{S}$ and let $G \in I^q$ for any $1 < q \leq \infty$. Then $\int_{-\infty}^{\infty} F'G = \int_{-\infty}^{\infty} F'(x)G(x) dx = - \int_{-\infty}^{\infty} F(x)G'(x) dx$. Hence, $\mathcal{S} \subset L'^p$. If $f \in L'^p \cap L'^q$ then $\|f\|'_p$ and $\|f\|'_q$ may well be different.

PROPOSITION 4.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$(a) \int_{-\infty}^{\infty} f(t) dt = 0, \quad (b) \int_{-\infty}^{\infty} |t|^\alpha f(t) dt \text{ exists for some } \alpha > 1/p.$$

Then $f \in L'^p$.

PROOF. Let $F(x) = \int_{-\infty}^x f(t) dt$. Then F is continuous on $\overline{\mathbb{R}}$. Let $M > 0$. By the second mean value theorem for integrals [6] there is $\xi \geq M$ such that

$$\int_M^\infty |F(x)|^p dx = \int_M^\infty \left| \int_x^\infty f(t) dt \right|^p dx = \int_M^\infty |x|^{-\alpha p} \left| \int_x^\xi t^\alpha f(t) dt \right|^p dx,$$

which is finite. Similarly, $\int_{-\infty}^{-M} |F(x)|^p dx < \infty$. $\square$

Condition (a) is here interpreted as a Henstock-Kurzweil integral but can be interpreted as a Lebesgue or wide Denjoy integral. See [6]. A sufficient condition for (b) is that $f(t) = O(|t|^{-\beta})$ as $|t| \rightarrow \infty$ for some $\beta > 1 + 1/p$. For example, let $f(x) = \sin(x)/|x|$. Then condition (a) is satisfied as a Henstock-Kurzweil or improper Riemann integral. From (b) we see that $f \in L^p$ for all $p > 1$. And, let $g(x) = x(|x| + 1)^{-\gamma} = O(|x|^{1-\gamma})$ as $|x| \rightarrow \infty$. Condition (a) is satisfied if $\gamma > 2$. Then $g \in L^p$ for all $p > (\gamma - 2)^{-1}$.

The Dirac distribution is not in L^p since $\delta = H'$ and for no $1 \leq p < \infty$ is $H \in L^p$. However, the Heaviside step function is regulated, i.e., it has a left and right limit at each point. The Dirac distribution then has a regulated primitive integral. See [18]. Differences of translated Dirac distributions may be in L^p . Let $F = \chi_{(a,b)}$. Then $F \in L^p$ for each $1 \leq p < \infty$. Hence, $f = F' = \tau_a \delta - \tau_b \delta \in L^p$. Take any $1 < q \leq \infty$ and let $g \in L^q$. Define $G(x) = \int_0^x g(t) dt$. Then $\int_{-\infty}^\infty fG = - \int_a^b G'(x) dx = - \int_a^b g(t) dt$.

Let $F(x) = |x|^{-\gamma} e^{-|x|}$. If $0 < \gamma < 1/p$ then $F \in L^p$. Let $f(x) = F'(x) = -\operatorname{sgn}(x)(1 + \gamma/|x|)|x|^{-\gamma} e^{-|x|}$ for $x \neq 0$. Since $f(x) \sim -\gamma \operatorname{sgn}(x)|x|^{-(\gamma+1)}$ as $x \rightarrow 0$ it follows that $f \in L^p$ but $f \notin L^1_{loc}$. Note that f is integrable in the principal value sense. Similarly, if $G(x) = \log|x| e^{-|x|}$ then $G' \in L^p$ for each $1 \leq p < \infty$ and $G'(x) \sim 1/x$ as $x \rightarrow 0$. This last is also integrable in the principal value sense.

Let $F(x) = \sin(\exp(|x|^3))/(x^2+1)$. Then $F \in C^\infty(\mathbb{R}) \cap L^p$ for each $1 \leq p < \infty$. Let $f(x) = F'(x)$. Then $f \in L^p$ but for no $1 \leq p \leq \infty$ is $f \in L^p$. Note that f is Henstock-Kurzweil integrable, improper Riemann integrable, and that $f(x) = O(\exp(|x|^3))$ as $|x| \rightarrow \infty$.

Let $F(x) = x^2 \sin(x^{-4})$ with $F(0) = 0$. Then $F \in C(\mathbb{R}) \cap L^p$ for each $1 \leq p < \infty$. The derivative exists at each point and

$$F'(x) = f(x) = \begin{cases} 2x \sin(x^{-4}) - 4x^{-3} \cos(x^{-4}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\sim \begin{cases} -4x^{-3} \cos(x^{-4}), & x \rightarrow 0 \\ x^{-3}(2 - 4 \cos(x^{-4})), & |x| \rightarrow \infty. \end{cases}$$

It follows that f is in each space L'^p but $\int_0^1 |f(x)|^p dx$ diverges for each $1 \leq p < \infty$ so f is not in any L^p space. However, f is integrable in the Henstock-Kurzweil and improper Riemann sense.

Let $F(x) = x^{-\gamma}H(x)H(1-x)$ and suppose $0 < \gamma < 1/p$. Then $F \in L^p$. And, $F = F_1 - F_2$ where $F_1(x) = x^{-\gamma}H(x)$ and $F_2(x) = x^{-\gamma}H(x-1)$. Define $f = F' \in L'^p$. To find an explicit formula for f let $\phi \in \mathcal{S}$. Then

$$\langle F'_1, \phi \rangle = - \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^{\infty} x^{-\gamma} \phi'(x) dx = \lim_{\varepsilon \rightarrow 0^+} \left[\varepsilon^{-\gamma} \phi(0) - \gamma \int_{\varepsilon}^{\infty} x^{-(\gamma+1)} \phi(x) dx \right].$$

And, F'_1 is the Hadamard finite part of the divergent integral $\int_0^{\infty} x^{-(\gamma+1)} \phi(x) dx$.

See [21: §2.5] for details. To find F'_2 let $p_{\gamma}(x) = x^{\gamma}$. Then

$$\langle F'_2, \phi \rangle = - \int_1^{\infty} x^{-\gamma} \phi'(x) dx = \phi(1) - \gamma \int_1^{\infty} x^{-(\gamma+1)} \phi(x) dx.$$

This shows that $F'_2 = \tau_1 \delta - \gamma(\tau_1 H)p_{-(\gamma+1)}$. Notice that the pointwise derivative of F is $F'(x) = -\gamma x^{-(\gamma+1)}$ for $0 < x < 1$, $F'(x) = 0$ for $x < 0$ and $x > 1$, and $F'(x)$ does not exist for $x = 0, 1$. Hence, this pointwise derivative has a non-integrable singularity at $x = 0$, i.e., it is not in L^1_{loc} .

