VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0287-6
Math. Slovaca **64** (2014), No. 6, 1475–1496

GENERALIZED ORLICZ-LORENTZ SEQUENCE SPACES AND CORRESPONDING OPERATOR IDEALS

Manjul Gupta — Antara Bhar

(*Communicated by Lajos Molnar*)

ABSTRACT. In this paper we introduce generalized or vector-valued Orlicz-Lorentz sequence spaces $l_{p,q,M}(X)$ on Banach space X with the help of an Orlicz function M and for different positive indices p and q. We study their structural properties and investigate cross and topological duals of these spaces. Moreover these spaces are generalizations of vector-valued Orlicz sequence spaces $l_M(X)$ for $p = q$ and also Lorentz sequence spaces for $M(x) = x^q$ for $q \geq 1$. Lastly we prove that the operator ideals defined with the help of scalar valued sequence spaces $l_{p,q,M}$ and additive s-numbers are quasi-Banach operator ideals for $p < q$ and Banach operator ideals for $p \geq q$. The results of this paper are more general than the work of earlier mathematicians, say A. Pietsch, M. Kato, L. R. Acharya, etc.

1. Introduction

The study of vector-valued sequence spaces(VVSS) which is a generalization of scalar-valued sequence spaces (SVSS) was motivated by the work of A. Grothendieck in his study on nuclear spaces, cf. [5]. Since then this theory has developed considerably in different directions, cf. [5], [11], [14] etc. and references given therein. The present work is an endeavor in this direction; indeed we introduce the vector-valued Orlicz-Lorentz sequence spaces which include as particular cases the vector-valued Orlicz sequence spaces as well as Lorentz sequence spaces. We find their Köthe and topological duals and study the interrelationships amongst these spaces. Finally, we introduce the operator ideals with the help of the corresponding scalar sequence spaces and s-numbers. The results of this paper generalize the work of A. Pietsch [19], M. Kato [10], L. R. Acharya [1].

2010 Mathematics Subject Classification: Primary 46A45, 47B06, 47L20.
Keywords: Lorentz sequence spaces, s-numbers of operators, Orlicz function and Orlicz sequence spaces, operator ideals.

2. Preliminaries

An *Orlicz function* M is a continuous, convex function defined from $[0, \infty)$ to itself such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Such function M always has the integral representation

$$M(x) = \int_0^x p(t)\,\mathrm{d}t$$

where $p(t)$, known as the *kernel* of M, is right continuous, non-decreasing function for $t > 0$. Let us note that an Orlicz function M is always increasing and $M(x) \to \infty$ as $x \to \infty$. Also $tp(t) \to \infty$ as $t \to \infty$ and $tp(t) = 0$ for $t = 0$, cf. [8: p. 139]. However $p(t) > 0$ for $t = 0$ is equivalent to the fact that the Orlicz sequence space l_M (see the definition below) is isomorphic to l_1, cf. [9: p. 309]. Therefore, we assume here that the karnel $p(t)$ has value 0 for $t = 0$ and obviously $p(t) \to \infty$ as $t \to \infty$.

Given an Orlicz function M with kernel p, define $q(s) = \sup\{t : \ p(t) \leq s\}$, $s \geq 0$. Then q possesses the same properties as p and the function N defined as $N(x) = \int_0^x q(t)\,\mathrm{d}t$, is an Orlicz funcion. The functions M and N are called *mutually complementary functions*. For such functions M and N we have the Young's inequality: $xy \leq M(x) + N(y)$, for $xy \geq 0$ and also $M(\alpha x) \leq \alpha M(x)$ for $0 < \alpha < 1$; cf. [9].

An Orlicz function M is said to satisfy the Δ_2-*condition for small* x *or at* 0, if for each $k > 1$, there exists $R_k > 0$ and $x_k > 0$ such that

$$M(kx) \leq R_k M(x) \qquad \text{for all} \quad x \in (0, x_k].$$

Let X and Y be vector spaces over the same field $\mathbb{K}$ of real or complex numbers, forming a dual system $\langle X, Y\rangle$ with respect to the bilinear functional $\langle x, y\rangle$. We denote by $\Omega(X)$ the vector space of all sequences formed by the elements of X, with respect to the operations of pointwise addition and scalar multiplication; and by $\phi(X)$, the space of all finitely non zero sequences from $\Omega(X)$. *A vector-valued sequence space* or *a generalized sequence space* $\Lambda(X)$ is a subspace of $\Omega(X)$ containing $\phi(X)$. The symbol δ_i^x stands for the sequence $\{0, 0, \dots, 0, x, 0, 0, \dots\}$, where x is placed at the ith coordinate.The notation $\overline{x}^{(n)}$ denotes the *nth section* of $\overline{x}$ given by $\{x_1, x_2, \dots, x_n, 0, 0, \dots\}$.

A subset M of $\Lambda(X)$ is said to be *normal* if for $\{x_i\} \in M$ and $\{\alpha_i\} \in \mathbb{K}$, with $|\alpha_i| \leq 1$, $i \geq 1$, the sequence $\{\alpha_i x_i\} \in M$.

The *generalized Köthe dual* of $\Lambda(X)$ is the space

$$\Lambda^{\times}(Y) = \Big\{\overline{y} = \{y_i\} \in Y : \ \sum_{i\geq 1} |\langle x_i, y_i\rangle| < \infty \ \text{ for all } \ \{x_i\} \in \Lambda(X)\Big\}.$$

The generalized Köthe dual of $\Lambda^{\times}(Y)$ is denoted by $\Lambda^{\times\times}(X)$. The space $\Lambda(X)$ is said to be *perfect* if $\Lambda(X) = \Lambda^{\times\times}(X)$. A vector-valued sequence space $\Lambda(X)$ equipped with a Hausdorff locally convex topology T is called

(i) a *GK-space* if the maps $P_{n,\Lambda(X)} \colon \Lambda(X) \to X$, $P_{n,\Lambda(X)}(\overline{x}) = x_n$, for each $n \geq 1$, are continuous;

(ii) a *GAK-space* if $\Lambda(X)$ is a GK-space and for each $\{x_i\} \in \Lambda(X)$, $\overline{x}^{(n)} \to \overline{x}$ as $n \to \infty$, in T;

(iii) a *GAD-space* if $\overline{x} \in \overline{\phi(X)}$, for every $\overline{x} \in \Lambda(X)$ i.e. $\overline{\phi(X)} = \Lambda(X)$.

Every perfect sequence space $\Lambda(X)$ is normal; cf. [14].

Let us mention here that if the dual system is $\langle X, X^* \rangle$ where X is a Banach space and X^* is its topological dual, then we may interchangeably use the notations $\langle x, f \rangle$ or $f(x)$ for $x \in X$ and $f \in X^*$ in the sequel.

In case $X = \mathbb{K}$, the field of scalars, we write ω for $\Omega(X)$, ϕ for $\phi(X)$ and λ for $\Lambda(X)$. If e_n's are the nth unit vectors in ω, i.e. $e^n = \{\delta_{nj}\}_{j=1}^{\infty}$, where δ_{nj} is the Kronecker delta, ϕ is clearly the subspace of ω spanned by e_n's, $n \geq 1$. A sequence space λ is said to be *symmetric* if $\overline{\alpha}_{\sigma} = \{\alpha_{\sigma(i)}\} \in \lambda$ whenever $\overline{\alpha} \in \lambda$ and $\sigma \in \Pi$, where Π *is the collection of all permutations of* $\mathbb{N}$. The Köthe dual $\lambda^{\times}$ of a symmetric sequence space λ is symmetric; cf. [9: p. 91].

For scalar valued sequence space λ, its δ-*dual* is defined as

$$\lambda^{\delta} = \Big\{\overline{\alpha} \in \omega : \sum_{i \geq 1} |\alpha_i \beta_{\rho(i)}| < \infty \ \text{ for all } \overline{\beta} \in \lambda \ \text{ and } \ \rho \in \Pi\Big\}.$$

$\lambda^{\times}$ coincides with λ^{δ}, if λ is symmetric. cf. [9: p. 52].

For a scalar-valued sequence space λ and a Banach space X we define

$$\lambda(X) = \Big\{\{x_n\} : \ x_n \in X, \ n \geq 1 \text{ and } \{\|x_n\|\} \in \lambda\Big\}.$$

In case, λ equipped with the norm $\|\cdot\|_{\lambda}$, is a Banach space, it turns out that $\lambda(X)$ is also a Banach space with respect to the norm given by

$$\|\overline{x}\|_{\lambda(X)} = \|\{\|x_n\|\}\|_{\lambda}.$$

cf. [1], [11]. As particular cases, we have $l_{\infty}(X)$ for $\lambda = l_{\infty}$ and $c_0(X)$ corresponding to $\lambda = c_0$.

