

SOME NEW RESULTS FOR THE MULTIVARIABLE HUMBERT POLYNOMIALS

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ABSTRACT. In this paper, we present some miscellaneous properties of the multivariable Humbert polynomials whose special cases include some well-known multivariable polynomials such as Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials. We give recurrence relations, addition formula and integral representation for them. Then, we obtain some partial differential equations for the products of the multivariable Humbert polynomials and some other multivariable polynomials. Furthermore, some special cases of the results presented in this study are also indicated.

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1. Introduction

The Generalized Humbert polynomials whose special cases include Legendre, Tchebycheff, Gegenbauer, Pincherle, Kinney and Humbert polynomials are defined by

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C) t^n \quad (1.1)$$

where m is a positive integer and other parameters are unrestricted in general [9] (see also [12: p. 77, 86] and [11, 13]).

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Aktaş et al. [1] derived a multivariable extension of the generalized Humbert polynomials, generated by

$$\prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n, \quad (1.2)$$

$$|m_i x_i t - y_i t^{m_i}| < |C_i|, \quad C_i \neq 0, \quad i = 1, 2, \dots, r,$$

where $\mathbf{x} = (x_1, \dots, x_r)$, $\mathbf{y} = (y_1, \dots, y_r)$, $\mathbf{C} = (C_1, \dots, C_r)$, $\mathbf{m} = (m_1, \dots, m_r)$, $m_i = 1, 2, \dots$ ($i = 1, 2, \dots, r$) and other parameters are unrestricted. For the multivariable Humbert polynomials in (1.2), the following explicit representation holds true [1]

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C})$$

$$= \sum_{m_1 k_1 + \dots + m_r k_r + n_1 + \dots + n_r = n} \frac{(\alpha_1)_{n_1 + k_1} \dots (\alpha_r)_{n_r + k_r}}{n_1! \dots n_r! k_1! \dots k_r!} C_1^{-\alpha_1 - n_1 - k_1} \dots C_r^{-\alpha_r - n_r - k_r}$$

$$\times m_1^{n_1} \dots m_r^{n_r} (-1)^{k_1 + \dots + k_r} x_1^{n_1} \dots x_r^{n_r} y_1^{k_1} \dots y_r^{k_r}$$

where

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} \quad (k \in \mathbb{N}_0 := \{0, 1, 2, 3, \dots\})$$

denotes the Pochhammer symbol.

We should remark that some special cases of these polynomials give some well-known multivariable polynomials.

It is clear that the case

$$C_i = 1, \quad m_i = 1, \quad y_i = 0, \quad i = 1, 2, \dots, r,$$

of the polynomials given by (1.2) is reduced to the Chan-Chyan-Srivastava multivariable polynomials which are the multivariable extension of the classical Lagrange polynomials (see [6: p. 267]), generated by [4]

$$\prod_{i=1}^r \left\{ (1 - x_i t)^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n,$$

$$\alpha_i \in \mathbb{C} \quad (i = 1, 2, \dots, r), \quad |t| < \min \left\{ |x_1|^{-1}, \dots, |x_r|^{-1} \right\}.$$

On the other hand, getting $C_i = 1$, $m_i = i$, $x_i = 0$, $y_i = -x_i$, $i = 1, 2, \dots, r$, in (1.2), we have the multivariable Lagrange-Hermite polynomials presented by Altın et al. [2]

$$\prod_{i=1}^r \left\{ (1 - x_i t^i)^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n, \quad (1.3)$$

$$\alpha_i \in \mathbb{C} \quad (i = 1, 2, \dots, r), \quad |t| < \min \left\{ |x_1|^{-1}, |x_2|^{-1/2}, \dots, |x_r|^{-1/r} \right\}.$$

It is clear that the case $r = 2$ of the polynomials defined by (1.3) reduced to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli et al. [5].