Let $E \subset \mathbb{R}$ be a set of finite measure. Then $\chi_E \in L^q$ for each $1 \leq q \leq \infty$. Define $g \in L^q$ by $g = \chi_E$ and define $G \in I^q$ by $G(x) = \int_0^x g(t) dt$. Let $1 \leq p < \infty$

with conjugate exponent q . Let $f \in L'^p$ with primitive $F \in L^p$. Then $\int_{-\infty}^{\infty} fG = - \int_E F$. In particular, if E is an interval with endpoints $a < b$ then $G(x) = 0$ for $x \leq a$, $G(x) = x - a$ for $a \leq x \leq b$ and $G(x) = b - a$ for $x \geq b$. And, $G - G(a) \in I^q$ so that $\int_{-\infty}^{\infty} fG = - \int_a^b F$. In general, $\int_a^b f$ does not exist.

Let $\mathcal{B}_c$ be the functions in $C(\overline{\mathbb{R}})$ that vanish at $-\infty$. Define $\mathcal{A}_c = \{f \in \mathcal{S}' \mid f = F' \text{ for some } F \in \mathcal{B}_c\}$. If $f \in \mathcal{A}_c$ then it has a unique primitive $F \in \mathcal{B}_c$ such that $F' = f$. The continuous primitive integral of f is $\int_a^b f = F(b) - F(a)$ for all $-\infty \leq a < b \leq \infty$. See [16] for details. The next two examples are distributions that have a continuous primitive integral.

Let $\sigma: \mathbb{R} \rightarrow [0, 1]$ be a continuous function such that $\sigma'(x) = 0$ for almost all $x \in \mathbb{R}$. Define $F(x) = \exp(-x^2)\sigma(x)$ then $F \in L^p$ for each $1 \leq p < \infty$. The pointwise derivative is $F'(x) = 0$ for almost all $x \in \mathbb{R}$. For each $1 \leq q \leq \infty$, if $\psi \in L^q$ we have the Lebesgue integral $\int_{-\infty}^{\infty} F'(x)\psi(x) dx = 0$. Now define $f = F' \in L'^p$. Then for each $G \in I^q$ with q conjugate to p we have the L^p primitive integral $\int_{-\infty}^{\infty} fG = - \int_{-\infty}^{\infty} F(x)G'(x) dx$. This need not be 0.

Let $\omega: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuous function such that the pointwise derivative $\omega'(x)$ exists for no $x \in \mathbb{R}$. Define $F(x) = \exp(-x^2)\omega(x)$ then $F \in L^p$ for each $1 \leq p < \infty$. The pointwise derivative $F'(x)$ exists nowhere so for no function ψ does the Lebesgue integral $\int_{-\infty}^{\infty} F'(x)\psi(x) dx$ exist. Now define $f = F' \in L'^p$. Then for each $G \in I^q$ with q conjugate to p we have the L^p primitive integral $\int_{-\infty}^{\infty} fG = - \int_{-\infty}^{\infty} F(x)G'(x) dx$. This last exists as a Lebesgue integral.

5. Convolution

The convolution of functions f and g is $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy$. In the literature, various conditions have been imposed on f and g for existence of the convolution. For instance, for each $1 \leq p \leq \infty$ with q conjugate, the convolution is a bounded linear operator $*$: $L^p \times L^q \rightarrow L^\infty$ such that $f * g$ is uniformly continuous and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. If $1 \leq p, q, r \leq \infty$ such that $1/p + 1/q = 1 + 1/r$. Then it is a bounded linear operator $*$: $L^p \times L^q \rightarrow L^r$ such that $\|f * g\|_r \leq \|f\|_p \|g\|_q$. See [9] for the L^p theory of convolutions.

Many of the L^p results have analogues in L'^p . Use the notation $\tilde{\phi}(x) = \phi(-x)$ for function ϕ . First we consider the convolution as $*$: $L'^p \times I^q \rightarrow L^\infty$. This can be defined directly using the integral $f * G(x) = \int_{-\infty}^{\infty} \widetilde{\tau_x f} G$, where $\langle \widetilde{\tau_x f}, \phi \rangle = \langle f, \widetilde{\tau_x \phi} \rangle$ for distribution f and test function ϕ . We will write this as $f * G(x) = \int_{-\infty}^{\infty} f(x-y)G(y) dy$ even though it is an integral in L'^p . To distinguish from Lebesgue integrals, in this section if $f \in L'^p$ then we will always write its primitive as $F \in L^p$. Further in this section we define the convolution $*$: $L'^p \times L^q \rightarrow L'^r$ and $*$: $L^p \times L'^q \rightarrow L'^r$ using a limiting procedure and the density of the compactly supported smooth functions in L^q .

THEOREM 5.1. *For $1 \leq p < \infty$, let $f \in L'^p$ with primitive $F \in L^p$. Let $G \in I^q$ such that $G(x) = \int_0^x g(t) dt$ for some $g \in L^q$, where q is conjugate to p . Define the convolution $f * G(x) = \int_{-\infty}^{\infty} f(x-y)G(y) dy$. The convolution has the following properties.*

- (a) $*$: $L'^p \times I^q \rightarrow L^\infty$ such that $f * G = F * g = g * F$, $\|f * G\|_\infty \leq \|f\|'_p \|G\|_{I,q}$ and $f * G$ is uniformly continuous on $\mathbb{R}$.
- (b) $f * k = 0$ for each $k \in \mathbb{R}$.
- (c) Suppose that $f \in L'^1$ and $g \in L^\infty$. If $h \in L^1$ such that $\int_{-\infty}^{\infty} |y h(y)| dy < \infty$ then $f * (G * h) = (f * G) * h$.
- (d) For each $z \in \mathbb{R}$, $\tau_z(f * G) = (\tau_z f) * G = f * (\tau_z G)$.
- (e) For each $f \in L'^p$ define $\Phi_f: I^q \rightarrow L^\infty$ by $\Phi_f[G] = f * G$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| = \|f\|'_p$. For each $G \in I^q$ define $\Psi_G: L'^p \rightarrow L^\infty$ by $\Psi_G[f] = f * G$. Then Ψ_G is a bounded linear operator. For $1 < p < \infty$ we have $\|\Psi_G\| = \|G\|_{I,q}$. For $p = 1$ we have $\|\Psi_G\| \leq \|G\|_{I,\infty}$.
- (f) $\text{supp}(f * G) \subset \text{cl}(\text{supp}(F) + \text{supp}(g))$.

Proof. Most parts of the theorem are proved by reverting to equivalent L^p results. These are proved in [9].

(a) Note that $\tau_x(DF) = D(\tau_x F)$ so $f * G(x) = \int_{-\infty}^{\infty} f(x-y)G(y) dy = \int_{-\infty}^{\infty} F(x-y)g(y) dy$ by Definition 3.5. The Hölder inequality now gives $\|f * G\|_\infty \leq \|F\|_1 \|g\|_\infty$. Uniform continuity follows [9: p. 241].