Corresponding to an Orlicz function M and the Banach space X, the set $\tilde{l}_M(X)$ is defined by

$$\tilde{l}_M(X) = \Big\{\overline{x} \in \Omega(X) : \sum_{i \geq 1} M(\|x_i\|) < \infty\Big\}.$$

If M and N are mutually complementary functions, the *vector-valued Orlicz sequence space* is defined as

$$l_M(X) = \Big\{\overline{x} \in \Omega(X) : \ \sum_{i \geq 1} f_i(x_i) \ \text{ converges for all } \ \{f_i\} \in \tilde{l}_N(X^*)\Big\}.$$

An equivalent way of defining $l_M(X)$ is

$$l_M(X) = \Big\{\overline{x} \in \Omega(X) :\ \sum_{i\geq 1} M\Big(\frac{\|x_i\|}{\rho}\Big) < \infty \text{ for some } \rho > 0\Big\}.$$

Two equivalent norms on l_M are given by

$$\|\overline{x}\|_{(M)} = \sup\Big\{\big|\sum_{i\geq 1} f_i(x_i)\big| :\ \sum_{i\geq 1} N(\|f_i\|) \leq 1\Big\},$$

and

$$\|\overline{x}\|_M = \inf\Big\{\rho > 0 :\ \sum_{i\geq 1} M(\frac{\|x_i\|}{\rho}) \leq 1\Big\};$$

indeed,

$$\|\overline{x}\|_M \leq \|\overline{x}\|_{(M)} \leq 2\|\overline{x}\|_M \qquad \text{for} \quad \overline{x} \in l_M(X)$$

cf. [24]. For $X = \mathbb{K}$, we write $l_M(X)$ as l_M. If M and N are mutually complementary Orlicz functions and M satisfies Δ_2-condition at 0, then $(l_M)^\times = l_N$, cf. [9].

A sequence $\widetilde{M} = \{M_n\}$ of Orlicz functions known as *Musielak-Orlicz* function has been studied in cf. [3], [13]. Let p_n be the kernel of M_n for all $n \in \mathbb{N}$. $\widetilde{M}$ is said *to satisfy* L1 *condition* if $p_n(x) \geq p_{n+1}(x)$ for all $x \in [0, \infty)$. Given a Musielak-Orlicz function $\widetilde{M}$, a *convex modular* $\varrho_{\widetilde{M}}$ on ω is defined as

$$\varrho_{\widetilde{M}}(\{\alpha_n\}) = \sup_{\sigma\in\Pi} \sum_{n=1}^{\infty} M_n(\alpha_{\sigma(n)}).$$

Corresponding to a convex modular $\varrho_{\widetilde{M}}$, we have modular space defined in the literature as

$$\lambda_{\widetilde{M}} = \big\{\overline{\alpha} = \{\alpha_n\} \in \omega :\ \varrho_{\widetilde{M}}(\beta\overline{\alpha}) < \infty \text{ for some } \beta > 0\big\}$$

This space becomes a normed space under the Luxemburg norm

$$\|\overline{\alpha}\| = \inf\big\{\beta > 0 :\ \varrho_{\widetilde{M}}\big(\tfrac{\overline{\alpha}}{\beta}\big) \leq 1\big\}.$$

Let us note that a modular sequence space $\lambda_{\widetilde{M}}$ is a symmetric sequence space.

The *decreasing rearrangement of the absolute values of a sequence* $\overline{\alpha} = \{\alpha_n\}$ in l_∞ (space of all bounded sequences), is given by $\{t_n(\overline{\alpha})\}$, where

$$t_n(\overline{\alpha}) = \inf\big\{\rho > 0 :\ \text{card}\{k :\ |\alpha_k| > \rho\} < n\big\}.$$

Here *card* A denotes the cardinality of the set A. The sequence $\{t_n(\overline{\alpha})\}$ satisfies the following properties, cf. [19].

(i) $\|\overline{\alpha}\|_\infty = t_1(\overline{\alpha}) \geq t_2(\overline{\alpha}) \geq \dots \geq 0$ for $\overline{\alpha} \in l_\infty$.

(ii) $t_{m+n-1}(\overline{\alpha} + \overline{\beta}) \leq t_m(\overline{\alpha}) + t_n(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in l_\infty$.

(iii) $t_{m+n-1}(\overline{\alpha}\ \overline{\beta}) \leq t_m(\overline{\alpha}) t_n(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in l_\infty$.

Here $\overline{\alpha}\ \overline{\beta} = \{\alpha_n\ \beta_n\}$.

For $\overline{x} = \{x_n\} \in l_\infty(X)$, we denote by

$$t_n(\overline{x}) = t_n(\{x_n\}) = t_n(\{\|x_n\|\}), \qquad n \in \mathbb{N}.$$

For $0 < p, q \leq \infty$, the *Lorentz sequence space* $l_{p,q}$ is given by

$$l_{p,q} = \Big\{\overline{\alpha} = \{\alpha_n\} \in l_\infty : \ \{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{\alpha})\} \in l_q\Big\}.$$

For $\overline{\alpha} \in l_{p,q}$, let us consider the real-valued function $\|\cdot\|_{p,q}$ as follows

$$\|\overline{\alpha}\|_{p,q} = \begin{cases} \Big\{ \sum\limits_{n\geq 1} (n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{\alpha}))^q \Big\}^{\frac{1}{q}} & \text{for } 0 < q < \infty \\ \sup\limits_{n\geq 1} n^{\frac{1}{p}} t_n(\overline{\alpha}) & \text{for } q = \infty \end{cases}$$

For a convex modular $\varrho_{\widetilde{M}}$ defined on ω, it has been proved in [3], that

$$\sum_{n\geq 1} M_n(t_n(\overline{\alpha})) = \varrho_{\widetilde{M}}(\overline{\alpha}) \tag{2.1}$$

for $\overline{\alpha} \in \omega$ if and only if $\widetilde{M}$ satisfies L1 condition.

In view of (2.1) $(l_{p,q}, \ \|\cdot\|_{p,q})$ is Banach spaces for $p \geq q$. However for $p < q$, it is a quasi-Banach space. Further, they are symmetric sequence spaces, cf. [17].

Throughout this paper, we denote by X and Y, the Banach spaces over the complex field $\mathbb{C}$ and by $L(X,Y)$, the *class of all bounded linear maps from X to Y*. Let L be the *class of all bounded linear operators between any pair of Banach spaces.*

A map $s\colon L \to \omega^+$, where ω^+ is the class of sequences of non-negative real numbers, is called an *s-number function* if it satisfies the following conditions:

(i) $\|S\| = s_1(S) \geq s_2(S) \geq \cdots \geq 0$, $s(S) = \{s_n(S)\}$, $S \in L$,

(ii) $s_n(S+T) \leq s_n(S) + \|T\|$ for $S, T \in L(X,Y)$ and $n \in \mathbb{N}$,

(iii) $s_n(RST) \leq \|R\| s_n(S) \|T\|$ for $T \in L(X_0, X)$, $S \in L(X,Y)$, $R \in L(Y, Y_0)$, and $n \in \mathbb{N}$,

(iv) if $\operatorname{rank} S < n$, then $s_n(S) = 0$,

(v) if $\dim X \geq n$, then $s_n(I_X) = 1$, where I_X denotes the identity map of X.

An s-number function is called *additive* if the condition (ii) is replaced by

(ii)′ $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$ for $S, T \in L(X,Y)$ and $m, n = 1, 2, \ldots,$

If the condition (iii) is replaced by

(iii)′ $s_{m+n-1}(RT) \leq s_m(R) s_n(T)$ for $R \in L(Y_0, Y)$ and $T \in L(X, Y_0)$, $m, n = 1, 2, \ldots,$

then the s-number function is called *multiplicative.*

For a subset A of L, we write $A(X,Y) = A \cap L(X,Y)$ where X and Y are Banach spaces. The collection A is said to be an *operator ideal* if it satisfies the following conditions:

(i) A contains all finite rank operators,

(ii) $T + S \in A(X,Y)$ for $S, T \in A(X,Y)$,

(iii) if $T \in A(X,Y)$ and $S \in L(Y,Z)$, then $ST \in A(X,Z)$ and also if $T \in L(X,Y)$ and $S \in A(Y,Z)$, then $ST \in A(X,Z)$.

The collection $A(X,Y)$, for a given pair of Banach spaces E and F, is called a *component* of A.

An *ideal quasi norm* is a real valued function f defined on an operator ideal A, which satisfies the following properties:

(i) $0 \le f(T) < \infty$, for each $T \in A$ and $f(T) = 0$ if and only if $T = 0$,

(ii) there exists a constant $\sigma \ge 1$ such that $f(S+T) \le \sigma[f(S) + f(T)]$ for $S, T \in A(X,Y)$, where $A(X,Y)$ is any component of A,

(iii) a) $f(RS) \le \|R\| f(S)$ for $S \in A(X,Z)$, $R \in L(Z,Y)$ and
b) $f(RS) \le \|S\| f(R)$ for $S \in L(X,Z)$, $R \in A(Z,Y)$

A *quasi-normed operator ideal* is an operator ideal equipped with an ideal quasi-norm and a *quasi-Banach operator ideal* is a quasi-normed operator ideal of which each component is complete with respect to the ideal quasi-norm.

