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UNIFORM SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED PARTIAL Λ-VARIATION

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ABSTRACT. The sufficient and necessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the uniformly convergence of Cesàro means of negative order of cubic partial sums of double Walsh-Fourier series of functions of bounded partial Λ-variation.

1. Classes of functions of bounded generalized variation

In 1881 Jordan [17] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [2, 18, 25, 27]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [16].

Let f be a real function of two variable of period 1 with respect to each variable. Given intervals $\Delta = (a, b)$, $J = (c, d)$ and points x, y from $I := [0, 1)$ we denote

$$f(\Delta, y) := f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(\Delta, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E.

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For the sequence of positive numbers $\Lambda^1 = \{\lambda_n^1\}_{n=1}^\infty$, $\Lambda^2 = \{\lambda_n^2\}_{n=1}^\infty$ and $I^2 := [0,1)^2$ we denote

$$\Lambda^1 V_1(f; I^2) = \sup_y \sup_{E \in \Omega} \sum_i \frac{|f(\Delta_i, y)|}{\lambda_i^1} \qquad (E = \{\Delta_i\}),$$

$$\Lambda^2 V_2(f; I^2) = \sup_x \sup_{F \in \Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j^2} \qquad (F = \{J_j\}),$$

$$\left(\Lambda^1 \Lambda^2\right) V_{1,2}(f; I^2) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(\Delta_i, J_j)|}{\lambda_i^1 \lambda_j^2}.$$

Definition 1. We say that the function f has Bounded $\left(\Lambda^1, \Lambda^2\right)$-variation on I^2 and write $f \in \left(\Lambda^1, \Lambda^2\right) BV\left(I^2\right)$, if

$$\left(\Lambda^1, \Lambda^2\right) V(f; I^2) := \Lambda^1 V_1(f; I^2) + \Lambda^2 V_2(f; I^2) + \left(\Lambda^1 \Lambda^2\right) V_{1,2}(f; I^2) < \infty.$$

We say that the function f has Bounded Partial Λ-variation and write $f \in P\Lambda BV\left(I^2\right)$ if

$$P\Lambda BV(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty.$$

The notion of Λ-variation was introduced by Waterman [25] in one dimensional case, by Sahakian [23] in two dimensional case. The notion of bounded partial variation (PBV) was introduced by Goginava [8] and the notion of bounded partial Λ-variation, by Goginava and Sahakian [13].

2. Walsh function

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. The set of all integers by$\mathbf{Z}$ and the set of dyadic rational numbers in the unit interval $I := [0,1)$ by $\mathbf{Q}$. In particular, each element of $\mathbf{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbf{N}$, $0 \le p \le 2^n$. By a dyadic interval in I we mean one of the form $I_N^l := [l2^{-N}, (l+1)\, 2^{-N})$ for some $l \in \mathbf{N}, 0 \le l < 2^N$. Given $N \in \mathbf{N}$ and $x \in I$, let $I_N(x)$ denote a dyadic interval of length 2^{-N} which contains the point x. Denote $I_N := [0, 2^{-N})$ and $\overline{I}_N := I \setminus I_N$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ -1 & \text{if } x \in [1/2, 1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \qquad n \ge 1.$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_k(x) = r_{n_1}(x) \ldots r_{n_s}(x).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that [15, 24]

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0, 2^{-n}), \\ 0 & \text{if } x \in [2^{-n}, 1), \end{cases} \tag{1}$$

$$D_{2^n+m}(x) = D_{2^n}(x) + w_{2^n}(x)\, D_m(x), \qquad 0 \le m < 2^n, \tag{2}$$

$$D_{2^n-m}(x) = D_{2^n}(x) - w_{2^n-1}(x)\, D_m(x), \qquad 0 \le m < 2^n \tag{3}$$

and

$$D_n(t) = w_n(t) \sum_{j=0}^{\infty} n_j w_{2^j}(t) D_{2^j}(t), \tag{4}$$

where $n = \sum\limits_{j=0}^{\infty} n_j 2^j$. Denote for $n \in \mathbf{P}$, $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

Given $x \in I$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{5}$$

where each $x_k = 0$ or 1, will be called a dyadic expansion of x. If $x \in I \setminus \mathbf{Q}$, then (5) is uniquely determined. For the dyadic expansion $x \in \mathbf{Q}$ we choose the one for which $\lim\limits_{k\to\infty} x_k = 0$.

The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$\rho(x, y) := x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k|\, 2^{-(k+1)}.$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbf{N}\}$ on the unit square I^2,

We say that $f(x, y)$ is continuous at (x, y) if

$$\lim_{h,\delta\to 0+} f(x \dotplus h, y \dotplus \delta) = f(x, y). \tag{6}$$

Denote by $C\left(I^2\right)$ the space of functions $f\colon I^2 \to R$ that are uniformly continuous from the dyadic topology of I^2 to the usual topology of R, 1-periodic with respect to each variable and endowed with the supremum norm (see [24]).

If $f \in L^1\left(I^2\right)$, then

$$\hat{f}\left(n,m\right) = \int\limits_{I^2} f(x,y) w_n(x) w_m(y)\,\mathrm{d}x\,\mathrm{d}y$$

is the (n,m)-th Walsh-Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N} f(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}\left(m,n\right) w_m(x) w_n(y).$$

The Cesàro $(C;\alpha)$-means of cubic partial sums of double Walsh-Fourier series are defined as follows

$$\sigma_n^{\alpha} f(x,y) = \frac{1}{A_{n-1}^{\alpha}} \sum_{i=1}^{n} A_{n-i}^{\alpha-1} S_{i,i} f(x,y),$$

where

$$A_0^{\alpha} = 1, \quad A_n^{\alpha} = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \qquad \alpha \neq -1, -2, \ldots .$$

The Cesàro $(C;\alpha,\beta)$-means of rectangular partial sums of double Walsh-Fourier series are defined by

$$\sigma_{n,m}^{\alpha,\beta} f(x,y) = \frac{1}{A_{n-1}^{\alpha} A_{m-1}^{\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j} f(x,y).$$

It is well-known that [28]

$$A_n^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}, \tag{7}$$

$$A_n^{\alpha} \sim n^{\alpha} \tag{8}$$

and

$$\sigma_n^{\alpha} f(x,y) = \int\limits_{I^2} f\left(s,t\right) K_n^{\alpha}\left(x \dotplus s, y \dotplus t\right)\,\mathrm{d}s\,\mathrm{d}t, \tag{9}$$

where

$$K_n^{\alpha}(x,y) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k(x) D_k\left(y\right). \tag{10}$$