(b) See Definition 3.5.

(c) Note that $G * h$ exists on $\mathbb{R}$ by the Hölder inequality since $G(y)/(|y| + 1)$ is bounded and $(|y| + 1)h(y)$ is in L^1 . By dominated convergence $(G * h)'(x) = g * h(x)$ for each $x \in \mathbb{R}$. Hence, $\|(G * h)'\|_\infty \leq \|g\|_\infty \|h\|_1$. Thus, $G * h \in I^\infty$ (modulo a constant). We then have $f * (G * h) = F * (G * h)' = F * (g * h) = (F * g) * h = (f * G) * h$. Associativity for Lebesgue integrals is proved in [9: p. 240].

(d) $\tau_z(f * G) = \tau_z(F * g) = (\tau_z F) * g = (\tau_z f * G) = F * (\tau_z g) = f * (\tau_z G)$. Translation for functions is proved in [9: p 240].

(e) Note that

$$\|\Phi_f\| = \sup_{\|G\|_{I,q}=1} \|f * G\|_\infty = \sup_{\|g\|_q=1} \|F * g\|_\infty \leq \sup_{\|g\|_q=1} \|F\|_p \|g\|_q = \|F\|_p.$$

We get equality in the Hölder inequality [12: p. 46] by taking $g(y) = \operatorname{sgn}(F(-y))$ when $p = 1$ and otherwise $g(y) = \|F\|_p^{1-p} \operatorname{sgn}(F(-y))|F(-y)|^{p/q}$. Then $F * g(0) = \|F\|_p$. The proof for Ψ is similar.

(f) See [9: p. 240]. □

Note that if $F \in L^p$ and $G \in L^q$ then $F * G$ need not exist as a function at any point. For example, let $F(x) = |x|^{-4/(3p)}$ for $|x| > 1$ and $F(x) = 0$, otherwise. Take $g(x) = |x|^{-(1/q+1/(3p))}$ for $|x| > 1$ and $g(x) = 0$, otherwise. For each $1 \leq p < \infty$ we have $F \in L^p$ and $g \in L^q$. Let $G(x) = \int_0^x g$. Then the Lebesgue integral defining $F * G(x)$ diverges for each $x \in \mathbb{R}$. Hence, we cannot define $f * G$ as $(F * G)'$.

From [12: p. 46], we get the equality $\|F * g\|_\infty = \|F\|_1 \|g\|_\infty$ if and only if $|g(y)| = \operatorname{sgn}(F(x - y))$ for some x and almost all y . Hence, in (e) the operator norm of Ψ_G need not equal $\|G\|_{L^1, \infty}$.

For tempered distributions, the convolution can be defined as $*$: $\mathcal{S}' \times \mathcal{S} \rightarrow C^\infty$ by $T * \phi(x) = \langle T, \tau_x \tilde{\phi} \rangle$. See [9], where another equivalent definition is also given. Let $\phi \in \mathcal{S}$. If $f \in L'^p$ with primitive $F \in L^p$ then according to this definition,

$$\begin{aligned} f * \phi(x) &= -\langle F, (\tau_x \tilde{\phi})' \rangle = - \int_{-\infty}^{\infty} F(y) \frac{\partial \phi(x - y)}{\partial y} dy \\ &= \int_{-\infty}^{\infty} F(y) \phi'(x - y) dy = F * \phi'(x). \end{aligned}$$

Since $\mathcal{S} \subset L^q$ for each $1 \leq q \leq \infty$ and $\mathcal{S}$ is closed under differentiation, if $\Phi \in \mathcal{S}$ then the function $x \mapsto \int_0^x \Phi'(t) dt$ is in L^q for each $1 \leq q \leq \infty$. The definition in Theorem 5.1 then gives $f * \Phi(x) = F * \Phi'(x)$, since convolution with a constant is zero. Our definition then agrees with the usual one given above when we convolve with the integral of a test function in $\mathcal{S}$.

If $f \in L'^p$ and $g \in L^q$ then the integral $\int_{-\infty}^{\infty} f(y)g(x - y) dy$ need not exist.

However, C_c^∞ , the set of smooth functions with compact support, is dense in L^q and we can use the definition in Theorem 5.1 to define $f * g$ with a sequence in C_c^∞ that converges to a function in L^q .

DEFINITION 5.2. Let $p, q, r \in [1, \infty)$ such that $1/p + 1/q = 1 + 1/r$. Let $f \in L'^p$ with primitive $F \in L^p$ and let $g \in L^q$. Let $(g_n) \subset C_c^\infty$ such that $\|g_n - g\|_q \rightarrow 0$. Define $f * g$ to be the unique element in L'^r such that $\|f * g - f * g_n\|_r' \rightarrow 0$. Define $F * g'$ to be the unique element in L'^r such that $\|F * g' - F * g_n'\|_r \rightarrow 0$.

To see that the definition makes sense, suppose $\text{supp}(g_n) \subset [a_n, b_n]$. Let $x \in (\alpha, \beta)$. Then $(F * g_n)'(x) = \frac{d}{dx} \int_{\alpha-b_n}^{\beta-a_n} F(y)g_n(x-y) dy = \int_{-\infty}^{\infty} F(y)g'_n(x-y) dy$ by dominated convergence since $F \in L^1_{\text{loc}}$ and $g'_n \in L^\infty$. Then, from Theorem 5.1(a),

$$\|f * g_n\|'_r = \|F * g'_n\|'_r = \|F * g_n\|_r \leq \|F\|_p \|g_n\|_q.$$

From this it follows that $\|f * g_n - f * g_m\|'_r = \|f * (g_n - g_m)\|'_r \leq \|f\|'_p \|g_n - g_m\|_q \rightarrow 0$ as $m, n \rightarrow \infty$. Hence, $(f * g_n)$ is a Cauchy sequence in the complete space L'^r . It therefore converges to an element of L'^r which we label $f * g$. Similarly with $(F * g'_n)$. This also shows that $f * g$ and $F * g'$ are independent of the choice of sequence (g_n) .

The following theorem shows that convolutions in $L'^p \times L^q$ and $L^p \times L'^q$ share many of the properties of convolutions in $L^p \times L^q$. Part (b) extends Young's inequality [9: p. 241]. Part (g) shows that convolutions with L^p functions can be used to approximate distributions in L'^p .