3. The vector-valued sequence spaces $l_{p,q,M}(X)$ and $h_{p,q,M}(X)$

Corresponding to a Banach space X, an Orlicz function M and $0 < p, q \le \infty$, let us introduce

$$l_{p,q,M}(X) = \Big\{\overline{x} = \{x_n\} \in l_\infty(X) : \sum_{n\ge1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho}\Big) < \infty \text{ for some } \rho > 0\Big\}$$

and

$$h_{p,q,M}(X) = \Big\{\overline{x} = \{x_n\} \in l_\infty(X) : \sum_{n\ge1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\delta}\Big) < \infty \text{ for every } \delta > 0\Big\}$$

For $\overline{x} \in l_{p,q,M}(X)$, we define

$$\|\overline{x}\|_{p,q,M} = \inf\Big\{\rho > 0 : \sum_{n\ge1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho}\Big) \le 1\Big\}$$

We prove the following theorem:

Theorem 3.1. *The space $l_{p,q,M}(X)$ equipped with $\|\cdot\|_{p,q,M}$ is a quasi-Banach space for $p<q$ and Banach space for $p\ge q$. Further, for $\overline{x}\in l_{p,q,M}(X)$,*

$$\sum_{n\ge1} M\left(\frac{n^{1/p-1/q}t_n(\overline{x})}{\|\overline{x}\|_{p,q,M}}\right)\le 1. \tag{3.1}$$

P r o o f. One can easily check that $l_{p,q,M}(X)$ is a vector space with usual coordinate wise addition and scalar multiplication.

For showing that $\|\cdot\|_{p,q,M}$ is a quasi-norm, let us note that $\|\overline{x}\|_{p,q,M}\ge 0$ for each $\overline{x}\in l_{p,q,M}(X)$ and $\|\overline{x}\|_{p,q,M}=0$ for $\overline{x}=0$. Now, if $\|\overline{x}\|_{p,q,M}=0$ for some $\overline{x}=\{x_n\}\in l_{p,q,M}(X)$, then for any given $\varepsilon>0$, we can find $\rho>0$ such that $\rho<\varepsilon$ and

$$\sum_{n\ge1} M\left(\frac{n^{\frac1p-\frac1q}t_n(\overline{x})}{\rho}\right)\le 1$$

If $\overline{x}\ne 0$, then $\|x_{n_0}\|\ne 0$ for some $n_0\in\mathbb{N}$ and so $t_{n_1}(\overline{x})=\|x_{n_0}\|$, for some $n_1\in\mathbb{N}$ would imply

$$M\left(\frac{n_1^{\frac1p-\frac1q}t_{n_1}(\overline{x})}{\varepsilon}\right)\le M\left(\frac{n_1^{\frac1p-\frac1q}t_{n_1}(\overline{x})}{\rho}\right)\le 1$$

for any $\varepsilon>0$. This leads to a contradiction and hence $\overline{x}=0$.

For proving triangular-type inequality, consider $\overline{x}=\{x_n\}$ and $\overline{y}=\{y_n\}$ in $l_{p,q,M}(X)$. Then for any given $\varepsilon>0$, there exist $\rho_1,\rho_2>0$, such that

$$\rho_1<\|\overline{x}\|_{p,q,M}+\frac{\varepsilon}{2}\qquad\text{with}\qquad\sum_{n\ge1} M\left(\frac{n^{\frac1p-\frac1q}t_n(\overline{x})}{\rho_1}\right)\le 1;$$

and

$$\rho_2<\|\overline{y}\|_{p,q,M}+\frac{\varepsilon}{2}\qquad\text{with}\qquad\sum_{n\ge1} M\left(\frac{n^{\frac1p-\frac1q}t_n(\overline{y})}{\rho_2}\right)\le 1.$$

If $\frac1p-\frac1q>0$, then using properties (i) and (ii) of $\{t_n(\overline{x})\}$, we get

$$\begin{aligned}
&\sum_{n\ge1} M\left(\frac{n^{\frac1p-\frac1q}t_n(\overline{x}+\overline{y})}{2^{\frac1p-\frac1q+1}(\rho_1+\rho_2)}\right)\\
&=\sum_{n\ge1} M\left(\frac{(2n)^{\frac1p-\frac1q}t_{2n}(\overline{x}+\overline{y})}{2^{\frac1p-\frac1q+1}(\rho_1+\rho_2)}\right)+\sum_{n\ge1} M\left(\frac{(2n-1)^{\frac1p-\frac1q}t_{2n-1}(\overline{x}+\overline{y})}{2^{\frac1p-\frac1q+1}(\rho_1+\rho_2)}\right)\\
&\le 2\sum_{n\ge1} M\left(\frac{n^{\frac1p-\frac1q}(t_n(\overline{x})+t_n(\overline{y}))}{2(\rho_1+\rho_2)}\right)
\end{aligned}$$

$$\leq \sum_{n\geq 1} M\left(\frac{\rho_1}{\rho_1+\rho_2}\frac{n^{\frac{1}{p}-\frac{1}{q}}t_n(\overline{x})}{\rho_1}+\frac{\rho_2}{\rho_1+\rho_2}\frac{n^{\frac{1}{p}-\frac{1}{q}}t_n(\overline{y})}{\rho_2}\right)\leq 1$$

Hence,

$$\begin{aligned}\|\overline{x}+\overline{y}\|_{p,q,M} &\leq 2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)\\ &\leq 2^{\frac{1}{p}-\frac{1}{q}+1}\big(\|\overline{x}\|_{p,q,M}+\|\overline{y}\|_{p,q,M}+\varepsilon\big).\end{aligned}$$

If $\frac{1}{p}-\frac{1}{q}\leq 0$, we consider $M_n(u)=M(n^{\frac{1}{p}-\frac{1}{q}}u)$ for $u>0$. Then $\{M_n\}$ defines a Musielak-Orlicz function satisfying L1 condition and so by (2.1) and the definition of $\|\cdot\|_{p,q,M}$, $l_{p,q,M}(X)$ becomes a normed space.

In order to show the completeness of the space $(l_{p,q,M}(X),\|\cdot\|_{p,q,M})$, consider a Cauchy sequence $\{\overline{x_n}\}$, $\overline{x_n}=\{x_n^k\}_{k\geq 1}$, $n\in\mathbb{N}$, in $l_{p,q,M}(X)$. Then for $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\|\overline{x_{n+j}}-\overline{x_n}\|_{p,q,M}=\inf\left\{\rho>0:\ \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}}t_k(\overline{x_{n+j}}-\overline{x_n})}{\rho}\right]\leq 1\right\}<\varepsilon$$

for each $n\geq n_0$ and each $j\in\mathbb{N}$. Hence

$$\sum_{k\geq 1}\left[\frac{k^{\frac{1}{p}-\frac{1}{q}}t_k(\overline{x_{n+j}}-\overline{x_n})}{\varepsilon}\right]\leq 1 \qquad \text{for all}\quad n\geq n_0,\quad j\in\mathbb{N}$$

$\Longrightarrow$ $\left\{\{k^{\frac{1}{p}-\frac{1}{q}}t_k(\overline{x_{n+j}}-\overline{x_n})/\varepsilon\}:\ k\in\mathbb{N}\right\}$ is a bounded set for $j\in\mathbb{N}$ and for all $n\geq n_0$. Thus $\{x_n^k\}$ is a Cauchy sequence in X, for each $k\in\mathbb{N}$ and so converges to z_k, say. Let $\overline{z}=\{z_k\}$. Then $t_k(\overline{x_{n+j}}-\overline{x_n})\to t_k(\overline{z}-\overline{x_n})$ as $j\to\infty$ and hence by continuity of M,

$$\sum_{k\geq 1} M\left(\frac{k^{\frac{1}{p}-\frac{1}{q}}t_k(\overline{z}-\overline{x_n})}{\varepsilon}\right)\leq 1 \qquad \text{for all}\quad n\geq n_0$$

$\Longrightarrow$ $\overline{z}\in l_{p,q,M}(X)$ and $\|\overline{z}-\overline{x_n}\|_{p,q,M}\to 0$ as $n\to\infty$.

Expression (3.1) is immediate from the definition of the quasi-norm $\|\cdot\|_{p,q,M}$. This completes the proof. □

Remark. Let us note that the spaces $l_{p,q,M}(X)$ are weighted vector-valued Orlicz sequence spaces. If the sequence $\{n^{\frac{1}{p}-\frac{1}{q}}\}$ is replaced by a strictly non-negative weight $\{\beta_n\}$, then the weighted Orlicz sequence spaces $l_{\overline{\beta},M}(X)$,which are defined as

$$l_{\overline{\beta},M}(X)=\left\{\overline{x}=\{x_n\}\in l_\infty(X):\ \{\beta_n\ t_n(\overline{x})\}\in l_M\right\}$$

are quasi-Banach or Banach spaces according as $\{\beta_n\}$ is increasing or decreasing respectively.