The dyadic partial moduli of continuity of a function $f \in C\left(I^2\right)$ in the C-norm are defined by

$$\omega_1\left(f;\delta\right) := \sup\left\{\left|f\left(x',y\right) - f\left(x'',y\right)\right| : \ x',x'',y \in I,\ \rho\left(x',x''\right) < \delta\right\},$$

$$\omega_2\left(f;\delta\right) := \sup\left\{\left|f\left(x,y'\right) - f\left(x,y''\right)\right| : x,y',y'' \in I,\ \rho\left(y',y''\right) < \delta\right\},$$

while the dyadic mixed modulus of continuity is defined as follows:

$$\omega_{1,2}(f;\delta_1,\delta_2) := \sup\big\{|f(x',y') - f(x'',y') - f(x',y'') + f(x'',y'')| : \\ x',x'',y',y'' \in I,\ \rho(x',x'') < \delta_1,\ \rho(y',y'') < \delta_2\big\}.$$

It is evident that

$$\omega_{1,2}(f;\delta_1,\delta_2) \le \omega_1(f;\delta_1) + \omega_2(f;\delta_2). \tag{11}$$

Given a function $f(x,y)$, periodic in both variables with period 1, for $0 \le j < 2^m$ and $0 \le i < 2^n$ and integers $m,n \ge 0$ we set

$$\begin{aligned}
\Delta_j^m f(x,y)_1 &= f\left(x \dotplus 2j2^{-m-1}, y\right) - f\left(x \dotplus (2j+1)\,2^{-m-1}, y\right), \\
\Delta_i^n f(x,y)_2 &= f\left(x, y \dotplus 2i2^{-n-1}\right) - f\left(x, y \dotplus (2i+1)\,2^{-n-1}\right), \\
\Delta_{ji}^{mn} f(x,y) &= \Delta_i^n\left(\Delta_j^m f(x,y)_1\right)_2 = \Delta_j^m\left(\Delta_i^n f(x,y)_2\right)_1 \\
&= f\left(x \dotplus 2j2^{-m-1}, y \dotplus 2i2^{-n-1}\right) \\
&\quad - f\left(x \dotplus (2j+1)\,2^{-m-1}, y \dotplus 2i2^{-n-1}\right) \\
&\quad - f\left(x \dotplus 2j2^{-m-1}, y \dotplus (2i+1)\,2^{-n-1}\right) \\
&\quad + f\left(x \dotplus (2j+1)\,2^{-m-1}, y \dotplus (2i+1)\,2^{-n-1}\right).
\end{aligned}$$

3. Formulation of problems

The well known Dirichlet-Jordan theorem (see [28]) states that the trigonometric Fourier series of a function $f(x),\ x \in [0,2\pi)$ of bounded variation converges at every point x to the value $[f(x+0) + f(x-0)]/2$.

Hardy [16] generalized the Dirichlet-Jordan theorem to the double trigonometric Fourier series. He proved that if function $f(x,y)$ has bounded variation in the sense of Hardy ($f \in BV$), then double trigonometric Fourier series of the function f converges at any point (x,y) to the value $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$. The author [8] has proved that in Hardy's theorem there is no need to require the boundedness of $V_{1,2}(f)$; moreover, it is proved that if f is continuous function and has bounded partial variation ($f \in PBV$) then its double trigonometric Fourier series converges uniformly on $[0,2\pi)^2$. Goginava and Sahakian [13] generalized this result and proved that the following theorem is true.

Theorem GS. *Let $f \in P\Lambda BV(I^2)$ with*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \qquad \frac{\lambda_j}{j} \downarrow 0.$$

Then double trigonometric Fourier series of the function f converges at any point (x,y) to $\sum f(x \pm 0, y \pm 0)$ in the sense of Pringsheim, where the quadrant limits $\sum f(x \pm 0, y \pm 0)$ exist. The convergence is uniformly on any compact K, where the function f is continuous.

Convergence of rectangular and spherical partial sums of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [23], Dyachenko [3–5], Bakhvalov [1], Sablin [22].

For the two-dimensional Walsh-Fourier series the convergence of partial sums of functions Harmonic bounded fluctuation and other bounded generalized variation were studied by Moricz [19, 20], Onnewer, Waterman [21], Waterman [26], Goginava [9] (see also in [6, 7, 12, 14]).

For the two-dimensional Walsh-Fourier series the summability by Cesàro metod of negative order for functions of partial bounded variation investigated by the author.

Theorem G1 ([10]). *Let $f \in C\left(I^2\right) \cap PBV(I^2)$ and $\alpha + \beta < 1,\ \alpha, \beta > 0$. Then the double Walsh-Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.*

Theorem G2 ([10]). *Let $\alpha + \beta \geq 1,\ \alpha, \beta > 0$. Then there exists a continuous function $f_0 \in PBV(I^2)$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,n}^{-\alpha,-\beta} f_0(0,0)$ of the doubleWalsh-Fourier series of f_0 diverges.*

In this paper we consider the following problem: *Let $\alpha \in (0,1)$.* Under what conditions on the sequence $\Lambda = \{\lambda_n\}$ the double Walsh-Fourier series of the function $f \in P\Lambda BV\left(I^2\right) \bigcap C\left(I^2\right)$ is uniformly $(C; -\alpha)$ summable. The solution is given in Theorem 1 bellow.

4. Main results

The main results of this paper are presented in the following propositions:

Theorem 1. *Let $\Lambda = \{\lambda_n : n \geq 1\},\ \alpha \in (0,1),\ \frac{\lambda_n \log(n+1)}{n^{1-\alpha}} \downarrow 0$ and $f \in P\Lambda BV\left(I^2\right) \bigcap C\left(I^2\right)$.*

a) *If*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{2-\alpha}} < \infty,$$

then cubic partial sums of double Walsh-Fourier series is uniformly $(C; -\alpha)$ converges to f.

b) *If*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{2-\alpha}} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV\left(I^2\right)$ for which $\sigma_{2^n}^{-\alpha} f\,(0,0)$ diverges.

Corollary 1. *Let* $\alpha \in (0,1)$.

a) *If* $f \in P\left\{\frac{n^{1-\alpha}}{\log^{1+\varepsilon}(n+1)}\right\} BV(I^2)$ *for some* $\varepsilon > 0$, *then the double Walsh--Fourier series of the function* f *is uniformly* $(C;-\alpha)$ *summable to* f.

b) *There exists a continuous function* $f \in P\left\{\frac{n^{1-\alpha}}{\log(n+1)}\right\} BV(I^2)$ *such that* $\sigma_{2^n,}^{-\alpha} f(0,0)$ *diverges.*

Corollary 2. *Let* $\alpha \in (0,1)$ *and* $f \in PBV\left(I^2\right) \bigcap C\left(I^2\right)$. *Then the double Walsh-Fourier series of the function* f *is uniformly* $(C;-\alpha)$ *summable to* f.