Moreover, the special case

$$C_i = 1, \quad x_i = 0, \quad y_i = -x_i, \quad i = 1, 2, \dots, r,$$

gives the Erkus-Srivastava multivariable polynomials generated by [7]

$$\prod_{i=1}^r \left\{ (1 - x_i t^{m_i})^{-\alpha_i} \right\} = \sum_{n=0}^{\infty} u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) t^n,$$

$$\alpha_i \in \mathbb{C} \quad (i = 1, 2, \dots, r), \quad |t| < \min \left\{ |x_1|^{-1/m_1}, |x_2|^{-1/m_2}, \dots, |x_r|^{-1/m_r} \right\}.$$

Getting $C_i = 1$, $\alpha_i = \nu_i$, $m_i = 2$, $y_i = 1$, $i = 1, 2, \dots, r$, in (1.2) gives a multivariable analogue of Gegenbauer polynomials. It is defined by

$$\prod_{i=1}^r \left\{ (1 - 2x_i t + t^2)^{-\nu_i} \right\} = \sum_{n=0}^{\infty} C_n^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) t^n \quad (1.4)$$

where

$$C_n^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) = \sum_{2k_1 + \dots + 2k_r + n_1 + \dots + n_r = n} \frac{(\nu_1)_{n_1+k_1} \dots (\nu_r)_{n_r+k_r}}{n_1! \dots n_r! k_1! \dots k_r!} 2^{n_1 + \dots + n_r} (-1)^{k_1 + \dots + k_r} x_1^{n_1} \dots x_r^{n_r}.$$

In its special cases when $\nu_i = \frac{1}{2}$, $i = 1, 2, \dots, r$ and $\nu_i = 1$, $i = 1, 2, \dots, r$, (1.4) reduces to multivariable analogues of Legendre polynomials and Tchebycheff polynomials, respectively.

The main objective of this paper is to obtain several recurrence relations, addition formula and integral representation for the multivariable Humbert polynomials given by (1.2). We also give various partial differential equations satisfied by products of the multivariable Humbert polynomials and some well-known multivariable polynomials such as Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials. Furthermore, we present some special cases of our results.

Recall that the multivariable Humbert polynomials satisfy the following equation:

$$\sum_{j=1}^r \left(x_j \frac{\partial}{\partial x_j} + m_j y_j \frac{\partial}{\partial y_j} \right) P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = n P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}). \quad (1.5)$$

On the other hand, Chan et al. [4] showed that Chan-Chyan-Srivastava multivariable polynomials verify the partial differential equation

$$\sum_{i=1}^r x_i \frac{\partial}{\partial x_i} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) = n g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r). \quad (1.6)$$

Then Lagrange-Hermite and Erkus-Srivastava multivariable polynomials hold the equations, respectively ([2], [7])

$$\sum_{i=1}^r i x_i \frac{\partial h_n^{(\alpha_1, \dots, \alpha_r)}}{\partial x_i} = n h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \quad (1.7)$$

and

$$\sum_{i=1}^r m_i x_i \frac{\partial u_n^{(\alpha_1, \dots, \alpha_r)}}{\partial x_i} = n u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r). \quad (1.8)$$

2. Recurrence relations and addition formula

In this section, we obtain some recurrence relations and addition formula for the multivariable Humbert polynomials defined by (1.2). We also give some special cases of our results. Now, we begin with the following theorem.

THEOREM 2.1. *The multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C})$ hold that*

$$m_j \frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = - \frac{\partial}{\partial x_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}), \quad (2.1)$$

$$\begin{aligned} m_j \alpha_j P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) &= C_j \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad - m_j x_j \frac{\partial}{\partial x_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad + y_j \frac{\partial}{\partial x_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} -\alpha_j P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) &= C_j \frac{\partial}{\partial y_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad - m_j x_j \frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad + y_j \frac{\partial}{\partial y_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned} \quad (2.3)$$

for $n \geq m_j$, $j = 1, 2, \dots, r$.