Theorem 3.2. *$h_{p,q,M}(X)$ is a closed subspace of $l_{p,q,M}(X)$. If M satisfies Δ_2-condition at 0, then $l_{p,q,M}(X) = h_{p,q,M}(X)$.*

Proof. Clearly, $h_{p,q,M}(X)$ is a subspace of $l_{p,q,M}(X)$. For proving that it is closed in $l_{p,q,M}(X)$, consider $\overline{x} = \{x_n\} \in \bar{h}_{p,q,M}(X)$, the closure of $h_{p,q,M}(X)$ in $l_{p,q,M}(X)$. Then there exists a sequence $\{\overline{y_n}\} = \{\{y_n^k\}\}$, $n \geq 1$, in $h_{p,q,M}(X)$ such that $\|\overline{y_n} - \overline{x}\|_{p,q,M} \to 0$ as $n \to \infty$.

Choose any $\delta > 0$. Then for $\delta_1 = \min\{2^{\frac{1}{q}-\frac{1}{p}}\delta\ ,\ \delta\}$, we can find $n_0 \in \mathbb{N}$ such that

$$\|\overline{y_n} - \overline{x}\|_{p,q,M} < \frac{\delta_1}{2} \qquad \text{for all} \quad n \geq n_0. \tag{3.2}$$

Then for $\frac{1}{p} - \frac{1}{q} \geq 0$, we have

$$\begin{aligned}
\sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} t_k(\overline{x})}{\delta}\right] &\leq 2\sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} (t_k(\overline{x} - \overline{y_{n_0}}) + t_k(\overline{y_{n_0}}))}{\delta_1}\right] \\
&\leq \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} (t_k(\overline{x} - \overline{y_{n_0}}))}{\delta_1/2}\right] + \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} t_k(\overline{y_{n_0}})}{\delta_1/2}\right] \\
&\leq \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} (t_k(\overline{x} - \overline{y_{n_0}}))}{\|\overline{x} - \overline{y_{n_0}}\|_{p,q,M}}\right] + \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} (t_k(\overline{y_{n_0}})}{\delta_1/2}\right] \\
&< \infty
\end{aligned}$$

For the case $\frac{1}{p} - \frac{1}{q} < 0$, we have

$$\sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} t_k(\overline{x})}{\delta}\right] \leq \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} (t_k(\overline{x} - \overline{y_{n_0}}))}{\delta/2}\right] + \sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} t_k(\overline{y_{n_0}})}{\delta/2}\right] < \infty$$

by (3.2). Thus $\overline{x} \in h_{p,q,M}(X)$ and so the subspace $h_{p,q,M}(X)$ is closed.

Let us now assume that M satisfies Δ_2-condition at 0. Consider $\overline{x} \in l_{p,q,M}(X)$. Then $\sum\limits_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho_0}\Big) < \infty$ for some $\rho_0 > 0$. In order to show that $\overline{x} \in h_{p,q,M}(X)$, take any $\eta > 0$. If $\eta \geq \rho_0$ we clearly have $\sum\limits_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\eta}\Big) < \infty$. So, assume $\eta < \rho_0$ and write $K = \frac{\rho_0}{\eta}$. Then by Δ_2-condition of M, we can find $R_K > 0$ and $x_K > 0$ such that

$$\begin{aligned}
&M(Kx) \leq R_K M(x) \qquad \text{for all} \quad x \in\ (0, x_K] \\
\Longrightarrow &\sum_{n\geq n_0} M\left(K\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho_0}\right) \leq R_K \sum_{n\geq n_0} M\left(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho_0}\right) < \infty
\end{aligned}$$

for some $n_0 \in \mathbb{N}$.

Thus $\sum\limits_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\eta}\Big) < \infty$, for any $\eta > 0$, and so $\overline{x} \in h_{p,q,M}(X)$, i.e $h_{p,q,M}(X) = l_{p,q,M}(X)$.
This completes the proof. □

Proposition 3.3. *Let M be an Orlicz function; $0 < p, q \leq \infty$ and $Y = h_{p,q,M}(X) \cap c_0(X)$. Then Y equipped with the subspace topology of $h_{p,q,M}(X)$ is a GAD-space.*

Proof. Clearly $\phi(X) \subset Y$. Let $\overline{x} \in Y$. Then for any given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$, such that

$$\sum_{n\geq n_0} M\bigg(\frac{n^{1/p-1/q} t_n(\overline{x})}{\varepsilon}\bigg) \leq 1.$$

For given $n \in \mathbb{N}$, let

$$I_n = \big\{i \in \mathbb{N} : \ \|x_i\| > 1/n\big\}$$

and

$$\overline{v_n} = \sum_{i\in I_n} \delta_i^{x_i}.$$

Since $\overline{x} \in c_0(X)$, I_n is finite and so $\overline{v_n} \in \phi(X)$. Write $k_n = \operatorname{card} I_n$. Then choose $m_0 \in \mathbb{N}$, such that,

$$\sum_{n>m_0} M\bigg(\frac{n^{1/p-1/q}\ t_n(\overline{x})}{\varepsilon}\bigg) \leq \frac{1}{2}.$$

Choose n so large that

$$\frac{1}{n}\sum_{i=1}^{m_0} M\bigg(\frac{i^{1/p-1/q}}{\varepsilon}\bigg) \leq \frac{1}{2}$$

Hence,

$$\sum_{i\geq 1} M\bigg(\frac{i^{1/p-1/q}\ t_{k_n+i}(\overline{x})}{\varepsilon}\bigg)$$
$$\leq \frac{1}{n}\sum_{i=1}^{m_0} M\bigg(\frac{i^{1/p-1/q}}{\varepsilon}\bigg) + \sum_{i\geq m_0+1} M\bigg(\frac{i^{1/p-1/q}\ t_i(\overline{x})}{\varepsilon}\bigg) \leq \frac{1}{2} + \frac{1}{2} = 1.$$

$\implies \|\overline{x} - \overline{v_n}\|_{p,q,M} \leq \varepsilon$ for sufficiently large n. Thus Y is a GAD-space.
This completes the proof. □

Noting that for $0 < p \leq q \leq \infty$, $l_{p,q,M}(X) \subseteq c_0(X)$, we derive from the above proposition, the following

Proposition 3.4. *Let M be an Orlicz function satisfying Δ_2-condition at 0, and $0 < p \le q \le \infty$. Then $l_{p,q,M}$ is a GAD-space.*

Remarks. It would be interesting to know whether the space $h_{p,q,M}(X)$ is a GAK-space, i.e., the nth section $\overline{x}^{(n)} = \{x_1, x_2, \dots, x_n, 0, 0, 0, 0, \dots\}$ of an element $\overline{x} = \{x_i\}$ of $h_{p,q,M}$ converges to $\overline{x}$ with respect to its quasi-norm. However, if $p, q > 0$ with $1/p - 1/q \ge 0$ and $\overline{x} \in h_{p,q,M}(X)$ is such that $\|x_1\| > \|x_2\| > \|x_3\| > \dots$, then $t_n(\overline{x}) = \|x_n\|$ and in this case, one can establish $\|\overline{x} - \overline{x}^{(n)}\|_{p,q,M} \to 0$ as $n \to \infty$.

4. Duals of the space $l_{p,q,M}(X)$, $1 \le p \le q \le \infty$

Let us first note that the spaces $l_{p,q,M}(X)$ are symmetric sequence spaces, since the decreasing rearrangement of $\overline{x}$ would be the same as that of $\overline{x}_\pi$ for any permutation π of $\mathbb{N}$ and M is an increasing function. Hence the δ-dual of the scalar valued sequence space $l_{p,q,M}$ would coincide with its cross-dual. For the results on the duals of vector-valued sequence spaces $l_{p,q,M}(X)$, we first consider the scalar case as proved in

Theorem 4.1. *Let M and N be mutually complementary Orlicz functions such that M satisfies Δ_2-condition at 0. Then $(l_{p_1,q_1,M})^\times \supseteq l_{p_2,q_2,N}$, where $1/p_1 + 1/p_2 = 1$ and $1/q_1 + 1/q_2 = 1$. Equality holds when $1/p_1 - 1/q_1 \ge 0$.*

P r o o f. In order to show that $l_{p_2,q_2,N} \subset (l_{p_1,q_1,M})^\times$, consider $\overline{\beta} \in l_{p_2,q_2,N}$. Then

$$\sum_{n\ge1} N\left(\frac{n^{\frac{1}{p_2}-\frac{1}{q_2}} t_n(\overline{\beta})}{\delta_0}\right) < \infty \qquad \text{for some} \quad \delta_0 > 0.$$

If $\overline{\alpha} \in l_{p_1,q_1,M}$, then $\sum\limits_{n\ge1} M\left(\frac{n^{\frac{1}{p_1}-\frac{1}{q_1}} t_n(\overline{\alpha})}{\rho}\right) < \infty$ for all $\rho > 0$. Consequently,

$$\begin{aligned}\sum_{n\ge1} |\alpha_n \beta_n| &\le \sum_{n\ge1} t_n(\overline{\alpha}) t_n(\overline{\beta}) \\ &\le \sum_{n\ge1} M\left(\frac{n^{\frac{1}{p_1}-\frac{1}{q_1}} t_n(\overline{\alpha})}{1/\delta_0}\right) + \sum_{n\ge1} N\left(\frac{n^{\frac{1}{p_2}-\frac{1}{q_2}} t_n(\overline{\beta})}{\delta_0}\right) < \infty\end{aligned}$$

It follows that $\overline{\beta} \in (l_{p_1,q_1,M})^\times$.