5. Auxiliary results

Set

$$n^{(k)} := 2^{n_{k+1}} + 2^{n_{k+2}} + \cdots + 2^{n_r},\ n_{k+1} > n_{k+2} > \cdots > n_r \geq 0,\ n^{(0)} := n.$$

Lemma 1. *Let* $\alpha \in (0,1)$. *Then*

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) = \sum_{l=1}^{r-1}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) D_{2^{n_l}}(x) A_{n^{(l-1)}-1}^{-\alpha} - \sum_{l=1}^{r}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x).$$

P r o o f o f L e m m a 1. From (2) and (7) can write

$$\begin{aligned}\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) &= \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) + \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_{j+2^{n_1}}(x) \\ &= \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) + D_{2^{n_1}}(x) A_{n^{(1)}-1}^{-\alpha} \\ &\quad + w_{2^{n_1}}(x) \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_j(x).\end{aligned}$$

Iterating this equality gives

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) = \sum_{l=1}^{r}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) \sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) + \sum_{l=1}^{r-1}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) A_{n^{(l)}-1}^{-\alpha} D_{2^{n_l}}(x). \qquad (12)$$

From (3) we can write

$$\sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) = \sum_{j=1}^{2^{n_l}} A_{n^{(l)}+2^{n_l}-j}^{-\alpha-1} D_j(x) = \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_{2^{n_l}-j}(x)$$

$$= D_{2^{n_l}}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} - w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x)$$

$$l = 1, 2, \ldots, r. \tag{13}$$

Combining (12) and (13) we obtain

$$\begin{aligned}
\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) &= \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) D_{2^{n_l}}(x) A_{n^{(l-1)}-1}^{-\alpha} \\
&\quad - \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x) \\
&\quad - \left(\prod_{k=1}^{r-1} w_{2^{n_k}}(x) \right) D_{2^{n_r}}(x) A_{n^{(r-1)}-1}^{-\alpha} \\
&= \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) D_{2^{n_l}}(x) A_{n^{(l-1)}-1}^{-\alpha} \\
&\quad - \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x).
\end{aligned}$$

Lemma 1 is proved. □

Lemma 2. *Let $\alpha \in (0,1)$. Then*

$$\left| K_n^{-\alpha}(x) \right| \leq \frac{c(\alpha)}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|} 2^{-l\alpha} D_{2^l}(x).$$

Proof of Lemma 2. From Lemma 1 we can write

$$\begin{aligned}
\left| \sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) \right| &\leq \sum_{l=1}^{r} D_{2^{n_l}}(x) A_{n^{(l-1)}}^{-\alpha} + \sum_{k=1}^{r} \sum_{j=1}^{2^{n_k}-1} \left| A_{n^{(k)}+j}^{-\alpha-1} \right| \left| D_j(x) \right| \\
&:= B_1 + B_2.
\end{aligned} \tag{14}$$

From (8) we have

$$B_1 \leq c(\alpha) \sum_{l=0}^{|n|} 2^{-l\alpha} D_{2^l}(x). \tag{15}$$

For B_2 we can write

$$\begin{aligned} B_2 &= \sum_{k=1}^{r}\sum_{m=1}^{n_k}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j(x)\right| \\ &= \sum_{k=1}^{r}\sum_{m=1}^{n_{k+1}}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j(x)\right| \\ &\quad+\sum_{k=1}^{r}\sum_{m=n_{k+1}+1}^{n_k}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j(x)\right|. \end{aligned}$$

From (4) and (8) we have

$$\begin{aligned} B_2 &\leq c(\alpha)\left\{\sum_{k=1}^{r}2^{n_{k+1}(-\alpha-1)}\sum_{m=1}^{n_{k+1}}2^m\sum_{l=0}^{m}D_{2^l}(x)\right. \\ &\quad\left.+\sum_{k=1}^{r}\sum_{m=n_{k+1}+1}^{n_k}2^{m(-\alpha-1)}2^m\sum_{l=0}^{m}D_{2^l}(x)\right\} \\ &\leq c(\alpha)\sum_{k=1}^{n_1}2^{-\alpha k}\sum_{l=0}^{k}D_{2^l}(x) \\ &\leq c(\alpha)\sum_{l=0}^{n_1}2^{-\alpha l}D_{2^l}(x). \end{aligned} \tag{16}$$

Combining (14)–(16) we complete the proof of Lemma 2. □

Corollary 3. *Let $\alpha \in (0,1)$. Then*

$$\left|K_n^{-\alpha}(x)\right| \leq c\min\left\{\frac{1}{A_{n-1}^{-\alpha}}\frac{1}{x^{1-\alpha}},n\right\}.$$

Lemma 3. *Let $(x,y)\in I^2$ and $\alpha\in(0,1)$. Then*

$$\begin{aligned} &\left|\sum_{j=1}^{n}A_{n-j}^{-\alpha-1}D_j(x)D_j(y)\right| \\ \leq\ & c(\alpha)\left\{\sum_{j=0}^{n_1}2^{-j\alpha}D_{2^j}(x)\left(\sum_{s=0}^{n_1}D_{2^s}(y)\right)+\sum_{j=0}^{n_1}2^{-j\alpha}D_{2^j}(y)\left(\sum_{l=0}^{n_1}D_{2^l}(x)\right)\right\}. \end{aligned}$$

P r o o f o f L e m m a 3. From (7) and (2) we have

$$
\begin{aligned}
&\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y) \\
&= \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) D_j(y) + \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_{j+2^{n_1}}(x) D_{j+2^{n_1}}(y) \\
&= \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) D_j(y) + D_{2^{n_1}}(x) D_{2^{n_1}}(y) A_{n^{(1)}-1}^{-\alpha} \\
&\quad + D_{2^{n_1}}(x) w_{2^{n_1}}(y) \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_j(y) + w_{2^{n_1}}(x) D_{2^{n_1}}(y) \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_j(x) \\
&\quad + w_{2^{n_1}}(x) w_{2^{n_1}}(y) \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_j(x) D_j(y).
\end{aligned}
$$