THEOREM 5.3. *Let $p, q, r \in [1, \infty)$ such that $1/p + 1/q = 1 + 1/r$. Let $f \in L'^p$ with primitive $F \in L^p$ and let $g \in L^q$. Then $*$: $L'^p \times L^q \rightarrow L'^r$ and $*$: $L^p \times L'^q \rightarrow L'^r$ with the following properties.*

- (a) $f * g = (F * g)' = F * g'$.
- (b) $\|f * g\|'_r = \|F * g'\|'_r \leq \|F\|_p \|g\|_q$.
- (c) If $h \in L^1$, then $f * (g * h) = (f * g) * h$ and $F * (g' * h) = (F * g') * h$.
- (d) For each $z \in \mathbb{R}$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g) = (\tau_z F) * g' = F * \tau_z g'$.
- (e) For each $f \in L'^p$ define $\Phi_f: L^q \rightarrow L'^r$ by $\Phi_f[g] = f * g$. Then Φ_f is a bounded linear operator and $\|\Phi_f\| \leq \|f\|'_p$.
For each $g \in L^q$ define $\Psi_g: L'^p \rightarrow L'^r$ by $\Psi_g[f] = f * g$. Then Ψ_g is a bounded linear operator and $\|\Psi_g\| \leq \|g\|_q$.
For each $F \in L^p$ define $A_F: L'^q \rightarrow L'^r$ by $A_F[g'] = F * g'$ for each $g \in L^q$. Then A_F is a bounded linear operator and $\|A_F\| \leq \|F\|_p$.
For each $g \in L^q$ define $B_{g'}: L^p \rightarrow L'^r$ by $B_{g'}[F] = F * g'$. Then $B_{g'}$ is a bounded linear operator and $\|B_{g'}\| \leq \|g'\|'_q$.
- (f) $\text{supp}(f * g) \subset \text{cl}(\text{supp}(f) + \text{supp}(g))$.
- (g) Let $1 \leq p < \infty$. Let $g \in L^1$. Define $F_t(x) = F(x/t)/t$ for $t > 0$. Let $a = \int_{-\infty}^{\infty} F_t(x) dx = \int_{-\infty}^{\infty} F(x) dx$. Then $\|F_t * g' - ag'\|'_p \rightarrow 0$ as $t \rightarrow 0$.

Proof.

(a) Take $(g_n) \subset C_c^\infty$ such that $\|g_n - g\|_q \rightarrow 0$. Then by Theorem 5.1(a) and the paragraph following Definition 5.2, $\|f * g_n - (F * g)'\|'_r = \|F * g'_n - (F * g)'\|'_r = \|F * (g_n - g)\|_r \leq \|F\|_p \|g_n - g\|_q \rightarrow 0$ as $n \rightarrow \infty$.

(b) $\|f * g\|'_r = \|(F * g)'\|'_r = \|F * g'\|'_r = \|F * g\|_r \leq \|F\|_p \|g\|_q$.

(c) $(f * g) * h = (F * g)' * h = [(F * g) * h]' = [F * (g * h)]' = F' * (g * h) = f * (g * h)$. The L^p result used is in [9: p. 240]. The other case is similar.

(d) $\tau_z(f * g) = \tau_z D(F * g) = D[\tau_z(F * g)] = D[(\tau_z F) * g] = (\tau_z F) * g' = [D(\tau_z F)] * g = (\tau_z f) * g = D[F * \tau_z g] = f * \tau_z g = F * D(\tau_z g) = F * \tau_z g'$. The L^p results used are in [9: p. 240].

(e) Note that

$$\|\Phi_f\| = \sup_{\|g\|_q=1} \|f * g\|'_r = \sup_{\|g\|_q=1} \|F * g\|_r \leq \sup_{\|g\|_q=1} \|F\|_p \|g\|_q = \|F\|_p.$$

The other cases are similar.

(f) Zemanian [21: 5.4(2)] uses the definition

$$\langle F * g, \phi \rangle = \int_{-\infty}^{\infty} F(y) \int_{-\infty}^{\infty} g(x) \phi(x + y) dx dy = \int_{-\infty}^{\infty} F \widetilde{g * \phi}$$

for $\phi \in \mathcal{S}$. The integral exists for $F \in L^p$ and $g \in L^q$ since $\phi \in L^1$ so $g * \phi \in L^q$ (Young's inequality). Whereas, Definition 5.2 and (a) of this theorem give

$$\begin{aligned} \langle f * g, \phi \rangle &= -\langle F * g, \phi' \rangle = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(y) g(x - y) dy \phi'(x) dx \\ &= - \int_{-\infty}^{\infty} F(y) \int_{-\infty}^{\infty} g(x - y) \phi'(x) dx dy \\ &= - \int_{-\infty}^{\infty} F(y) \int_{-\infty}^{\infty} g(x) \phi'(x + y) dx dy. \end{aligned}$$

The Fubini-Tonelli theorem justifies reversing the order of integration. This proves the equivalence of our Definition 5.2 with Zemanian's definition. The support result then follows from [21: Theorem 5.4-2, Theorem 5.3-1] and the fact that the support of a function is the closure of the set on which it does not vanish.

(g) $\|F_t * g' - ag'\|'_p = \|D(F_t * g - ag)\|'_p = \|F_t * g - ag\|_p \rightarrow 0$ as $t \rightarrow 0$. See [9: p. 242]. $\square$

If $F, G \in L^1$ then $F * G \in L^1$. Hence, we can define a Banach algebra by defining $F' \star G' = (F * G)'$.

THEOREM 5.4. *Let $f, g \in L'^1$ with respective primitives $F, G \in L^1$. Define the product $\star: L'^1 \times L'^1 \rightarrow L'^1$ by $f \star g = (F * G)'$. Then L'^1 is a Banach algebra isometrically isomorphic to the convolution algebra on L^1 .*

Proof. The proof is elementary. For example, to show commutativity, $f \star g = F' \star G' = (F * G)' = (G * F)' = G' \star F' = g \star f$. $\square$

This product is not compatible with the convolution defined in Theorem 5.3. For example, let $g(x) = -2x \exp(-x^2)$. Then $g \in L^1$. Define $G(x) = \exp(-x^2)$. Then $g(x) = G'(x)$ for all x and $G \in L^1$ so $g \in L'^1$. Let $F = \chi_{(0,1)} \in L^1$ and $f = F' = \delta_0 - \delta_1 \in L'^1$. A calculation shows $f * g(x) = 2(x-1) \exp(-[x-1]^2) - 2x \exp(-x^2)$ while $f \star g(x) = \exp(-x^2) - \exp(-[x-1]^2)$.

6. Fourier transform in L'^1

If $F \in L^1$ then its Fourier transform is $\hat{F}(s) = \int_{-\infty}^{\infty} e^{-isx} F(x) dx$. This

Lebesgue integral converges for each $s \in \mathbb{R}$. It is known that $\hat{F}$ is continuous on $\overline{\mathbb{R}}$ and vanishes at $\pm\infty$ (Riemann-Lebesgue lemma). This defines a linear operator $\hat{\cdot}: L^1 \rightarrow C(\overline{\mathbb{R}})$. Also, each tempered distribution has a Fourier transform that is also a tempered distribution. If $T \in \mathcal{S}'$ then $\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle$. The most important properties of Fourier transforms in L^1 and $\mathcal{S}'$ can be found in [9].

Since the complex exponential is in I^∞ we can define the Fourier transform in L'^1 using the integral definition. It then shares many of the properties of L^1 transforms.