Proof. By differentiating each member of the generating function (1.2) with respect to x_j and y_j ($j = 1, 2, \dots, r$), we obtain

$$m_j \sum_{n=1}^{\infty} \frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n = - \sum_{n=0}^{\infty} \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^{n+m_j}. \quad (2.4)$$

If we replace n by $n - m_j$ on the right hand side of (2.4) and compare the coefficients of t^n , then we have

$$m_j \frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = - \frac{\partial}{\partial x_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}), \quad n \geq m_j,$$

where $\frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = 0$ for $n = 1, 2, \dots, m_j$, $j = 1, 2, \dots, r$. It completes the proof of (2.1).

In order to prove (2.2), if we differentiate each member of the generating function (1.2) with respect to x_j , we find

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \\ &= m_j \alpha_j t (C_j - m_j x_j t + y_j t^{m_j})^{-\alpha_j - 1} \prod_{\substack{i=1 \\ i \neq j}}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i} \right\}, \end{aligned}$$

from which, by (1.2), we can write

$$\begin{aligned} & m_j \alpha_j \sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^{n+1} \\ &= (C_j - m_j x_j t + y_j t^{m_j}) \sum_{n=0}^{\infty} \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n. \end{aligned}$$

After making necessary calculations, comparing the coefficients of t^n , we have

$$\begin{aligned} m_j \alpha_j P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) &= C_j \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad - m_j x_j \frac{\partial}{\partial x_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad + y_j \frac{\partial}{\partial x_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned}$$

for $n \geq m_j$, and

$$\begin{aligned} & m_j \alpha_j P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &= C_j \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) - m_j x_j \frac{\partial}{\partial x_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \end{aligned}$$

for $n = 1, 2, \dots, m_j - 1$, $j = 1, 2, \dots, r$. This proves the equality (2.2).

For the proof of (2.3), it is enough to use derivative of (1.2) with respect to y_j similar to proof of (2.2). $\square$

For the purpose of illustration of the usefulness of these results, we choose some special cases here.

For $r = 1$, (2.1), (2.2) and (2.3) reduce to recurrence relations for the Generalized Humbert polynomials $P_n(m, x, y, p, C)$ given by (1.1).

Getting $C_j = 1$, $m_j = 1$, $y_j = 0$ ($i = 1, 2, \dots, r$) in (2.2) gives the recurrence relation for the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$ [8]

$$\begin{aligned} & \alpha_j g_{n-1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \frac{\partial}{\partial x_j} g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) - x_j \frac{\partial}{\partial x_j} g_{n-1}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \end{aligned}$$

which further for $r = 2$ lead to recurrence relation for the Lagrange polynomials.

In (2.3), setting $C_j = 1$, $m_j = j$, $x_j = 0$, $y_j = -x_j$ ($j = 1, 2, \dots, r$), we get the recurrence relation for $h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$

$$\begin{aligned} & \alpha_j h_{n-j}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \frac{\partial}{\partial x_j} h_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) - x_j \frac{\partial}{\partial x_j} h_{n-j}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r). \end{aligned} \quad (2.5)$$

For $r = 2$, (2.5) reduces to recurrence relation for the familiar (two-variable) Lagrange-Hermite polynomials.

For $C_j = 1$, $x_j = 0$, $y_j = -x_j$ ($j = 1, 2, \dots, r$), (2.3) gives the following relation for the Erkus-Srivastava multivariable polynomials $u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$

$$\begin{aligned} & \alpha_j u_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) \\ &= \frac{\partial}{\partial x_j} u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r) - x_j \frac{\partial}{\partial x_j} u_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r). \end{aligned}$$

If we take $C_j = 1$, $m_j = 2$, $y_j = 1$, $\alpha_j = \nu_j$ ($j = 1, 2, \dots, r$) in (2.2), we have the recurrence relation for the multivariable analogue of Gegenbauer polynomials defined by (1.4)

$$\begin{aligned} 2\nu_j C_{n-1}^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) &= \frac{\partial}{\partial x_j} C_n^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) \\ &\quad - 2x_j \frac{\partial}{\partial x_j} C_{n-1}^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) \\ &\quad + \frac{\partial}{\partial x_j} C_{n-2}^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r). \end{aligned}$$

The special cases of $\nu_j = \frac{1}{2}$ and $\nu_j = 1$ ($j = 1, 2, \dots, r$) give a recurrence relation for the multivariable Legendre and multivariable Tchebycheff polynomials, respectively.