Iterating this equality gives

$$
\begin{aligned}
&\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y) \\
= \; & \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) \sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) D_j(y) \\
& + \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(x) D_{2^{n_l}}(y) A_{n^{(l)}-1}^{-\alpha} \\
& + \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(x) w_{2^{n_l}}(y) \sum_{j=1}^{n^{(l)}} A_{n^{(l)}-j}^{-\alpha-1} D_j(y) \\
& + \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) w_{2^{n_l}}(x) D_{2^{n_l}}(y) \sum_{j=1}^{n^{(l)}} A_{n^{(l)}-j}^{-\alpha-1} D_j(x).
\end{aligned}
$$

Since (see (3))

$$
\sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) D_j(y) = \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_{2^{n_l}-j}(x) D_{2^{n_l}-j}(y)
$$

$$= D_{2^{n_l}}(x) D_{2^{n_l}}(y) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} - D_{2^{n_l}}(x) w_{2^{n_l}-1}(y) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(y)$$
$$- D_{2^{n_l}}(y) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x)$$
$$+ w_{2^{n_l}-1}(x) w_{2^{n_l}-1}(y) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x) D_j(y)$$

From (7) we obtain

$$\begin{aligned}
&\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y) \\
&= \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(x) D_{2^{n_l}}(y) A_{n^{(l-1)}-1}^{-\alpha} \\
&\quad - \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(x) w_{2^{n_l}-1}(y) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(y) \\
&\quad - \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(y) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x) \\
&\quad + \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) w_{2^{n_l}-1}(x) w_{2^{n_l}-1}(y) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x) D_j(y) \\
&\quad + \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) D_{2^{n_l}}(x) w_{2^{n_l}}(y) \sum_{j=1}^{n^{(l)}} A_{n^{(l)}-j}^{-\alpha-1} D_j(y) \\
&\quad + \sum_{l=1}^{r-1} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) w_{2^{n_k}}(y) \right) w_{2^{n_l}}(x) D_{2^{n_l}}(y) \sum_{j=1}^{n^{(l)}} A_{n^{(l)}-j}^{-\alpha-1} D_j(x) = \sum_{i=1}^{6} J_i.
\end{aligned} \tag{17}$$

From (8) and (4) it is easy to show that

$$|J_1| \leq c(\alpha) \sum_{l=0}^{n_1} 2^{-l\alpha} D_{2^l}(x) D_{2^l}(y), \tag{18}$$

$$
\begin{aligned}
|J_4| &\leq \sum_{l=1}^{r}\sum_{j=1}^{2^{n_l}-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(x)\right|\left|D_j(y)\right| \\
&= \sum_{l=1}^{r}\sum_{m=1}^{n_l}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(x)\right|\left|D_j(y)\right| \\
&= \sum_{l=1}^{r}\sum_{m=1}^{n_{l+1}}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(x)\right|\left|D_j(y)\right| \\
&\quad +\sum_{l=1}^{r}\sum_{m=n_{l+1}+1}^{n_l}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(x)\right|\left|D_j(y)\right| \\
&\leq c(\alpha)\sum_{l=1}^{r}2^{n_{l+1}(-\alpha-1)}\sum_{m=1}^{n_{l+1}}2^m\sum_{k=0}^{m}D_{2^k}(x)\sum_{s=0}^{m}D_{2^s}(x) \\
&\quad +c(\alpha)\sum_{l=1}^{r}\sum_{m=n_{l+1}+1}^{n_l}2^{m(-\alpha-1)}2^m\sum_{k=0}^{m}D_{2^k}(x)\sum_{s=0}^{m}D_{2^s}(x) \\
&\leq c(\alpha)\left\{\sum_{k=1}^{n_1}2^{-\alpha k}\sum_{l=0}^{k}D_{2^l}(x)\sum_{s=0}^{k}D_{2^s}(x)\right\},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
|J_2| &\leq \sum_{l=1}^{r}D_{2^{n_l}}(x)\sum_{j=0}^{2^{n_l}-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(y)\right| \\
&\leq \sum_{l=1}^{r}D_{2^{n_l}}(x)\sum_{m=1}^{n_{l+1}}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(y)\right| \\
&\quad +\sum_{l=1}^{r}D_{2^{n_l}}(x)\sum_{m=n_{l+1}+1}^{n_l}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(l)}+j}^{-\alpha-1}\right|\left|D_j(y)\right| \\
&\leq c(\alpha)\sum_{l=1}^{r}D_{2^{n_l}}(x)\,2^{-n_{l+1}\alpha}\sum_{s=0}^{n_{l+1}}D_{2^s}(y) \\
&\quad +c(\alpha)\sum_{l=1}^{r}D_{2^{n_l}}(x)\sum_{m=n_{l+1}+1}^{n_l}2^{-m\alpha}\sum_{s=0}^{m}D_{2^s}(y) \\
&\leq c(\alpha)\left(\sum_{k=0}^{n_1}2^{-\alpha k}\sum_{s=0}^{k}D_{2^s}(y)\right)\sum_{l=0}^{n_1}D_{2^l}(x) \\
&\leq c(\alpha)\left(\sum_{s=0}^{n_1}2^{-\alpha s}D_{2^s}(y)\right)\sum_{l=0}^{n_1}D_{2^l}(x).
\end{aligned}
\tag{20}
$$

Analogously, we can prove that

$$|J_3| \leq c(\alpha)\left(\sum_{l=0}^{n_1} 2^{-\alpha l} D_{2^l}(x)\right)\sum_{s=0}^{n_1} D_{2^s}(y). \tag{21}$$

From Lemma 2 we obtain

$$\begin{aligned} |J_5| &\leq c(\alpha)\sum_{l=1}^{r} D_{2^{n_l}}(x)\sum_{j=0}^{n_l} 2^{-j\alpha} D_{2^j}(y) \\ &\leq c(\alpha)\left(\sum_{j=0}^{n_1} 2^{-j\alpha} D_{2^j}(y)\right)\sum_{l=0}^{n_1} D_{2^l}(x). \end{aligned} \tag{22}$$

Analogously, we can prove that

$$|J_6| \leq c(\alpha)\left(\sum_{j=0}^{n_1} 2^{-j\alpha} D_{2^j}(x)\right)\sum_{l=0}^{n_1} D_{2^l}(y). \tag{23}$$

Combining (17)–(23) we complete the proof of Lemma 3. □

Corollary 4. *Let $\alpha \in (0,1)$. Then*

$$\left|\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y)\right| \leq c(\alpha)\begin{cases} \dfrac{1}{x^{1-\alpha}}\dfrac{1}{y} + \dfrac{1}{y^{1-\alpha}}\dfrac{1}{x}, & (x,y) \in \overline{I}_{n_1} \times \overline{I}_{n_1} \\ \dfrac{2^{n_1}}{x^{1-\alpha}}, & (x,y) \in \overline{I}_{n_1} \times I_{n_1} \\ \dfrac{2^{n_1}}{y^{1-\alpha}}, & (x,y) \in I_{n_1} \times \overline{I}_{n_1} \\ 2^{n_1(2-\alpha)}, & (x,y) \in I_{n_1} \times I_{n_1}. \end{cases}$$