For proving the equality with the given condition, if $\overline{\beta} \in (l_{p_1,q_1,M})^\times$, then $\sum\limits_{i\ge1} |\alpha_i \beta_i| < \infty$ for all $\{\alpha_i\} \in l_{p_1,q_1,M}$. As $l_{p_1,q_1,M}$ and $(l_{p_1,q_1,M})^\times$ both are symmetric sequence spaces, $\{t_n(\overline{\alpha})\} \in l_{p_1,q_1,M}$ for $\overline{\alpha} \in l_{p_1,q_1,M}$ and $\{t_n(\overline{\beta})\} \in (l_{p_1,q_1,M})^\times$ for $\overline{\beta} \in (l_{p_1,q_1,M})^\times$. Hence $\sum_{n\geq 1} t_n(\overline{\alpha})t_n(\overline{\beta}) < \infty$ for all $\overline{\alpha} \in l_{p_1,q_1,M}$.

Now if $\overline{\gamma} \in l_M$, then $\{t_n(\overline{\gamma})\} \in l_M$ as l_M is symmetric and normal and so $\{n^{\frac{1}{p_2}-\frac{1}{q_2}} t_n(\overline{\gamma})\} \in l_{p_1,q_1,M}$. Hence

$$\sum_{n\geq 1} n^{\frac{1}{p_2}-\frac{1}{q_2}} t_n(\overline{\gamma})t_n(\overline{\beta}) < \infty \qquad \text{for all} \quad \overline{\gamma} \in l_M.$$

$$\implies \left\{n^{\frac{1}{p_2}-\frac{1}{q_2}} t_n(\overline{\beta})\right\} \in l_M^\times = l_N \implies \overline{\beta} \in l_{p_2,q_2,N}. \qquad \text{(cf. Section 2).}$$

Thus $(l_{p_1,q_1,M})^\times = l_{p_2,q_2,N}$. □

This immediately leads to the following proposition:

PROPOSITION 4.2. *For positive reals p_1, p_2, q_1, q_2 with $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$ such that $q_1 < p_1$, the spaces $l_{p_1,q_1,M}$ are perfect sequence spaces.*

Proof. Indeed, in this case $(l_{p_2,q_2,N})^\times = l_{p_1,q_1,M}$ and $l_{p_2,q_2,N} \subseteq (l_{p_1,q_1,M})^\times$. Hence $l_{p_1,q_1,M} \subset (l_{p_1,q_1,M})^{\times\times} \subset (l_{p_2,q_2,N})^\times = l_{p_1,q_1,M}$. □

PROPOSITION 4.3. *Let X be a Banach space, M be an Orlicz function satisfying Δ_2-condition at 0 and p_1, p_2, q_1, q_2 are such that $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$ and $1/p_1 - 1/q_1 > 0$. Then*

$$(l_{p_1,q_1,M}(X))^\times = l_{p_2,q_2,N}(X^*)$$

Proof. For proving $l_{p_2,q_2,N}(X^*) \subseteq (l_{p_1,q_1,M}(X))^\times$, let us consider $\{f_i\} \in l_{p_2,q_2,N}(X^*)$ i.e., $\sum_{i\geq 1} N\Big(\frac{i^{1/p_2-1/q_2}\|f_{\phi(i)}\|}{\rho_0}\Big) < \infty$ for some $\rho_0 > 0$, where $\{\|f_{\phi(i)}\|\}$ is the decreasing rearrangement of $\{\|f_i\|\}$. Let $\{x_i\} \in l_{p_1,q_1,M}(X)$ be an arbitrary member and $\{\|x_{\psi(i)}\|\}$ be the decreasing rearrangement of $\{\|x_i\|\}$. Then

$$\sum_{i\geq 1} |\langle x_i, f_i\rangle| \leq \sum_{i\geq 1} \|x_{\psi(i)}\|\|f_{\phi(i)}\| < \infty.$$

Consequently, $\{f_i\} \in (l_{p_1,q_1,M}(X))^\times$.

For showing $(l_{p_1,q_1,M}(X))^\times \subseteq l_{p_2,q_2,N}(X^*)$, let us consider $\{g_i\} \in (l_{p_1,q_1,M}(X))^\times$. Then $\sum_{i\geq 1} |\langle x_i, g_i\rangle| < \infty$ for all $\{x_i\} \in l_{p_1,q_1,M}(X)$. Let $\{\alpha_i\}$ be any sequence in $l_{p_1,q_1,M}$. Choose $\{x_i\} \subseteq X$ with $\|x_i\| = 1$, for each i such that $\|g_i\| < |\langle x_i, g_i\rangle| + 1/2^i$ and so

$$\sum_{i\geq 1} |\alpha_i|\|g_i\| < \sum_{i\geq 1} |\langle \alpha_i x_i, g_i\rangle| + \sum_{i\geq 1} \frac{|\alpha_i|}{2^i} < \infty.$$

As $\{\alpha_i\} \in l_{p_1,q_1,M}$ is arbitrary, we infer $\{\|g_i\|\} \in l_{p_2,q_2,N}$, i.e., $\{g_i\} \in l_{p_2,q_2,N}(X^*)$ from Theorem 4.1
This completes the proof. □

Theorem 4.4. *For real numbers p_1, q_1, p_2, q_2 with $1 < p_1, q_1, p_2, q_2 < \infty$ and $1/p_1 + 1/p_2 = 1 = 1/q_1 + 1/q_2$, the dual of $l_{p_1,q_1,M}(X)$ is topologically isomorphic to $l_{p_2,q_2,N}(X^*)$ such that a sequence $\{f_i\} \in l_{p_2,q_2,N}(X^*)$ is identified with the linear functional F given by*

$$F(\{x_i\}) = \sum_{i\geq 1} \langle x_i, f_i\rangle \qquad \textit{for each} \quad \{x_i\} \in l_{p_1,q_1,M}(X). \tag{4.1}$$

P r o o f. Corresponding to $\{f_i\} \in l_{p_2,q_2,N}(X^*)$, define a linear functional F on $l_{p_1,q_1,M}(X)$ as in (+) where convergence of the series is being guaranted by Proposition4.3. For $n \in \mathbb{N}$, let

$$F_n(\{x_i\}) = \sum_{i=1}^{n} \langle x_i, f_i\rangle, \qquad \{x_i\} \in l_{p_1,q_1,M}(X).$$

Clearly, $\{F_n\}$ is a sequence of continuous linear functionals on $l_{p_1,q_1,M}(X)$ converging pointwise to F. Hence F is continuous by generalized version of Banach-Steinhaus theorem, cf. [21: p. 45]. Thus $F \in (l_{p_1,q_1,M}(X))^*$.

Further, for $\overline{x} \in l_{p_1,q_1,M}(X)$, we have

$$\begin{aligned} |F(\overline{x})| &\leq \sum_{i\geq 1} |\langle x_i, f_i\rangle| \\ &\leq \sum_{i\geq 1} t_i(\overline{x}) t_i(\overline{f}) \\ &\leq \|\overline{f}\|_{p_2,q_2,N} \sum_{i\geq 1} i^{1/p_1 - 1/q_1} t_i(\overline{x}) \frac{i^{1/p_2-1/q_2} t_i(\overline{f})}{\|\overline{f}\|_{p_2,q_2,N}} \\ &\leq \|\overline{f}\|_{p_2,q_2,N} \ \|\{i^{1/p_1-1/q_1} t_i(\overline{x})\}\|_{(M)} \end{aligned}$$

since $\sum\limits_{i\geq 1} N\Big(\frac{i^{1/p_2-1/q_2} t_i(\overline{f})}{\|\overline{f}\|_{p_2,q_2,N}}\Big) \leq 1$. Hence,

$$|F(\overline{x})| \leq 2\|\overline{f}\|_{p_2,q_2,N} \|\overline{x}\|_{p_1,q_1,M}$$

for any $\overline{x} \in l_{p_1,q_1,M}(X)$. Consequently,

$$\|F\| \leq 2\|\{f_i\}\|_{p_2,q_2,N}. \tag{4.2}$$

Conversely, let $F \in (l_{p_1,q_1,M}(X))^*$. For $i \in \mathbb{N}$, define $f_i \in X^*$, $i \in \mathbb{N}$, as