DEFINITION 6.1. Let $e_s(x) = e^{-isx}$. Let $f \in L'^1$ with primitive $F \in L^1$. For each $s \in \mathbb{R}$ define $\hat{f}(s) = \int_{-\infty}^{\infty} f e_s$.

Note that $e_s - 1 \in I^\infty$ since $e_s(x) = 1 - is \int_0^x e^{-ist} dt$. The integral $\int_{-\infty}^{\infty} f e_s$ is then well defined as in Definition 3.5.

If $f = F' \in L'^p$ and $G \in I^q$ for conjugate q are periodic then the formula $\int_{-\pi}^{\pi} F' G = - \int_{-\pi}^{\pi} F(x) G(x) dx$ can be used to define Fourier coefficients. We will leave a study of L'^p Fourier series for elsewhere.

THEOREM 6.2. *Let $f \in L'^1$ with primitive $F \in L^1$. Let $s \in \mathbb{R}$. Then*

- (a) $\hat{f}(s) = is\hat{F}(s)$.
- (b) $\hat{f}$ is continuous on $\mathbb{R}$.
- (c) $|\hat{f}(s)| = |s| \|\hat{F}\|_\infty \leq |s| \|F\|_1 = |s| \|f\|'_1$

- (d) $\hat{f}(s) = o(s)$ as $|s| \rightarrow \infty$.
- (e) Definition 6.1 agrees with the tempered distribution definition.
- (f) $\widehat{\tau_y f}(s) = e^{-isy} \hat{f}(s)$ for each $y \in \mathbb{R}$.
- (g) Let $K(x) = xF(x)$. If $F, K \in L^1$ then $\hat{f}$ is differentiable and $D\hat{f}(s) = i\hat{F}(s) + s\hat{K}(s)$.
- (h) Let $g \in L^1$. Then $\widehat{f * g}(s) = \hat{f}(s)\hat{g}(s)$.
- (i) Let $g \in L^1$ such that the function $s \mapsto sg(s)$ is also in L^1 .
Then $\int_{-\infty}^{\infty} \hat{f}(s)g(s) ds = \int_{-\infty}^{\infty} f\hat{g}$.

Proof. Most parts are proved by reverting to an L^1 result. For these, see [9: §8.3].

- (e) If $\phi \in \mathcal{S}$, then $\hat{\phi} \in \mathcal{S}$. Using the tempered distribution definition,

$$\langle \hat{f}, \phi \rangle = \langle F', \hat{\phi} \rangle = -\langle F, D\hat{\phi} \rangle = i \int_{-\infty}^{\infty} F(x) \int_{-\infty}^{\infty} se^{-ixs} \phi(s) ds dx \quad (6.1)$$

$$= i \int_{-\infty}^{\infty} s\phi(s) \int_{-\infty}^{\infty} e^{-ixs} F(s) dx ds = i \int_{-\infty}^{\infty} s\phi(s)\hat{F}(s) ds, \quad (6.2)$$

in agreement with Definition 6.1. Dominated convergence is used in (6.1) to differentiate under the integral. The Fubini-Tonelli theorem allows interchange of iterated integrals in (6.2).

- (f) Note that $(\tau_y F)' = \tau_y(F')$. Then $\widehat{\tau_y F}(s) = is\widehat{\tau_y F}(s) = ise^{-isy}\hat{F}(s) = e^{-isy}\hat{f}(s)$.

- (h) Use Theorem 5.3(a) with $p = q = r = 1$ to get $\widehat{f * g}(s) = (\widehat{F * g})'(s) = is\widehat{F * g}(s) = is\hat{F}(s)\hat{g}(s) = \hat{f}(s)\hat{g}(s)$.

- (i) Note that

$$\int_{-\infty}^{\infty} \hat{f}(s)g(s) ds = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} se^{-isx} F(x)g(s) dx ds \quad (6.3)$$

and

$$\int_{-\infty}^{\infty} f\hat{g} = - \int_{-\infty}^{\infty} F(x)\hat{g}'(x) dx = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x)e^{-isx} sg(s) ds dx. \quad (6.4)$$

The Fubini-Tonelli theorem gives the equality of the integrals in (6.3) and (6.4). $\square$

Thus, $\hat{f}: L'^1 \rightarrow C(\mathbb{R})$. The Riemann-Lebesgue lemma order relation $\hat{F}(s) = o(s)$ as $|s| \rightarrow \infty$ is known to be sharp for $F \in L^1$. Hence, $\hat{f}$ is a continuous function but need not be bounded.

7. Inner product and Fourier transform in L'^2

An important property of the L^p spaces is that L^2 is a Hilbert space with inner product $(f, g) = \int_{-\infty}^{\infty} f(x)g(x) dx$. If $f, g \in L'^2$ then $\int_{-\infty}^{\infty} fg$ need not exist. However, since the derivative $D: L^2 \rightarrow L'^2$ is a linear isometry, L'^2 is also a Hilbert space.

THEOREM 7.1. *L'^2 is a Hilbert space with inner product*

$$(f, g) = \frac{1}{4} \left(\|f + g\|_2'^2 - \|f - g\|_2'^2 \right) = (F, G),$$

where $f, g \in L'^2$ with primitives $F, G \in L^2$.

Proof. A necessary and sufficient condition for a Banach space to be a Hilbert space is that the norm satisfy the parallelogram identity $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all x and y in the space. See [20: I.5]. The L^2 norm then satisfies this identity and so does $\|\cdot\|_2'$. If $f, g \in L'^2$ with respective primitives $F, G \in L^2$ then

$$(f, g) = \frac{1}{4} \left(\|f + g\|_2'^2 - \|f - g\|_2'^2 \right) = \frac{1}{4} \left(\|F + G\|_2^2 - \|F - G\|_2^2 \right) = (F, G).$$

This completes the proof. $\square$

Using Theorem 7.1 we can define $\int_{-\infty}^{\infty} fg = (f, g) = \int_{-\infty}^{\infty} F(x)G(x) dx$ for $f, g \in L'^2$ with primitives $F, G \in L^2$.

Fourier transforms are defined in L^2 using the Plancherel theorem. If $F \in L^2$ let $(F_n) \subset L^2 \cap L^1$ such that $\|F_n - F\|_2 \rightarrow 0$. Each function $\hat{F}_n$ is well defined by the usual Fourier integral formula as an element of L^2 . There is a unique function $\hat{F} \in L^2$ such that $\|\hat{F}_n - \hat{F}\|_2 \rightarrow 0$. It follows that the Fourier transform $\hat{\cdot}: L^2 \rightarrow L^2$ is a unitary isomorphism, $\|F\|_2 = \|\hat{F}\|_2$ and if $G \in L^2$ then the Parseval equality is $(F, G) = \int_{-\infty}^{\infty} F(x)G(x) dx = [1/(2\pi)] \int_{-\infty}^{\infty} \hat{F}(s)\hat{G}(s) ds = (\hat{F}, \hat{G})$.