P r o o f o f C o r o l l a r y 4. Let $(x,y) \in I_{n_1} \times I_{n_1}$. Then from Lemma 3 we can write

$$\left|\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y)\right| \leq c(\alpha) 2^{n_1(2-\alpha)}. \tag{24}$$

Let $(x,y) \in I_{n_1} \times (I_b \backslash I_{b+1})$, $b = 0,1,\dots,n_1-1$. Then from (1) Lemma and 3 we have

$$\begin{aligned} \left|\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y)\right| &\leq c(\alpha)\left\{2^{n_1(1-\alpha)}2^b + 2^{b(1-\alpha)}2^{n_1}\right\} \\ &\leq c(\alpha) 2^{b(1-\alpha)}2^{n_1} \leq \frac{c(\alpha)2^{n_1}}{y^{1-\alpha}}. \end{aligned} \tag{25}$$

Analogously, we can prove that

$$\left|\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) D_j(y)\right| \leq \frac{c(\alpha)2^{n_1}}{x^{1-\alpha}}, (x,y) \in \overline{I}_{n_1} \times I_{n_1}. \tag{26}$$

Let $(x,y)\in(I_a\setminus I_{a+1})\times(I_b\setminus I_{b+1})$, $a,b=0,1,\ldots,n_1-1$. Then from (1) and Lemma 3 we obtain

$$\begin{aligned}&\left|\sum_{j=1}^{n}A_{n-j}^{-\alpha-1}D_j(x)D_j(y)\right|\\ \leq\ & c(\alpha)\left\{2^{a(1-\alpha)}2^b+2^{b(1-\alpha)}2^a\right\}\\ \leq\ & c(\alpha)\left\{\frac{1}{x^{1-\alpha}}\frac{1}{y}+\frac{1}{y^{1-\alpha}}\frac{1}{x}\right\}. \qquad (27)\end{aligned}$$

Combining (24)–(27) we complete the proof of Corollary 4. □

6. Proofs of main results

P r o o f o f T h e o r e m 1. Let $n:=2^N+n'$, $0\leq n'<2^N$. Then from (9) and (10) we can write

$$\begin{aligned}&\sigma_n^{-\alpha}f(x,y)-f(x,y)\\ =\ & \frac{1}{A_{n-1}^{-\alpha}}\left(\int\limits_{I_{N-1}\times I_{N-1}}+\int\limits_{\overline{I_{N-1}\times I_{N-1}}}\right)\\ & \left(\sum_{k=1}^{n}A_{n-k}^{-\alpha-1}D_k(u)D_k(v)\Delta f(x,y,u,v)\,\mathrm{d}u\,\mathrm{d}v\right)\\ :=\ & L_1+L_2, \qquad (28)\end{aligned}$$

where

$$\Delta f(x,y,u,v):=f(x\dotplus u,y\dotplus v)-f(x,y).$$

From (1), (11) and Corollary 4 we have

$$|L_1|\leq c(\alpha)\left\{\omega_1\left(f;\frac{1}{2^{N-1}}\right)+\omega_2\left(f;\frac{1}{2^{N-1}}\right)\right\}. \qquad (29)$$

For L_2 we have

$$\begin{aligned}L_2=&\left(\int\limits_{I_{N-1}\times\overline{I}_{N-1}}+\int\limits_{\overline{I}_{N-1}\times I_{N-1}}+\int\limits_{\overline{I}_{N-1}\times\overline{I}_{N-1}}\right)\\ &\left(\frac{1}{A_{n-1}^{-\alpha}}\sum_{k=1}^{n}A_{n-k}^{-\alpha-1}D_k(u)D_k(v)\Delta f(x,y,u,v)\,\mathrm{d}u\,\mathrm{d}v\right):=L_{21}+L_{22}+L_{23}. \qquad (30)\end{aligned}$$

We can write

$$L_{21} = \int\limits_{I_{N-1}\times\overline{I}_{N-1}} \frac{1}{A_{n-1}^{-\alpha}} \sum_{k=1}^{2^{N-1}} A_{n-k}^{-\alpha-1} D_k(u) D_k(v) \Delta f(x,y,u,v)\, \mathrm{d}u\, \mathrm{d}v$$

$$+ \int\limits_{I_{N-1}\times\overline{I}_{N-1}} \frac{1}{A_{n-1}^{-\alpha}} \sum_{k=2^{N-1}+1}^{2^{N}} A_{n-k}^{-\alpha-1} D_k(u) D_k(v) \Delta f(x,y,u,v)\, \mathrm{d}u\, \mathrm{d}v$$

$$+ \int\limits_{I_{N-1}\times\overline{I}_{N-1}} \frac{1}{A_{n-1}^{-\alpha}} \sum_{k=2^{N}+1}^{n} A_{n-k}^{-\alpha-1} D_k(u) D_k(v) \Delta f(x,y,u,v)\, \mathrm{d}u\, \mathrm{d}v$$

$$:= L_{211} + L_{212} + L_{213}. \tag{31}$$

It is proved in [11] that

$$L_{211} = o(1) \tag{32}$$

uniformly with respect to x, y as $n \to \infty$.

From (1) and (2) we get

$$L_{212} = \int\limits_{I_{N-1}\times\overline{I}_{N-1}} \frac{D_{2^{N-1}}(u)\, w_{2^{N-1}}(v)}{A_{n-1}^{-\alpha}} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(v)$$
$$\times \Delta f(x,y,u,v)\, \mathrm{d}u\, \mathrm{d}v$$
$$+ \int\limits_{I_{N-1}\times\overline{I}_{N-1}} \frac{w_{2^{N-1}}(u)\, w_{2^{N-1}}(v)}{A_{n-1}^{-\alpha}} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{n-1}}^{-\alpha-1} D_k(u) D_k(v)$$
$$\times \Delta f(x,y,u,v)\, \mathrm{d}u\, \mathrm{d}v.$$