COROLLARY 2.1.1. *As a consequence of Theorem 2.1, we can give*

$$\begin{aligned} m_j \alpha_j P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) &= C_j \frac{\partial}{\partial x_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad - m_j x_j \frac{\partial}{\partial x_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad - m_j y_j \frac{\partial}{\partial y_j} P_{n-1}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}), \quad n \geq 1 \end{aligned}$$

and

$$\begin{aligned} -\alpha_j P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) &= C_j \frac{\partial}{\partial y_j} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad + x_j \frac{\partial}{\partial x_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &\quad + y_j \frac{\partial}{\partial y_j} P_{n-m_j}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}), \quad n \geq m_j. \end{aligned}$$

THEOREM 2.2. *For the multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C})$, we have the following equalities*

$$\begin{aligned} \text{(i)} \quad &\sum_{n_1+n_2+\dots+n_p=n} P_{n_1}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \dots P_{n_p}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \\ &= C_j^{-p} \sum_{k=0}^n \sum_{l=0}^{[(n-k)/m_j]} \frac{(p)_{k+l} (m_j x_j)^k (-y_j)^l}{C_j^{k+l} k! l!} P_{n-k-m_j l}^{(p\alpha_1, \dots, p\alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ \text{(ii)} \quad &P_n^{\left(\sum_{i=1}^k \alpha_i^1, \dots, \sum_{i=1}^k \alpha_i^r\right)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \\ &= \sum_{n_1+n_2+\dots+n_k=n} P_{n_1}^{(\alpha_1^1, \dots, \alpha_1^r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) \dots P_{n_k}^{(\alpha_k^1, \dots, \alpha_k^r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}). \end{aligned}$$

Proof. (i) Getting $p\alpha_i$ ($p \in \mathbb{N}$) instead of α_i in generating function (1.2), we have

$$F(\mathbf{x}, \mathbf{y}; t) := \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-p\alpha_i} \right\} = \sum_{n=0}^{\infty} P_n^{(p\alpha_1, \dots, p\alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n.$$

If we differentiate $F(\mathbf{x}, \mathbf{y}; t)$ with respect to x_j ($j = 1, 2, \dots, r$), p times, we obtain

$$\begin{aligned}
 & \frac{\partial^p}{\partial x_j^p} F(\mathbf{x}, \mathbf{y}; t) \\
 &= (p\alpha_j)_p (m_j t)^p (C_j - m_j x_j t + y_j t^{m_j})^{-p(\alpha_j+1)} \prod_{\substack{i=1 \\ i \neq j}}^r (C_i - m_i x_i t + y_i t^{m_i})^{-p\alpha_i} \\
 &= (p\alpha_j)_p (m_j t)^p (C_j - m_j x_j t + y_j t^{m_j})^{-p} \sum_{n=0}^{\infty} P_n^{(p\alpha_1, \dots, p\alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \\
 &= (p\alpha_j)_p (m_j t)^p C_j^{-p} \\
 &\quad \times \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{\lfloor (n-k)/m_j \rfloor} \frac{(p)_{k+l} (m_j x_j)^k (-y_j)^l}{C_j^{k+l} k! l!} P_{n-k-m_j l}^{(p\alpha_1, \dots, p\alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n.
 \end{aligned} \tag{2.6}$$

On the other hand, (2.6) can be rewritten again as follows

$$\begin{aligned}
 & \frac{\partial^p}{\partial x_j^p} F(\mathbf{x}, \mathbf{y}; t) \\
 &= (p\alpha_j)_p (m_j t)^p \left(\sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \right)^p \\
 &= (p\alpha_j)_p (m_j t)^p \sum_{n=0}^{\infty} \left(\sum_{n_1+n_2+\dots+n_p=n} P_{n_1}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \dots \right. \\
 &\quad \left. \dots P_{n_p}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \right) t^n.
 \end{aligned}$$

By comparing the last two equalities, we obtain the desired relation.