The norm is as usual defined by $\|\hat{F}\|_2 = (\hat{F}, \hat{F})^{1/2}$. See [9] for details. If $F \in L^2$ then the defining sequence in $L^2 \cap L^1$ can be taken as $F_n = \chi_{(-n, n)}F$.

The derivative is a unitary mapping between L^2 and L'^2 . Suppose $F \in L^2$, $(F_n) \subset L^2$ such that $\|F_n - F\|_2 \rightarrow 0$. Then $\hat{F} \in L^2$ and $D\hat{F} \in L'^2$ is defined as a tempered distribution by $\langle D\hat{F}, \phi \rangle = -\langle \hat{F}, \phi' \rangle = -\langle F, \widehat{D\phi} \rangle$ for all $\phi \in \mathcal{S}$.

And, if $G \in I^2$, then $\int_{-\infty}^{\infty} (D\hat{F})G = - \int_{-\infty}^{\infty} \hat{F}(s)G'(s) ds$. Note that $\|F_n - F\|_2 = \|\hat{F}_n - \hat{F}\|_2 = \|D\hat{F}_n - D\hat{F}\|_2' \rightarrow 0$. If $g \in L^2$ the Parseval equality takes the form $(F, g) = \int_{-\infty}^{\infty} F(x)g(x) dx = (\hat{F}, \hat{g}) = [1/(2\pi)] \int_{-\infty}^{\infty} \hat{F}(s)\hat{g}(s) ds = (D\hat{F}, D\hat{g}) = [1/(2\pi)] \int_{-\infty}^{\infty} (D\hat{F})(D\hat{g}) = (F', g') = \int_{-\infty}^{\infty} F'g'$.

If $F \in L^2$ then $DF \in L'^2 \subset \mathcal{S}'$ so $\widehat{DF}$ is defined as a tempered distribution by $\langle \widehat{DF}, \phi \rangle = \langle DF, \hat{\phi} \rangle = -\langle F, D\hat{\phi} \rangle$.

Since the Fourier transform does not commute with the derivative, $D\hat{F}$ and $\widehat{DF}$ are not necessarily equal. For example, let $F = \chi_{(-1,1)} \in L^2 \cap L^1$. Then $\hat{F} \in L^2$ and $D\hat{F} \in L'^2$. We have

$$\hat{F}(s) = 2 \int_0^1 \cos(sx) ds = \begin{cases} \frac{2}{s} \sin(s), & s \neq 0 \\ 2, & s = 0. \end{cases}$$

The pointwise derivative is then

$$D\hat{F}(s) = \begin{cases} -\frac{2}{s^2} \sin(s) + \frac{2}{s} \cos(s), & s \neq 0 \\ 0, & s = 0. \end{cases}$$

Whereas, $DF = \tau_{-1}\delta - \tau_1\delta \in L'^1$ so $\widehat{DF}(s) = is\hat{F}(s) = 2i\sin(s)$. Hence, $D\hat{F} \neq \widehat{DF}$. Note that $\widehat{DF} \in L^\infty$ but is not in any L^p or L'^p space for $1 \leq p < \infty$.

8. Higher derivatives

In [19], an integral that inverts higher derivatives of continuous or regulated primitives was introduced. Define $\mathcal{A}_c^n = \{f \in \mathcal{S}' \mid f = D^n F \text{ for some } F \in \mathcal{B}_c\}$ where $\mathcal{B}_c = \{F \in C(\mathbb{R}) \mid F(-\infty) = 0\}$ and $\mathcal{A}_r^n = \{f \in \mathcal{S}' \mid f = D^n F \text{ for some } F \in \mathcal{B}_r\}$ where $\mathcal{B}_r$ are the regulated and left continuous functions on $\mathbb{R}$ with limit 0 at $-\infty$. If $f \in \mathcal{A}_c^n$ or $f \in \mathcal{A}_r^n$ for integer $n \in \mathbb{N}$ then $\int_{-\infty}^{\infty} fG$ exists if G is an n -fold iterated integral of a function of bounded variation. If $f \in \mathcal{A}_c^n$ or $\mathcal{A}_r^n$ with primitive $F \in \mathcal{B}_c$ or $\mathcal{B}_r$ then $\|f\|_{a,n} = \|F\|_\infty$ makes $\mathcal{A}_c^n$ and $\mathcal{A}_r^n$ into Banach spaces that are isometrically isomorphic to $\mathcal{B}_c$ and $\mathcal{B}_r$, respectively. We also write $D^n F = F^{(n)}$.

Define $L^{(n),p} = \{f \in \mathcal{S}' \mid f = D^n F \text{ for some } F \in L^p\}$ and denote by $I^{n,q}$ the functions $G: \mathbb{R} \rightarrow \mathbb{R}$ with $G(x) = \int_0^x \dots \int_0^{x_{i+1}} \dots \int_0^{x_2} g(x_1) dx_1 \dots dx_i \dots dx_n$

for some $g \in L^q$. Let q be the conjugate exponent of p . Then $\int_{-\infty}^{\infty} fG = (-1)^n \int_{-\infty}^{\infty} F(x)G^{(n)}(x)dx$ if $G \in I^{n,q}$. It follows that for each $0 \leq m \leq n$, $\int_{-\infty}^{\infty} F^{(n)}G = (-1)^m \int_{-\infty}^{\infty} F^{(n-m)}G^{(m)}$.

Results for the $L^{(n),p}$ spaces are analogous to those of $L'^p = L^{(1),p}$. Changing the lower limits in the integrals defining a function $G \in I^{n,q}$ changes G by the addition of a polynomial of degree at most $n-1$. We define $\int_{-\infty}^{\infty} fP = 0$ for $f \in L^{(n),p}$ and P a polynomial of degree at most $n-1$. Theorem 3.2 holds without change and we define $\|f\|_p^{(n)} = \|F\|_p$ where $f \in L^{(n),p}$ and F is its unique primitive in L^p . If $G \in I^{n,q}$ is an n -fold iterated integral of $g \in L^q$, define $\|G\|_{nI,q} = \|G^{(n)}\|_q = \|g\|_q$. Then $L^{(n),p}$ is a Banach space with norm $\|\cdot\|_p^{(n)}$ isometrically isomorphic to L^p and $I^{n,q}$ is a Banach space with norm $\|\cdot\|_{nI,q}$ isometrically isomorphic to L^q . The results of Theorem 3.3 hold with these changes. It then follows that the remaining theorems in Section 3 hold with minor changes.