Since

$$w_{2^{N-1}}(v) = \begin{cases} 1 & \text{if } v \in I_N^{2l} \\ -1 & \text{if } v \in I_N^{2l+1} \end{cases} \tag{33}$$

and $t = v \dot{+} \frac{1}{2^N}$ is a one-to-one mapping of I_N^{2l} into I_N^{2l+1}, we have

$$\int\limits_{I_{N-1}^{l}} \Delta f(x,y,u,v)\, w_{2^{N-1}}(v)\, \mathrm{d}v$$

$$= \int\limits_{I_N^{2l}} \Delta f(x,y,u,v)\, w_{2^{N-1}}(v)\, \mathrm{d}v + \int\limits_{I_N^{2l+1}} \Delta f(x,y,u,v)\, w_{2^{N-1}}(v)\, \mathrm{d}v$$

$$= \int\limits_{I_N^{2l}} \Delta_0^{N-1} f\left(x \dotplus u, y \dotplus v\right)_2 \mathrm{d}v.$$

Consequently, for L_{212} we obtain

$$\begin{aligned}
L_{212} &= \frac{2^{N-1}}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} \left(\int\limits_{\overline{I}_{N-1}} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(v)\, w_{2^{N-1}}(v)\, \Delta f(x,y,u,v)\, \mathrm{d}v \right) \mathrm{d}u \\
&\quad + \frac{1}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} w_{2^{N-1}}(u) \left(\int\limits_{\overline{I}_{N-1}} w_{2^{N-1}}(u) \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} \right. \\
&\qquad \left. \times D_k(u)\, D_k(v)\, \Delta f(x,y,u,v)\, \mathrm{d}v \right) \mathrm{d}u \\
&= \frac{2^{N-1}}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} \sum_{l=1}^{2^{N-1}-1} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k\left(\frac{l}{2^{N-1}}\right) \\
&\qquad \times \left(\int\limits_{I_{N-1}^{l}} w_{2^{N-1}}(v)\, \Delta f(x,y,u,v)\, \mathrm{d}v \right) \mathrm{d}u \\
&\quad + \frac{1}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} w_{2^{N-1}}(u) \sum_{l=1}^{2^{N-1}-1} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(u)\, D_k\left(\frac{l}{2^{N-1}}\right) \\
&\qquad \times \left(\int\limits_{I_{N-1}^{l}} \Delta f(x,y,u,v)\, w_{2^{N-1}}(v)\, \mathrm{d}v \right) \mathrm{d}u \\
&= \frac{2^{N-1}}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} \sum_{l=1}^{2^{N-1}-1} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k\left(\frac{l}{2^{N-1}}\right) \\
&\qquad \times \left(\int\limits_{I_N^{2l}} \Delta_0^{N-1} f\left(x \dotplus u, y \dotplus v\right)_2 \mathrm{d}v \right) \mathrm{d}u \\
&\quad + \frac{1}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}} w_{2^{N-1}}(u) \sum_{l=1}^{2^{N-1}-1} \sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(u)\, D_k\left(\frac{l}{2^{N-1}}\right) \\
&\qquad \times \left(\int\limits_{I_N^{2l}} \Delta_0^{N-1} f\left(x \dotplus u, y \dotplus v\right)_2 \mathrm{d}v \right) \mathrm{d}u.
\end{aligned} \tag{34}$$

Since (see (1))

$$\begin{aligned}
&\sum_{k=1}^{2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(u) D_k\left(\frac{l}{2^{N-1}}\right) \\
=\ & \sum_{k=1}^{n-2^{N-1}} A_{n-k-2^{N-1}}^{-\alpha-1} D_k(u) D_k\left(\frac{l}{2^{N-1}}\right) \\
& -w_{2^{N-1}}\left(\frac{l}{2^{N-1}}\right) D_{2^{N-1}}(u) \sum_{k=1}^{n-2^N} A_{n-k-2^N}^{-\alpha-1} D_k\left(\frac{l}{2^{N-1}}\right) \\
& -w_{2^{N-1}}\left(\frac{l}{2^{N-1}}\right) w_{2^{N-1}}(u) \sum_{k=1}^{n-2^N} A_{n-k-2^N}^{-\alpha-1} D_k(u) D_k\left(\frac{l}{2^{N-1}}\right),
\end{aligned}$$

from Corollary 3 and Corollary 4 we obtain

$$\left|\sum_{k=1}^{n} A_{n-k}^{-\alpha-1} D_k(u) D_k\left(\frac{l}{2^{N-1}}\right)\right| \leq \frac{c(\alpha) 2^{N(2-\alpha)}}{l^{1-\alpha}},$$

$$u \in I_{N-1}, \quad l = 1, 2, \ldots, 2^{N-1}-1, \tag{35}$$

$$\left|\sum_{k=1}^{n} A_{n-k}^{-\alpha-1} D_k\left(\frac{l}{2^{N-1}}\right)\right| \leq \frac{c(\alpha) 2^{N(1-\alpha)}}{l^{1-\alpha}}. \tag{36}$$

Consequently, by (34), (36) and from the conditon of the theorem we have

$$\begin{aligned}
|L_{212}| &\leq \frac{c(\alpha) 2^{N(2-\alpha)}}{A_{n-1}^{-\alpha}} \int\limits_{I_{N-1}\times I_N} \sum_{l=1}^{2^{N-1}-1} \frac{1}{l^{1-\alpha}} \left|\Delta_l^{N-1} f\left(x \dotplus u, y \dotplus v\right)_2\right| du \\
&\leq c(\alpha) \min_{1\leq k<2^{N-1}} \left\{\omega_2\left(f; \frac{1}{2^N}\right) k^\alpha + \frac{\Lambda V_2\left(f; I^2\right) \lambda_k}{k^{1-\alpha}}\right\} \\
&= o(1)
\end{aligned} \tag{37}$$

uniformly with respect to x, y as $n \to \infty$.

Analogously, we can prove that

$$L_{213} = o(1) \tag{38}$$

uniformly with respect to x, y as $n \to \infty$.

Combining (31), (32), (37) and (38) we conclude that

$$L_{21} = o(1) \tag{39}$$

uniformly with respect to x, y as $n \to \infty$.

Analogously, we can prove that

$$L_{22}=o(1) \tag{40}$$

uniformly with respect to x,y as $n\to\infty$.