$$f_i(x) = F(\delta_i^x)$$

Now, for showing $\{f_i\} \in l_{p_2,q_2,N}(X^*)$ we take any $\{\alpha_i\} \in l_{p_1,q_1,M}$. Choose $\{x_i\} \subseteq X$ with $\|x_i\| = 1$ and $\|f_i\| < f_i(x_i) + 1/2^i$, for all $i \in \mathbb{N}$. Let $\{\beta_i\} \subset \mathbb{C}$

be such that $|f_i(\alpha_i x_i)| = f_i(\alpha_i\beta_i x_i)$ for all $i \in \mathbb{N}$. Clearly $|\beta_i| = 1$ for all $i \in \mathbb{N}$, and so $\{\alpha_i\beta_i x_i\} \in l_{p_1,q_1,M}(X)$. Consider

$$\begin{aligned}\sum_{i\geq 1} |\alpha_i|\|f_i\| &< \sum_{i\geq 1} f_i(\alpha_i\beta_i x_i) + \sum_{i\geq 1}\frac{|\alpha_i|}{2^i}\\ &= \sum_{i\geq 1} F(\delta_i^{\alpha_i\beta_i x_i}) + K\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} F(\delta_i^{\alpha_i\beta_i x_i}) + K\\ &= F(\{\alpha_i\beta_i x_i\}) + K\end{aligned}$$

where $K = \sum\limits_{i\geq 1}\frac{|\alpha_i|}{2^i}$. Thus $\sum\limits_{i\geq 1}|\alpha_i|\|f_i\| < \infty$ for all $\{\alpha_i\} \in l_{p_1,q_1,M}$. Hence $\{f_i\} \in l_{p_2,q_2,N}(X^*)$ by Theorem 4.1

In order to show that F has the form as given in (4.1), consider for $\{x_i\} \in l_{p_1,q_1,M}(X)$

$$\begin{aligned}\sum_{i\geq 1}|\langle x_i, f_i\rangle| &= \sum_{i\geq 1}|F(\delta_i^{x_i})|\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} F(\delta_i^{\beta_i x_i}) = F(\{\beta_i x_i\})\end{aligned}$$

where β_i are chosen as above. Hence $\sum\limits_{i\geq 1}|\langle x_i, f_i\rangle| < \infty$. Consequently,

$$\sum_{i\geq 1}\langle x_i, f_i\rangle \ \text{ is unconditionally convergent.} \tag{4.3}$$

Again for $0 < p_1 \leq q_1 \leq \infty$, $l_{p_1,q_1,M}(X)$ is a GAD-space. Now write $t_i(\overline{x}) = \|x_{\phi(i)}\|$ for some $\phi \in \Pi$ and $\overline{u_n} = \sum\limits_{i=1}^{n}\delta_{\phi(i)}^{x_{\phi(i)}}$, for $n \in \mathbb{N}$. Thus $\overline{u_n} \in \phi(X)$ and $\|\overline{x} - \overline{u_n}\|_{p_1,q_1,M} \to 0$ as $n \to \infty$ by Proposition 4.3. Then,

$$F(\{x_i\}) = F\Big(\lim_{n\to\infty}\overline{u_n}\Big) = \sum_{i\geq 1}\langle x_i, f_i\rangle$$

in view of (4.3).

Thus the mapping $R\colon l_{p_2,q_2,N}(X^*) \to (l_{p_1,q_1,M}(X))^*$ defined by $R(\overline{f}) = F$, with $\overline{f} = \{f_i\}$, $f_i(x) = F(\delta_i^x)$, $i \in \mathbb{N}$, is a topological isomorphism from (4.2), (4.1) and the open mapping theorem, cf. [21: p. 47].
This completes the proof. □

Remark. For positive reals p, q with $q < p$ and $M(x) = x^q$, the spaces $l_{p,q,M}$ are particular cases of the classical Lorentz sequence spaces $d(x, q)$ where $x = \{x_i\} = \{i^{q/p-1}\}$, studied in [4], [20], [22]; cf. [9] and the references given there in for details.

5. Interrelationships among the spaces $l_{p,q,M}(X)$, $l_M(X)$ and $l_p(X)$

It is clear from the definition of $l_{p,q,,M}(X)$ that for $p = q$, $l_{p,q,M} = l_M(X)$. However, for $p \neq q$, we have:

Proposition 5.1.

(a) *If $q < p$, $l_M(X)$ is a proper subspace of $l_{p,q,M}(X)$, such that the inclusion map is continuous; and*

(b) *if $q > p$, then $l_{p,q,M}(X)$ is a proper subspace of $l_M(X)$ and again the inclusion map from $l_{p,q,M}(X)$ to $l_M(X)$ is continuous.*

Proof.

(a) If $q < p$, then $\frac{n^{\frac{1}{p}-\frac{1}{q}} t_n(\overline{x})}{\rho} \leq \frac{t_n(\overline{x})}{\rho}$ for each $n \in \mathbb{N}$ and $\rho > 0$. Therefore

$$\|\overline{x}\|_{p,q,M} \leq \|\overline{x}\|_M \qquad \text{for all} \quad \overline{x} \in l_M(X).$$

Thus the inclusion map is continuous. For showing that $l_M(X)$ is strictly contained in $l_{p,q,M}(X)$, consider $X = \mathbb{R}$, $M(x) = x^2$, $p = 2$, $q = 1$. Then the sequence $\left\{\frac{1}{\sqrt{n}}\right\} \notin l_M$; but it belongs to $l_{2,1,M}$.

(b) This part can be disposed off as above, for in this case, we have

$$\|\overline{x}\|_M \leq \|\overline{x}\|_{p,q,M} \qquad \text{for all} \quad \overline{x} \in l_{p,q,M}(X).$$

Again, the containment is proper; for example, consider $X = \mathbb{R}$, $M(x) = x^2$, $p = 1$, $q = 2$, $\left\{\frac{1}{n}\right\} \in l_M$, but $\left\{\frac{1}{n}\right\} \notin l_{1,2,M}$. $\square$

Let us now note in the following example that the spaces $l_{p,q,M}$ for $q = \infty$ are different from the modular sequence spaces l_M^α studied in [7], where for a given strictly increasing sequence $\overline{\alpha} = \{\alpha_n\}$ of positive real numbers, l_M^α is defined as

$$l_M^{\overline{\alpha}} = \left\{\overline{\beta} \in \omega : \ \{\alpha_n \beta_n\} \in l_M\right\}.$$

***Example* 5.2.** Consider $l_{1,\infty,M}$ and $l_M^\alpha$for $\alpha = \{j\}$ and $M(x) = x$. Define a sequence $\beta = \{\beta_j\}$ as follows

$$\beta_j = \begin{cases} 2^{-2^n} & j = 2^{2^n} \\ 0 & \text{otherwise} \end{cases}$$

Then $\beta \in l_{1,\infty,M}$ as $t_n(\beta) = 2^{-2^n}$ and $nt_n(\beta) \leq 1/n^2$ for all $n \in \mathbb{N}$. However $\beta \notin l_M^\alpha$.

We end this section with some examples illustrating the relationship of Orlicz sequences spaces $l_M(X)$ with the vector-valued sequence spaces $l_p(X)$. Let us begin with

***Example* 5.3.** Let us consider the Orlicz function $M(x) = x^\alpha(|\log\ x| + 1)$ for $x > 0$ and $M(0) = 0$, where $\alpha > 1$. Then $M(x)$ satisfies Δ_2-condition at 0 and $l_M(X) = h_M(X)$, for all Banach spaces X. For $\overline{x} = \{x_n\} \in l_M(X)$, we have

$$\sum_{n\geq 1} M(\|x_n\|) = \sum_{n\geq 1} \|x_n\|^\alpha(|\log \|x_n\|| + 1) < \infty$$

and so $\overline{x} = \{x_n\} \in l_\alpha(X) \subset l_q(X)$, for each $q \geq \alpha$. Hence $l_M(X) \subset l_q(X)$, for each $q \geq \alpha$. For the proper containment of $l_M(X)$ in $l_\alpha(X)$, consider the Banach space $X = C[0,1]$ equipped with the sup norm and the sequence $\overline{f} = \{f_n\} \subset C[0,1]$, defined for $1/\alpha < p < 2/\alpha$ as follows

$$f_1(t) = t,$$
$$f_n(t) = \frac{t^n}{n^{1/\alpha}(\log\ n)^p}, \qquad n = 2,3,\ldots$$

Then

$$\sum_{n\geq 1} \|f_n\|^\alpha = 1 + \sum_{n\geq 2} \frac{1}{n(\log\ n)^{p\alpha}} < \infty$$

However,

$$\sum_{n\geq 1} M(\|f_n\|) = 1 + \sum_{n\geq 2} \frac{(1/\alpha)\log n + p\log\log n + 1}{n(\log n)^{p\alpha}}$$
$$> 1/\alpha \sum_{n\geq 2} \frac{1}{n(\log n)^{p\alpha-1}} = \infty.$$

Thus $\overline{f} \notin l_M(C[0,1])$. Hence $l_M(C[0,1])$ is properly contained in $l_q([C[0,1])$ for $q \geq \alpha > 1$.