The convolution is $f * G(x) = \int_{-\infty}^{\infty} F^{(n)}(x-t)G(t)dt = F * G^{(n)}(x)$ for $f \in L^{(n),p}$ and $G \in I^{n,q}$ and q conjugate to p . Analogous properties to Theorem 5.1 can now be seen to hold in $L^{(n),p}$. The second integral condition in (c) is replaced by $\int_{-\infty}^{\infty} |y^n h(y)|dy < \infty$. Convolutions are defined in $L^{(n),1} \times L^p$ and in $L^1 \times L^{(n),p}$ by replacing Definition 5.2 with:

DEFINITION 8.1. Let $p, q, r \in [1, \infty)$ such that $1/p + 1/q = 1 + 1/r$. Let $f \in L^{(n),p}$ with primitive $F \in L^p$ and let $g \in L^q$. Let $(g_k) \subset L^q \cap I^{(n),q}$ such that $\|g_k - g\|_q \rightarrow 0$. Define $f * g$ to be the unique element in $L^{(n),r}$ such that $\|f * g - f * g_k\|_r^{(n)} \rightarrow 0$. Define $F * g^{(n)}$ to be the unique element in $L^{(n),r}$ such that $\|F * g^{(n)} - F * g_k^{(n)}\|_r^{(n)} \rightarrow 0$.

Theorem 5.3 now holds with minor changes.

Using Definition 6.1, the Fourier transform of $f \in L^{(n),1}$ with primitive $F \in L^1$ is $\hat{f}(s) = \int_{-\infty}^{\infty} f e_s = (is)^n \hat{F}(s)$. Then $\hat{f}$ is continuous with $|\hat{f}(s)| = |s^n| \|\hat{F}\|_{\infty} \leq |s^n| \|F\|_1 = |s^n| \|f\|_1^{(n)}$. The Riemann-Lebesgue lemma becomes $\hat{f}(s) = o(s^n)$ as $|s| \rightarrow \infty$. The rest of Theorem 6.2 holds with minor changes. In (i) the second condition on g is that the function $s \mapsto s^n g(s)$ is in L^1 .

The space $L^{(n),2}$ is a Hilbert space with inner product defined as per Definition 7.1: $(f, g) = (F, G) = \int_{-\infty}^{\infty} F(x)G(x) dx$ where $f, g \in L^{(n),2}$ with respective primitives $F, G \in L^2$. The Fourier transform in $L^{(n),2}$ is defined similarly and the Parseval equality continues to hold.

Finally, we have the connection between $L^{(n),1}$ and $\mathcal{A}_c^m$.

PROPOSITION 8.2. *Let $n \in \mathbb{N}$.*

- (a) $L^{(n),1}$ is a subspace of $\mathcal{A}_c^{n+1}$.
- (b) It is not closed.
- (c) The norms $\|\cdot\|_1^{(n)}$ and $\|\cdot\|_{a,n+1}$ are not equivalent.

Proof.

(a) If $f \in L^{(n),1}$ then there is $F \in L^1$ such that $f = F^{(n)}$. Let $G(x) = \int_{-\infty}^x F(t) dt$. Then $G \in AC(\overline{\mathbb{R}}) \subset \mathcal{B}_c$. For almost all $x \in \mathbb{R}$ we have $G'(x) = F(x)$ so $G^{(n+1)} = F^{(n)}$ and $f \in \mathcal{A}_c^{n+1}$.

(b) It is not closed since $AC(\overline{\mathbb{R}})$ is dense in $\mathcal{B}_c$. See [17: Proposition 3.3].

(c) Let $F_m(x) = \sin(mx)\chi_{(0,2\pi)}(x)$. Let $f_m = F_m^{(n)} \in L^{(n),1}$. Then $\|f_m\|_1^{(n)} = \|F_m\|_1 = \int_0^{2\pi} |\sin(mx)| dx = 4$. And, $\|f_m\|_{a,n+1} = \sup_{0 \leq x \leq 2\pi} \left| \int_0^x \sin(mt) dt \right| = \int_0^{\pi/m} \sin(mt) dt = 2/m \rightarrow 0$ as $m \rightarrow \infty$. The two norms are then not equivalent. $\square$

For no $1 < p < \infty$ is $L^{(n),p}$ a subset of any of the $\mathcal{A}_c^m$ spaces since for each $1 < p < \infty$ there is a function $F \in L^p$ for which $\int_0^x F(t) dt$ is not bounded.

9. Half plane Poisson integral

As an application we show the half plane Poisson integral can be defined for distributions in $L^{(n),p}$ and has essentially the same behaviour with respect to $\|\cdot\|_p^{(n)}$ as the Poisson integral of $F \in L^p$ has with $\|\cdot\|_p$. The upper half plane is $\Pi^+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. The Poisson kernel is $\Phi_y(x) = (y/\pi)(x^2 + y^2)^{-1}$. If $F \in L^p$ define $U_y(x) = \Phi_y * F(x) = (y/\pi) \int_{-\infty}^{\infty} F(\xi)[(x - \xi)^2 + y^2]^{-1} d\xi$. The following results are well known.

THEOREM 9.1. *Let $1 \leq p < \infty$. Let $F \in L^p$. For each $(x, y) \in \Pi^+$ define $U_y(x) = \Phi_y * F(x)$.*

- (a) U_y is harmonic in Π^+ .
- (b) $\|U_y\|_p \leq \|F\|_p$
- (c) $\|U_y - F\|_p \rightarrow 0$ as $y \rightarrow 0^+$.

For a proof see, for example, [3].

We have the following analogue for distributions that are the n th derivative of an L^p function.

THEOREM 9.2. *Let $1 \leq p < \infty$. Let $F \in L^p$. For each $(x, y) \in \Pi^+$ define $U_y(x) = \Phi_y * F(x)$. Let $f = F^{(n)} \in L^{(n),p}$.*

- (a) *For each $(x, y) \in \Pi^+$ the following integral is well defined*

$$u_y(x) = \Phi_y * f(x) = (y/\pi) \int_{-\infty}^{\infty} f(\xi) [(x - \xi)^2 + y^2]^{-1} d\xi.$$

- (b) u_y is harmonic in Π^+ .
- (c) $\|u_y\|_p^{(n)} \leq \|f\|_p^{(n)}$
- (d) $\|u_y - f\|_p^{(n)} \rightarrow 0$ as $y \rightarrow 0^+$.

Proof. Let q be the conjugate of p .

(a) The Poisson kernel is real analytic in Π^+ so each derivative is a continuous function. Note that $\partial^m \Phi_y(x)/\partial x^m = O(x^{-m-2})$ as $|x| \rightarrow \infty$. Hence, each derivative of Φ_y is in L^1 . And, for each $y > 0$ there is a polynomial, p_y , of degree at most $n - 1$ such that $\Phi_y + p_y \in I^{n,1}$. It follows from Definition 8.1 that

$$\begin{aligned} u_y(x) &= \frac{(-1)^n y}{\pi} \int_{-\infty}^{\infty} F(\xi) \frac{\partial^n \Phi_y(\xi - x)}{\partial \xi^n} d\xi = \frac{y}{\pi} \int_{-\infty}^{\infty} F(\xi) \frac{\partial^n \Phi_y(\xi - x)}{\partial x^n} d\xi \\ &= \frac{\partial^n U_y(x)}{\partial x^n}. \end{aligned}$$

The growth estimate $\partial^n \Phi_y(x)/\partial x^n = O(x^{-n-2})$ as $|x| \rightarrow \infty$ and dominated convergence allows the derivative to be moved outside the integral.