We write

$$\begin{aligned}
L_{23}=&\int\limits_{\overline{I}_{N-1}\times\overline{I}_{N-1}}\frac{1}{A_{n-1}^{-\alpha}}\sum_{k=1}^{2^{N-1}}A_{n-k}^{-\alpha-1}D_k\left(u\right)D_k\left(v\right)\Delta f\left(x,y,u,v\right)\,\mathrm{d}u\,\mathrm{d}v\\
&+\int\limits_{\overline{I}_{N-1}\times\overline{I}_{N-1}}\frac{1}{A_{n-1}^{-\alpha}}\sum_{k=2^{N-1}+1}^{2^{N}}A_{n-k}^{-\alpha-1}D_k\left(u\right)D_k\left(v\right)\Delta f\left(x,y,u,v\right)\,\mathrm{d}u\,\mathrm{d}v\\
&+\int\limits_{\overline{I}_{N-1}\times\overline{I}_{N-1}}\frac{1}{A_{n-1}^{-\alpha}}\sum_{k=2^{N}+1}^{n}A_{n-k}^{-\alpha-1}D_k\left(u\right)D_k\left(v\right)\Delta f\left(x,y,u,v\right)\,\mathrm{d}u\,\mathrm{d}v\\
&:=L_{231}+L_{232}+L_{233}.
\end{aligned} \tag{41}$$

It is proved in [11] that

$$L_{231}=o(1) \tag{42}$$

uniformly with respect to x,y as $n\to\infty$.

Since

$$\begin{aligned}
&\int\limits_{I_N^l\times I_N^s}\Delta f\left(x,y,u,v\right)w_{2^N}\left(u\right)w_{2^N}\left(v\right)\,\mathrm{d}u\,\mathrm{d}v\\
&=\int\limits_{I_{N+1}^{2l}\times I_{N+1}^{2s}}\Delta_{00}^{N,N}f\left(x\dotplus u,y\dotplus v\right)\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}$$

for L_{233} we can write (see (1) and (2))

$$\begin{aligned}
L_{233}&=\frac{1}{A_{n-1}^{-\alpha}}\int\limits_{\overline{I}_{N-1}\times\overline{I}_{N-1}}w_{2^N}\left(u\right)w_{2^N}\left(v\right)\sum_{k=1}^{n'}A_{n'-k}^{-\alpha-1}D_k\left(u\right)D_k\left(v\right)\\
&\quad\times\Delta f\left(x,y,u,v\right)\,\mathrm{d}u\,\mathrm{d}v\\
&=\frac{1}{A_{n-1}^{-\alpha}}\sum_{l=1}^{2^N-1}\sum_{s=1}^{2^N-1}\sum_{k=1}^{n'}A_{n'-k}^{-\alpha-1}D_k\left(\frac{l}{2^N}\right)D_k\left(\frac{s}{2^N}\right)\\
&\quad\times\int\limits_{I_{N+1}\times I_{N+1}}\Delta_{ls}^{N,N}f\left(x\dotplus u,y\dotplus v\right)\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

Consequently, from Corollary 4 we obtain

$$
\begin{aligned}
|L_{233}| &\le c(\alpha)\, n^{\alpha} \int\limits_{I_{N+1}\times I_{N+1}} \sum_{l=1}^{2^N-1}\sum_{s=1}^{2^N-1} \left\{ \frac{2^{N(2-\alpha)}}{l^{1-\alpha}s} + \frac{2^{N(2-\alpha)}}{s^{1-\alpha}l} \right\} \\
&\quad \times \left| \Delta_{ls}^{N,N} f\left(x \dotplus u, y \dotplus v\right) \right| \mathrm{d}u\,\mathrm{d}v \\
&\le c(\alpha)\, 2^{2N} \int\limits_{I_{N+1}\times I_{N+1}} \sum_{l=1}^{2^N-1}\sum_{s=1}^{l} \frac{1}{l^{1-\alpha}s} \left| \Delta_{ls}^{N,N} f\left(x \dotplus u, y \dotplus v\right) \right| \mathrm{d}u\,\mathrm{d}v \\
&\quad + c(\alpha)\, 2^{2N} \int\limits_{I_{N+1}\times I_{N+1}} \sum_{l=1}^{2^N-1}\sum_{s=l+1}^{2^N-1} \frac{1}{s^{1-\alpha}l} \left| \Delta_{ls}^{N,N} f\left(x \dotplus u, y \dotplus v\right) \right| \mathrm{d}u\,\mathrm{d}v \\
&\quad := R_1 + R_2. \qquad (43)
\end{aligned}
$$

Since

$$
\begin{aligned}
& \sum_{l=1}^{2^N-1}\sum_{s=l+1}^{2^N-1} \frac{1}{s^{1-\alpha}l} \left| \Delta_{ls}^{N,N} f\left(x \dotplus u, y \dotplus v\right) \right| \\
= \quad & \left(\sum_{l=1}^{m}\sum_{s=l+1}^{m} + \sum_{l=1}^{m}\sum_{s=m+1}^{2^N-1} + \sum_{l=m+1}^{2^N-1}\sum_{s=l+1}^{2^N-1} \right) \\
& \left(\frac{1}{s^{1-\alpha}l} \left| \Delta_{ls}^{N,N} f\left(x \dotplus u, y \dotplus v\right) \right| \right) \\
\le \quad & c(\alpha) \left\{ \omega_{1,2}\left(f; \frac{1}{2^N}, \frac{1}{2^N}\right) m^{\alpha} \log m \right. \\
& \qquad + \sum_{l=1}^{m} \frac{1}{l} \sup_x \sum_{s=m+1}^{2^N-1} \frac{\left|\Delta_s^N f\left(x, y \dotplus v\right)_2\right|}{\lambda_s} \frac{\lambda_s}{s^{1-\alpha}} \\
& \qquad \left. + \sum_{l=m+1}^{2^N-1} \frac{1}{l} \sup_x \sum_{s=l+1}^{2^N-1} \frac{\left|\Delta_s^N f\left(x, y \dotplus v\right)_2\right|}{\lambda_s} \frac{\lambda_s}{s^{1-\alpha}} \right\} \\
\le \quad & c(\alpha) \left\{ \omega_{1,2}\left(f; \frac{1}{2^N}, \frac{1}{2^N}\right) m^{\alpha} \log m + \frac{\Lambda V_2(f)\, \lambda_m}{m^{1-\alpha}} \log m \right. \\
& \qquad \left. + \Lambda V_2\left(f; I^2\right) \sum_{l=m+1}^{\infty} \frac{\lambda_l}{l^{2-\alpha}} \right\},
\end{aligned}
$$

From the condition of the Theorem 1 we obtain

$$\begin{aligned} |R_1| &\leq c(\alpha) \min_{1\leq m<2^N} \left\{ \omega_{1,2}\left(f;\frac{1}{2^N},\frac{1}{2^N}\right) m^{\alpha}\log m \right. \\ &\quad \left. + \frac{\Lambda V_2\left(f;I^2\right)\lambda_m}{m^{1-\alpha}}\log m + \Lambda V_2\left(f;I^2\right)\sum_{l=m+1}^{\infty}\frac{\lambda_l}{l^{2-\alpha}} \right\} \\ &= o(1) \end{aligned} \tag{44}$$

uniformly with respect to x, y as $n \to \infty$.