(ii) It is enough to use the generating function (1.2) in order to obtain the desired addition formula. $\square$

We can give some special cases of this theorem.

In the case of $r = 1$, these results are satisfied for the Generalized Humbert polynomials $P_n(m, x, y, p, C)$.

Getting $C_j = 1$, $x_j = 0$, $y_j = -x_j$, ($j = 1, 2, \dots, r$) in this theorem gives the following results, which are given in [8], for the Erkus-Srivastava multivariable polynomials $u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$

$$\begin{aligned}
 & u_n^{\left(\sum_{i=1}^k \alpha_i^1, \dots, \sum_{i=1}^k \alpha_i^r \right)}(x_1, \dots, x_r) \\
 &= \sum_{n_1+n_2+\dots+n_k=n} u_{n_1}^{(\alpha_1^1, \dots, \alpha_1^r)}(x_1, \dots, x_r) \dots u_{n_k}^{(\alpha_k^1, \dots, \alpha_k^r)}(x_1, \dots, x_r)
 \end{aligned}$$

and

$$\sum_{n_1+n_2+\dots+n_p=n} u_{n_1}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \dots u_{n_p}^{(\alpha_1, \dots, \alpha_{j-1}, \alpha_j+1, \alpha_{j+1}, \dots, \alpha_r)} \\ = \sum_{l=0}^{[n/m_j]} \frac{(p)_l x_j^l}{l!} u_{n-m_j l}^{(p\alpha_1, \dots, p\alpha_r)}(x_1, \dots, x_r).$$

The special cases of $m_j = j$ and $m_j = 1$ ($j = 1, 2, \dots, r$) give similar equalities for the Lagrange-Hermite and Chan-Chyan-Srivastava multivariable polynomials, respectively. We should remark that the result in (ii) reduces to a known relation in [10] for the Chan-Chyan-Srivastava multivariable polynomials $g_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$.

If we take $C_j = 1$, $m_j = 2$, $y_j = 1$, $\alpha_j = \nu_j$ ($j = 1, 2, \dots, r$) in this theorem, we have

$$\sum_{n_1+n_2+\dots+n_p=n} C_{n_1}^{(\nu_1, \dots, \nu_{j-1}, \nu_j+1, \nu_{j+1}, \dots, \nu_r)} \dots C_{n_p}^{(\nu_1, \dots, \nu_{j-1}, \nu_j+1, \nu_{j+1}, \dots, \nu_r)} \\ = \sum_{k=0}^n \sum_{l=0}^{[(n-k)/2]} \frac{(p)_{k+l} (2x_j)^k (-1)^l}{k!l!} C_{n-k-2l}^{(p\nu_1, \dots, p\nu_r)}(x_1, \dots, x_r)$$

and

$$C_n^{\left(\sum_{i=1}^k \nu_i^1, \dots, \sum_{i=1}^k \nu_i^r\right)}(x_1, \dots, x_r) \\ = \sum_{n_1+n_2+\dots+n_k=n} C_{n_1}^{(\nu_1^1, \dots, \nu_1^r)}(x_1, \dots, x_r) \dots C_{n_k}^{(\nu_k^1, \dots, \nu_k^r)}(x_1, \dots, x_r)$$

for the multivariable Gegenbauer polynomials. For the special cases of $\nu_j = \frac{1}{2}$ and $\nu_j = 1$ ($j = 1, 2, \dots, r$), these results are given for the multivariable Legendre polynomials and multivariable Tchebycheff polynomials, respectively.

3. Integral representations

The use of the identity [3]

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt, \quad \operatorname{Re}(v) > 0 \quad (3.1)$$

allows to obtain integral representation for the multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C})$.