Again for any Banach space X, it is easy to show that $l_1(X) \subseteq l_M(X)$. For proper containment, we consider $X = C[0,1]$ equipped with the sup norm and the sequence $\overline{f} = \{f_n\} \subset C[0,1]$ defined as follows

$$f_1(t) = t,$$
$$f_n(t) = \frac{t^n}{n}, \qquad \text{for} \quad n = 2,3,4\ldots$$

Then

$$\sum_{n\geq 1} M(\|f_n\|) = \sum_{n\geq 1} \frac{|\log(1/n)| + 1}{n^\alpha} \leq \sum_{n\geq 1} \frac{1}{n^\alpha} < \infty,$$

but

$$\sum_{n\geq 1} \|f_n\| = \sum_{n\geq 1} \frac{1}{n} = \infty.$$

Hence $l_1(X)$ is properly contained in $l_M(X)$.

Next, we consider examples of vector-valued Orlicz sequence spaces defined by Orlicz functions equivalent to the function $M(x) = x^2$, for any Banach space X.

***Example* 5.4.** Let, $M(x) = \frac{x^2}{\log(\mathrm{e}+x)}$, for $x \geq 0$. Then $M(x)$ is an Orlicz function satisfying Δ_2-condition at 0 and so $l_M(X) = h_M(X)$, for all Banach spaces X. If $\overline{x} = \{x_n\} \in l_2(X)$ then, $\sum\limits_{n\geq 1} \frac{\|x_n\|^2}{\log(\mathrm{e}+\|x_n\|)} \leq \sum\limits_{n\geq 1} \|x_n\|^2 < \infty$. Again,

$$\sum_{n\geq 1} M(\|x_n\|) \leq \sum_{n\geq 1} \|x_n\|^2 \leq \Big(\sum_{n\geq 1} \|x_n\|\Big)^2 < \infty$$

and so $l_1(X) \subset l_2(X) \subset l_M(X)$. Also, if $\overline{x} = \{x_n\} \in l_M(X)$, then

$$\lim_{n\to\infty} \frac{\|x_n\|^2}{\frac{\|x_n\|^2}{\log(\mathrm{e}+\|x_n\|)}} = 1$$

$\implies$ $\{x_n\} \in l_2(X)$ and so $l_M(X) \subset l_2(X)$. Thus $l_M(X) = l_2(X)$. Clearly, $l_1(X)$ is properly contained in $l_M(X)$; e.g. consider a sequence $\{x_n\}$ in X with $\|x_n\| = \frac{1}{n}$, $n \in \mathbb{N}$.

***Example* 5.5.** Let $M_1(x) = x(\mathrm{e}^x - 1)$, for $x \geq 0$. Then $M_1(x)$ is an Orlicz function satisfying Δ_2-condition. If $\overline{x} = \{x_n\} \in l_2(X)$, then $\overline{x} = \{x_n\} \in l_p(X)$, for any $p \geq 2$ and so

$$\begin{aligned}\sum_{n=1}^{k} M(\|x_n\|) &= \sum_{n=1}^{k} \|x_n\|\big(\|x_n\| + \|x_n\|^2/2! + \|x_n\|^3/3! + \dots\big)\\ &\leq \|\overline{x}\|_2^2 + \|\overline{x}\|_2^3/2! + \dots\\ &\leq \|\overline{x}\|_2\big(\mathrm{e}^{\|\overline{x}\|_2} - 1\big) < \infty, \qquad \text{true for every} \quad k \in \mathbb{N},\end{aligned}$$

where $\|\cdot\|_2$ denotes the norm of $l_2(X)$. Hence, $\{x_n\} \in l_M(X)$. Therefore, $l_2(X) \subset l_M(X)$.

On the other hand, $|\alpha|^k/(k-1)! \leq M(\alpha)$, for $\alpha \geq 0$, $k \in \mathbb{N}$ and $k > 1$. Thus, if $\{x_n\} \in l_M(X)$, then $\{x_n\} \in l_k(X)$, for $k = 2, 3, 4, \dots$.

Therefore, $l_M(X) = l_2(X)$ for all Banach spaces X.

6. The operator ideals $L^{(s)}_{p,q,M}$, $0 < p, q, \leq \infty$

Let M be an Orlicz function and X, Y be Banach spaces.

Definition 6.1. An operator T in $L(X,Y)$ is said to be of type $l_{p,q,M}$ if $\{s_n(T)\} \in l_{p,q,M}$.

Let us denote the collection of such mappings by $L^{(s)}_{p,q,M}$, i.e.,

$$L^{(s)}_{p,q,M} = \Big\{T \in L: \ \{s_n(T)\} \in l_{p,q,M}\Big\}$$

For $T \in L^{(s)}_{p,q,M}$, define

$$\|T\|_{p,q,M} = \inf \Big\{\rho > 0 : \sum_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T)}{\rho}\Big) \leq 1\Big\}$$

Then we have:

Theorem 6.1. *$L^{(s)}_{p,q,M}$ equipped with $\|\cdot\|_{p,q,M}$ is a quasi-Banach operator ideal if $p < q$ and Banach ideal for $p \geq q$.*

P r o o f. In order to show that $L^{(s)}_{p,q,M}$ is an operator ideal, let us first note that all finite rank operators are contained in $L^{(s)}_{p,q,M}$, since $s_n(T) = 0$ for $n \geq n_0$ if $\operatorname{rank} T < n_0$.

If $T_1, T_2 \in L^{(s)}_{p,q,M}(X,Y)$, then

$$\sum_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1)}{\rho_1}\Big) < \infty \qquad \text{for some} \quad \rho_1 > 0$$

and

$$\sum_{n\geq 1} M\Big(\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_2)}{\rho_2}\Big) < \infty \qquad \text{for some} \quad \rho_2 > 0.$$

Then for $\frac{1}{p} - \frac{1}{q} \geq 0$

$$\begin{aligned} &\sum_{n\geq 1} M\Big[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1+T_2)}{2^{\frac{1}{p}-\frac{1}{q}+1}(\rho_1+\rho_2)}\Big] \\ &\leq \sum_{n\geq 1} \frac{\rho_1}{\rho_1+\rho_2} M\Big[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1)}{\rho_1}\Big] + \sum_{n\geq 1} \frac{\rho_2}{\rho_1+\rho_2} M\Big[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_2)}{\rho_2}\Big] < \infty \end{aligned}$$

and for $\frac{1}{p}-\frac{1}{q}<0$

$$\begin{aligned}&\sum_{n\geq 1} M\left[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1+T_2)}{(\rho_1+\rho_2)}\right]\\ &\leq \sum_{n\geq 1}\frac{\rho_1}{\rho_1+\rho_2} M\left[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1)}{\rho_1}\right]+\sum_{n\geq 1}\frac{\rho_2}{\rho_1+\rho_2} M\left[\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_2)}{\rho_2}\right]<\infty\end{aligned}$$

$\Longrightarrow T_1+T_2\in L^{(s)}_{p,q,M}(X,Y)$.

Next, we show that for $T\in L^{(s)}_{p,q,M}(E,F)$, $R\in L(F,Y)$ and $S\in L(X,E)$, $RTS\in L^{(s)}_{p,q,M}(X,Y)$. Now

$$T\in L^{(s)}_{p,q,M}(E,F) \implies \sum_{n\geq 1} M\left(\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T)}{\rho_0}\right)<\infty,$$

for some $\rho_0>0$ and so

$$\sum_{n\geq 1} M\left(\frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(RTS)}{\|R\|\|S\|\rho_0}\right)<\infty$$

by the property (iii) of s-number function. Thus $RTS\in L^{(s)}_{p,q,M}(X,Y)$. Hence $L^{(s)}_{p,q,M}$ is an operator ideal.

The proof of quasi-norm/norm nature of the function $\|\cdot\|_{p,q,M}$ defined on $L^{(s)}_{p,q,M}$ is analogous to the one defined on $l_{p,q,M}(X)$ and so omitted.

However, for the completeness, consider a Cauchy sequence $\{T_n\}$ in a component $L^{(s)}_{p,q,M}(X,Y)$ of $L^{(s)}_{p,q,M}$. Then for given $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\|T_{n+j}-T_n\|_{p,q,M}<\varepsilon \qquad \text{for all} \quad n\geq n_0 \quad \text{and} \quad j\in\mathbb{N}$$

$\Longrightarrow$ exists $\rho>0$ such that $\rho<\varepsilon$ and

$$\sum_{k\geq 1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} s_k(T_{n+j}-T_n)}{\varepsilon}\right]\leq 1 \qquad \text{for all} \quad n\geq n_0,\ \ j\in\mathbb{N} \tag{6.1}$$

$\Longrightarrow \left\{\frac{k^{1/p-1/q} s_k(T_{n+j}-T_n)}{\varepsilon}, k\geq 1\right\}$ is a bounded sequence for each $n\geq n_0$ and $j\in\mathbb{N}$.