Analogously, we can prove that

$$|R_2| = o(1) \tag{45}$$

uniformly with respect to x, y as $n \to \infty$.

Combining (43)–(45) we have

$$L_{233} = o(1) \tag{46}$$

uniformly with respect to x, y as $n \to \infty$.

Analogously, we can prove that

$$L_{232} = o(1) \tag{47}$$

uniformly with respect to x, y as $n \to \infty$.

From (41), (42), (47) and (46) we obtain

$$L_{23} = o(1) \tag{48}$$

uniformly with respect to x, y as $n \to \infty$.

Combining (28), (29), (30), (39), (40) and (48) we complete the proof of part a) of Theorem 1.

Now, we prove part b) of Theorem 1. By the simple calculation we note that

$$\int_{[2^{-m},2^{-m+1})} D_v = \begin{cases} v2^{-m}, & v < 2^{m-1} \\ \left(2^m - v\right)2^{-m}, & 2^{m-1} \leq v < 2^m \\ 0, & v \geq 2^m. \end{cases} \tag{49}$$

Then from (1), (3) and (49) we can write

$$\begin{aligned} &\int_{[2^{-m},2^{-m+1})^2} \left|K_{2^N}^{-\alpha}(x,y)\right| \mathrm{d}x\,\mathrm{d}y \\ &= \int_{[2^{-m},2^{-m+1})^2} \left|\frac{1}{A_{2^N-1}^{-\alpha}}\sum_{v=0}^{2^N-1} A_v^{-\alpha-1} D_{2^N-v}(x) D_{2^N-v}(y)\right| \mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{[2^{-m},2^{-m+1})^2} \left| \frac{1}{A_{2^N-1}^{-\alpha}} \sum_{v=0}^{2^N-1} A_v^{-\alpha-1} D_v(x) D_v(y) \right| \mathrm{d}x\,\mathrm{d}y \\
&\geq c(\alpha)\, 2^{N\alpha} \sum_{v=1}^{2^N-1} \left|A_v^{-\alpha-1}\right| \int\limits_{[2^{-m},2^{-m+1})^2} D_v(x) D_v(y)\, \mathrm{d}x\,\mathrm{d}y \\
&\geq \frac{c(\alpha)\, 2^{N\alpha}}{2^{2m}} \sum_{v=2^{m-1}}^{2^m-1} \left|A_v^{-\alpha-1}\right| (2^m - v) \\
&\geq \frac{c(\alpha)\, 2^{N\alpha}}{2^{m\alpha}} \qquad (m = 1, 2, \ldots, N). \qquad (50)
\end{aligned}
$$

Consider the function φ_N^m defined by

$$
\varphi_N^m(x) := \begin{cases} 2^{N+1}x - 2j, & x \in \left[2j2^{-N-1}, (2j+1)\,2^{-N-1}\right) \\ -\left(2^{N+1}x - 2j - 2\right), & x \in \left[(2j+1)\,2^{-N-1}, (2j+2)\,2^{-N-1}\right) \\ & j = 2^{m-1}, \ldots, 2^m - 1. \end{cases}
$$

Let

$$
f_N(x,y) := \sum_{m=1}^{N} t_{2^m} \varphi_N^m(x) \varphi_N^m(y) \operatorname{sgn}\left(K_{2^N}^{-\alpha}(x,y)\right),
$$

where

$$
t_n := \left(\sum_{j=1}^{n} \frac{1}{\lambda_j} \right)^{-1}.
$$

It is easy to show that $f_N \in P\Lambda BV\left(I^2\right)$. Indeed, let $y \in \left[2^{m-N-1}, 2^{m-N}\right)$ for some $m = 1, 2, \ldots, N$. Then from the construction of the function f_N we can write

$$
\sum_i \frac{\left|f_N\left(\Delta_i, y\right)\right|}{\lambda_i} \leq c t_{2^m} \sum_{i=1}^{2^m} \frac{1}{\lambda_i} \leq c < \infty.
$$

consequently

$$
\Lambda V_1\left(f_N\right) < \infty. \qquad (51)
$$

Analogously, we can prove that

$$
\Lambda V_2\left(f_N\right) < \infty. \qquad (52)
$$

Combining (51) and (52) we conclude that $f_N \in P\Lambda BV\left(I^2\right)$.

From (50) we can write

$$
\begin{aligned}
\sigma_{2^N}^{-\alpha} f_N(0,0) &= \int_{I^2} f_N(x,y) K_{2^N}^{-\alpha}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{m=1}^{N} t_{2^m} \int_{[2^{m-N-1},2^{m-N})^2} \varphi_N^m(x)\varphi_N^m(y) \left|K_{2^N}^{-\alpha}(x,y)\right| \mathrm{d}x\,\mathrm{d}y \\
&\geq c \sum_{m=1}^{N} t_{2^m} \int_{[2^{m-N-1},2^{m-N})^2} \left|K_{2^N}^{-\alpha}(x,y)\right| \mathrm{d}x\,\mathrm{d}y \\
&\geq c(\alpha) \sum_{m=1}^{N} t_{2^m} 2^{m\alpha}. \qquad (53)
\end{aligned}
$$

Let $\lambda_j := \gamma_j j^{1-\alpha}$. The from the condition of the Theorem 1 we obtain that $\gamma_j \geq \gamma_{j+1}$. Hence, we have

$$
\frac{1}{t_{2^m}} = \sum_{i=1}^{2^m} \frac{1}{\lambda_i} = \sum_{i=1}^{2^m} \frac{1}{i^{1-\alpha}\gamma_i} \leq c(\alpha) \frac{2^{m\alpha}}{\gamma_{2^m}},
$$

$$
t_{2^m} 2^{m\alpha} \geq c(\alpha) \gamma_{2^m}.
$$

Consequently, from (53) we have

$$
\begin{aligned}
\left|\sigma_{2^N}^{-\alpha} f_N(0,0)\right| &\geq c(\alpha) \sum_{m=1}^{N} \gamma_{2^m} \\
&= c(\alpha) \sum_{m=1}^{N} \frac{\lambda_{2^m}}{2^{m(1-\alpha)}} \to \infty \qquad \text{as} \quad N \to \infty.
\end{aligned}
$$

Applying the Banach-Steinhaus Theorem, we obtain that there exists a continuous function $f \in P\Lambda BV\left(I^2\right)$ such that

$$
\sup_n \left|\sigma_{2^n}^{-\alpha} f(0,0)\right| = \infty.
$$

Theorem 1 is proved. □

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