THEOREM 3.1. *The multivariable Humbert polynomials $P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C})$ have the following integral representation:*

$$P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) = \frac{1}{\prod_{j=1}^r \Gamma(\alpha_j)} \int_0^\infty \int_0^\infty \dots \int_0^\infty e^{-(C_1 \xi_1 + \dots + C_r \xi_r)} \prod_{j=1}^r (\xi_j)^{\alpha_j - 1} S_n^{(\xi_1, \dots, \xi_r)}(\mathbf{x}, \mathbf{y}) d\xi \quad (3.2)$$

where

$$S_n^{(\xi_1, \dots, \xi_r)}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^n \sum_{m_1 k_1 + \dots + m_r k_r = k} \frac{(m_1 x_1 \xi_1 + \dots + m_r x_r \xi_r)^{n-k} (-1)^{k_1 + \dots + k_r}}{(n-k)! k_1! \dots k_r!} \times (y_1 \xi_1)^{k_1} \dots (y_r \xi_r)^{k_r},$$

and $\operatorname{Re}(\alpha_j) > 0$ ($j = 1, 2, \dots, r$), $d\xi = d\xi_1 \dots d\xi_r$.

Proof. If we use the identity (3.1) in left hand side of the generating function (1.2), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} P_n^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) t^n \\ &= \prod_{i=1}^r \left\{ (C_i - m_i x_i t + y_i t^{m_i})^{-\alpha_i} \right\} \\ &= \frac{1}{\prod_{j=1}^r \Gamma(\alpha_j)} \int_0^\infty \int_0^\infty \dots \int_0^\infty e^{-(C_1 \xi_1 + \dots + C_r \xi_r)} e^{-(y_1 \xi_1 t^{m_1} + \dots + y_r \xi_r t^{m_r})} \prod_{j=1}^r (\xi_j)^{\alpha_j - 1} \\ & \quad \times e^{(m_1 x_1 \xi_1 + \dots + m_r x_r \xi_r) t} d\xi \\ &= \frac{1}{\prod_{j=1}^r \Gamma(\alpha_j)} \int_0^\infty \int_0^\infty \dots \int_0^\infty \left\{ e^{-(C_1 \xi_1 + \dots + C_r \xi_r)} e^{-(y_1 \xi_1 t^{m_1} + \dots + y_r \xi_r t^{m_r})} \left\{ \prod_{j=1}^r (\xi_j)^{\alpha_j - 1} \right\} \right. \\ & \quad \left. \times \sum_{n=0}^{\infty} \frac{(m_1 x_1 \xi_1 + \dots + m_r x_r \xi_r)^n t^n}{n!} d\xi \right\}. \end{aligned}$$

Simple calculations show that

$$\begin{aligned} e^{-(y_1 \xi_1 t^{m_1} + \dots + y_r \xi_r t^{m_r})} &= \sum_{k=0}^{\infty} \frac{(-1)^k (y_1 \xi_1 t^{m_1} + \dots + y_r \xi_r t^{m_r})^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{m_1 k_1 + \dots + m_r k_r = k} \frac{(-1)^{k_1 + \dots + k_r} (y_1 \xi_1)^{k_1} \dots (y_r \xi_r)^{k_r}}{k_1! \dots k_r!} \right) t^k. \end{aligned}$$

If we use this above and make necessary arrangements, we obtain the desired result. $\square$

Now, we choose some special cases of this theorem.

The case $C_j = 1$, $x_j = 0$, $y_j = -x_j$ ($j = 1, 2, \dots, r$) in this theorem gives the integral representation, which is presented in [8], for the Erkus-Srivastava multivariable polynomials $u_n^{(\alpha_1, \dots, \alpha_r)}(x_1, \dots, x_r)$. For $m_j = j$ and $m_j = 1$, $j = 1, 2, \dots, r$, it reduces to the integral representation for the Lagrange-Hermite and Chan-Chyan-Srivastava multivariable polynomials, respectively.