Hence for some constant $K>0$, we get

$$\|T_{n+j}\ -\ T_n\|<\varepsilon K \qquad \text{for all} \quad n\geq n_0,\ \ j\in\mathbb{N}.$$

Thus $\{T_n\}$ is a Cauchy sequence in $L(X,Y)$ which is complete. Hence there exists a $T\in L(X,Y)$ such that $\|T_n-T\|\to 0$ as $n\to\infty$.

As $s_k(T_n - T) \le \|T_n - T\|$, for all $k \ge 1$, we have $s_k(T_n - T) \to 0$ as $n \to \infty$. Also

$$\begin{aligned} &|s_k(T_{n+j} - T_n) - s_k(T - T_n)| \le \|T_{n+j} - T_n\| \\ \implies\ & s_k(T_{n+j} - T_n) \to s_k(T - T_n) \qquad \text{as} \quad j \to \infty. \end{aligned}$$

Hence, from (6.1),

$$\sum_{k\ge1} M\left[\frac{k^{\frac{1}{p}-\frac{1}{q}} s_k(T - T_n)}{\varepsilon}\right] \le 1 \qquad \text{for all} \quad n \ge n_0.$$

$\implies T - T_n \in L^{(s)}_{p,q,M}(X, Y)$ and $\|T - T_n\|_{p,q,M} < \varepsilon$ for all $n \ge n_0$. Consequently, $T \in L^{(s)}_{p,q,M}$ and $T_n \to T$ in $L^{(s)}_{p,q,M}$, implying that $L^{(s)}_{p,q,M}$ is a quasi-Banach operator ideal. $\square$

Finally, we prove a result concerning the composition of two such operator ideals.

Theorem 6.2. *Let M and N be two complementary Orlicz functions and s is a multiplicative s-number function. If $0 < p_1, p_2, p, q_1, q_2, q < \infty$ are such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then*

$$L^{(s)}_{p_1,q_1,M} \circ L^{(s)}_{p_2,q_2,N} \subset L^{(s)}_{p,q,1}$$

where $L^{(s)}_{p,q,1} = \left\{T \in L:\ \{n^{\frac{1}{p}-\frac{1}{q}} s_n(T)\} \in l_1\right\}$

Proof. Let $T \in L^{(s)}_{p_1,q_1,M} \circ L_{p_2,q_2,N}(X, Y)$. Then $T = T_1T_2$ where $T_1 \in L^{(s)}_{p_1,q_1,M}(Z, Y)$ and $T_2 \in L^{(s)}_{p_2,q_2,N}(X, Z)$ where G is a Banach space. So

$$\sum_{n\ge1} M\left[\frac{n^{\frac{1}{p_1}-\frac{1}{q_1}} s_n(T_1)}{\rho_1}\right] < \infty \qquad \text{for some} \quad \rho_1 > 0$$

and

$$\sum_{n\ge1} N\left[\frac{n^{\frac{1}{p_2}-\frac{1}{q_2}} s_n(T_2)}{\rho_2}\right] < \infty \qquad \text{for some} \quad \rho_2 > 0$$

Then for $\frac{1}{p} - \frac{1}{q} \ge 0$

$$\begin{aligned} &\sum_{n\ge1} \frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1T_2)}{2^{\frac{1}{p}-\frac{1}{q}} \rho_1\rho_2} \\ &\le 2\left[\sum_{n\ge1} M\left(\frac{n^{\frac{1}{p_1}-\frac{1}{q_1}} s_n(T_1)}{\rho_1}\right) + \sum_{n\ge1} N\left(\frac{n^{\frac{1}{p_2}-\frac{1}{q_2}} s_n(T_2)}{\rho_2}\right)\right] < \infty \end{aligned}$$

and for $\frac{1}{p} - \frac{1}{q} < 0$

$$\sum_{n\geq 1} \frac{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1T_2)}{\rho_1\rho_2}$$

$$\leq 2\Bigg[\sum_{n\geq 1} M\bigg(\frac{n^{\frac{1}{p_1}-\frac{1}{q_1}} s_n(T_1)}{\rho_1}\bigg) + \sum_{n\geq 1} N\bigg(\frac{n^{\frac{1}{p_2}-\frac{1}{q_2}} s_n(T_2)}{\rho_2}\bigg)\Bigg] < \infty$$

$\implies \{n^{\frac{1}{p}-\frac{1}{q}} s_n(T_1T_2)\} \in l_1$ or $T_1T_2 \in L^{(s)}_{p,q,1}$.
This completes the proof. □

Acknowledgement. The authors are thankful to the referees for their useful suggestions and are grateful to the one who helped us pointing out the valuable reference [3], for showing the norm character of the space $l_{p,q,M}(X)$ when $p \geq q$.

REFERENCES

[1] ACHARYA, L. R.: *Linear Operators and Approximation Quantites.* Dissertation, Indian Institute of Technology, Kanpur, 2008.

[2] CARL, B.—STEPHANI, I.: *Entropy, Compactness and the Approximation of Operators*, Cambridge Univ. Press, Cambridge, 1990.

[3] FORALWESKI, P.—HUDZIK, H.—SZYMASZKIEWICZ, L.: *On some geometric and topological prpperties of generalized Orlicz-Lorentz sequence spaces*, Math. Nachr. **281** (2008), 181–198..

[4] GARLING, D. J. H.: *On symmetric sequence spaces I*, Proc. Lond. Math. Soc. (3) **16** (1966), 85–106.

[5] GREGORY, D. A.: *Vector-valued Sequence Spaces.* Dissertation, Univ. of Michigan, Ann. Arbor, 1967.

[6] GROTHENDIECK, A.: *Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une, classes remarquable d'espaces liees a cette notion*, C. R. Math. Acad. Sci. Paris **233** (1951), 1556–1558.

[7] GUPTA MANJUL—PRADHAN SHESADEV: *On certain type of modular sequence spaces*, Turkish J. Math. **32** (2008), 293–303.

[8] LINDENSTRAUSS, J.—TZAFRIRI, L.: *Classical Banach Spaces I, Sequence Spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[9] KAMTHAN, P. K.—GUPTA, M.: *Sequence Spaces and Series.* Lect. Notes Pure Appl. Math. 65, Marcel Dekker, Inc., New York-Basel, 1981.

[10] KATO, M.: *On Lorentz Spaces $l_{p,q}\{E\}$*, Hiroshima Math. J. **6** (1976), 73–93.

[11] DE GRANDE-KIMPE, M.: *Generalized sequence spaces*, Bull. Soc. Math. Belgique **XXIII** (1971).

[12] KRASNOSELSKII, M. A.—RUTISKY, Y. B.: *Convex Functions and Orlicz Spaces*, Groningen, 1961.

[13] MUSIELAK, J.: *Orlicz Spaces and Modular Spaces.* Lecture Notes in Math. 1034, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[14] PATTERSON, J.: *Generalized Sequence Spaces and Matrix Transformations.* Dissertation, I.I.T. Kanpur, India, 1980.

[15] PIETSCH, A.: *Zur Fredholmchen Theorie in lokalkonvexen Räumen*, Studia Math. **22** (1963), 161–179.

[16] PIETSCH, A.: *Einigeneue Klassen von kompakten linearen Abbildungen*, Rev. Roumaine Math. Pures Appl. **8** (1963), 427–447.

[17] PIETSCH, A.: *s-numbers of operators in Banach spaces*, Studia Math. **51** (1974), 201–223.

[18] PIETSCH, A.: *Operator Ideals*, Math. Monograph. 16, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[19] PIETSCH, A.: *Eigenvalues and s-numbers*, Cambridge Stud. Adv. Math. 13, Cambridge Univ. Press, Cambridge, 1987.

[20] RUCKLE, W.: *The construction of sequence spaces that have schauder bases*, Canad. J. Math. **19** (1967), 828–838.

[21] RUDIN, W.: *Functional Analysis*, McGraw-Hill, Inc., New York, 1976.

[22] SARGENT, W. L. C: *Some sequence spaces related to the l^p spaces*, J. Lond. Math. Soc. (2) **35** (1960), 161–171.

[23] SCHMIDT, E.: *Zur Theorie der linearen nichtlinearnen Integralgleichungen*, Math. Ann. **63; 64** (1907), 433–476; 161–174.

[24] YILMAZ, Y.—KERMAL ÖZDEMIR, M.—SOLAK, I.—CANDAN, M.: *Operator on some vector-valued Orlicz sequence spaces*, Fen Derg. **17** (2005), 59–71.

Received 1. 9. 2011
Accepted 2. 7. 2012

Department of Mathematics
and Statistics
Indian Institute of Technology Kanpur
Kanpur–208016
INDIA
E-mail: manjul@iitk.ac.in
antarab@iitk.ac.in