(b) Since the Laplacian commutes with each derivative, we now see that u_y is harmonic in Π^+ .

- (c) From (b) and Theorem 9.1, $\|u_y\|_p^{(n)} = \|U_y\|_p \leq \|F\|_p = \|f\|_p^{(n)}$.

- (d) $\|u_y - f\|_p^{(n)} = \|U_y - F\|_p \rightarrow 0$ as $y \rightarrow 0^+$. □

The Poisson integral of distributions has been considered by other authors in the spaces $\mathcal{D}'_{L^p}(\mathbb{R})$ (end of Section 2) and weighted versions of these spaces.

See [1] and [7] for results and further references. However, the boundary values are then taken on in a weak sense, whereas here we have boundary values taken on in the norm $\|\cdot\|_p^{(n)}$. A deeper study would involve questions of uniqueness. The condition on $F \in L^1$ in Theorem 9.2 can be weakened to existence of the integral $\int_{-\infty}^{\infty} |F(x)|(x^2 + 1)^{-1} dx$.

10. Integration in $\mathbb{R}^n$

We briefly outline a method of extending the L^p integrals to $\mathbb{R}^n$.

Impose a Cartesian coordinate system on $\mathbb{R}^n$ and write $x = (x_1, \dots, x_n)$. Let $1 \leq p < \infty$ and let q be its conjugate. Let $F \in L^p(\mathbb{R}^n)$. Let $D = \partial^n / \partial x_1 \partial x_2 \dots \partial x_n$. Define $L'^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid f = DF \text{ for some } F \in L^p(\mathbb{R}^n)\}$. For the distributional derivative write $DF(x) = F_{12\dots n}(x)$. Let $G(x) = \int_0^{x_n} \dots \int_0^{x_1} g(y_1) dy_1 \dots dy_n$ for some $g \in L^q(\mathbb{R}^n)$. If $f = DF \in L'^p(\mathbb{R}^n)$ for unique $F \in L^p(\mathbb{R}^n)$, the integral is defined with iterated Lebesgue integrals as $\int_{\mathbb{R}^n} fG = (-1)^n \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x) G_{12\dots n}(x) dx_1 \dots dx_n$. The norm is $\|f\|_p' = \|F\|_p$ and $L'^p(\mathbb{R}^n)$ is isometrically isomorphic to $L^p(\mathbb{R}^n)$. Most of the results of the previous sections can now be extended to this integral. This method originates in work on integrals with continuous primitives by Ang, Schmitt, Vy [2] and Mikusiński, Ostaszewski [13]. Similarly, if $\alpha_i \in \mathbb{N}_0$ for $1 \leq i \leq m$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a multi-index then define the differential operator $\delta = \partial^{\alpha_1} / \partial x_1 \dots \partial^{\alpha_m} / \partial x_m$. Let $g \in L^q(\mathbb{R}^n)$ and G be an iterated integral of g , integrated α_i times in the i th coordinate. For $F \in L^p(\mathbb{R}^n)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ defined by $f = \delta F$ the integral is $\int_{\mathbb{R}^n} fG = (-1)^{|\alpha|} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x) \delta G(x) dx$.

REFERENCES

- [1] ALVAREZ, J.—GUZMÁN-PARTIDA, M.—PÉREZ-ESTEVA, S.: *Harmonic extensions of distributions*, Math. Nachr. **280** (2007), 1443–1466.
- [2] ANG, D. D.—SCHMITT, K.—VY, L. K.: *A multidimensional analogue of the Denjoy-Perron-Henstock-Kurzweil integral*, Bull. Belg. Math. Soc. Simon Stevin **4** (1997), 355–371.
- [3] AXLER, S.—BOURDON, P.—RAMEY, W.: *Harmonic Function Theory*, Springer-Verlag, New York, 2001.
- [4] BARROS-NETO, J.: *An Introduction to the Theory of Distributions*, Marcel Dekker, New York, 1973.
- [5] BARTLE, R. G.: *Elements of Integration*, Wiley, New York, 1966.

- [6] ČELIDZE, V. G.—DŽVARŠEĪŠVILI, A. G.: *The Theory of the Denjoy Integral and some Applications*, World Scientific, Singapore, 1989.
- [7] CICHOCKA, A.—KIERAT, W.: *An application of the Wiener functions to the Dirichlet problem of the Laplace equation*, *Integral Transforms Spec. Funct.* **7** (1998), 13–20.
- [8] CLARKSON, J. A.: *Uniformly convex spaces*, *Trans. Amer. Math. Soc.* **40** (1936), 396–414.
- [9] FOLLAND, G. B.: *Real Analysis*, Wiley, New York, 1999.
- [10] FRIEDLANDER, F. G.—JOSHI, M.: *Introduction to the Theory of Distributions*, Cambridge University Press, Cambridge, 1999.
- [11] LEADER, S.: *The Kurzweil-Henstock Integral and Its Differentials*, Marcel Dekker, New York, 2001.
- [12] LIEB, E. H.—LOSS, M.: *Analysis*, Amer. Math. Soc., Providence, RI, 2001.
- [13] MIKUSIŃSKI, P.—OSTASZEWSKI, K.: *Embedding Henstock integrable functions into the space of Schwartz distributions*, *Real Anal. Exchange* **14** (1988-89), 24–29.
- [14] RUDIN, W.: *Real and Complex Analysis*, McGraw-Hill, New York, 1987.
- [15] SCHWARTZ, L.: *Théorie des distributions*, Hermann, Paris, 1966.
- [16] TALVILA, E.: *The Distributional Denjoy Integral*, *Real Anal. Exchange* **33** (2008), 51–82.
- [17] TALVILA, E.: *Convolutions with the continuous primitive integral*, *Abstr. Appl. Anal.* **2009** (2009), Art. ID 307404.
- [18] TALVILA, E.: *The regulated primitive integral*, *Illinois J. Math.* **53** (2009), 1187–1219.
- [19] TALVILA, E.: *Integrals and Banach spaces for finite order distributions*, *Czechoslovak Math. J.* **62** (2012), 77–104.
- [20] YOSIDA, K.: *Functional Analysis*, Springer-Verlag, Berlin, 1980.
- [21] ZEMANIAN, A. H.: *Distribution Theory and Transform Analysis*, Dover, New York, 1987.

Received 28. 2. 2012

Accepted 14. 8. 2012

*Department of Mathematics & Statistics
University of the Fraser Valley
Abbotsford, BC V2S 7M8
CANADA*

E-mail: Erik.Talvila@ufv.ca