Setting $C_j = 1$, $m_j = 2$, $y_j = 1$, $\alpha_j = \nu_j$ ($j = 1, 2, \dots, r$) in (3.2), we obtain the integral representation for the multivariable Gegenbauer polynomials

$$C_n^{(\nu_1, \dots, \nu_r)}(x_1, \dots, x_r) = \frac{1}{\prod_{j=1}^r \Gamma(\nu_j)} \int_0^\infty \int_0^\infty \dots \int_0^\infty e^{-(\xi_1 + \dots + \xi_r)} \prod_{j=1}^r (\xi_j)^{\nu_j-1} S_n^{(\xi_1, \dots, \xi_r)}(\mathbf{x}, \mathbf{1}) d\xi$$

where

$$S_n^{(\xi_1, \dots, \xi_r)}(\mathbf{x}, \mathbf{1}) = \sum_{k=0}^n \sum_{2k_1 + \dots + 2k_r = k} \frac{(2x_1\xi_1 + \dots + 2x_r\xi_r)^{n-k} \xi_1^{k_1} \dots \xi_r^{k_r} (-1)^{k_1 + \dots + k_r}}{(n-k)!k_1! \dots k_r!},$$

$\text{Re}(\nu_j) > 0$ ($j = 1, 2, \dots, r$) and $d\xi = d\xi_1 \dots d\xi_r$.

The special cases of $\nu_j = \frac{1}{2}$ and $\nu_j = 1$ ($j = 1, 2, \dots, r$) give integral representations for the multivariable Legendre polynomials and multivariable Tchebycheff polynomials, respectively.

4. Partial differential equations for the products of families of some polynomials

In this section, we find partial differential equations for the product of the multivariable Humbert polynomials and some multivariable polynomials. We gave a method in [8] for finding partial differential equations satisfied by the products of any two of extended Jacobi, Chan-Chyan-Srivastava, Erkus-Srivastava and Lagrange-Hermite multivariable polynomials. Here, we use similar idea for the multivariable Humbert polynomials and multivariable polynomials given above.

Let us define two linear differential operators by

$$L = \sum_{i=1}^r \left(a_i(x_i) \frac{\partial}{\partial x_i} + b_i(y_i) \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad N = \sum_{j=1}^s \left(c_j(u_j) \frac{\partial}{\partial u_j} + d_j(v_j) \frac{\partial}{\partial v_j} \right).$$

THEOREM 4.1. *Let $\{R_n(\mathbf{x}, \mathbf{y})\}_{n=0}^{\infty}$ and $\{Q_n(\mathbf{u}, \mathbf{v})\}_{n=0}^{\infty}$ be polynomials satisfying*

$$L[R_n] = \lambda_n R_n = n R_n$$

and

$$N[Q_n] = \eta_n Q_n = n Q_n$$

where $\mathbf{u} = (u_1, \dots, u_s)$ and $\mathbf{v} = (v_1, \dots, v_s)$. Then the product polynomial $\{\Phi_n\}_{n=0}^{\infty} = \{R_{n-k}(\mathbf{x}, \mathbf{y}) Q_k(\mathbf{u}, \mathbf{v})\}_{k=0, n=0}^{n, \infty}$ holds the following partial differential equation:

$$L[w] + N[w] = nw. \quad (4.1)$$

Proof. Setting $w(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v}) = R_{n-k}(\mathbf{x}, \mathbf{y}) Q_k(\mathbf{u}, \mathbf{v})$, we obtain

$$L[w] = \lambda_{n-k} w = (n-k)w$$

and

$$N[w] = \eta_k w = kw.$$

Thus, we have

$$L[w] + N[w] = nw$$

which completes the proof. $\square$

Now, we apply this theorem for the multivariable Humbert polynomials and some other multivariable polynomials.

Remark 1.

(i) Let $\{R_n(\mathbf{x}, \mathbf{y})\}_{n=0}^{\infty}$ and $\{Q_n(\mathbf{u}, \mathbf{v})\}_{n=0}^{\infty}$ be the multivariable Humbert polynomials so that

$$\{\Phi_n\}_{n=0}^{\infty} = \left\{ P_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) P_k^{(\beta_1, \dots, \beta_s)}(\mathbf{m}, \mathbf{u}, \mathbf{v}, \mathbf{C}) \right\}_{k=0, n=0}^{n, \infty}.$$

Then from (1.5), we get

$$L = \sum_{i=1}^r \left(x_i \frac{\partial}{\partial x_i} + m_i y_i \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad N = \sum_{j=1}^s \left(u_j \frac{\partial}{\partial u_j} + m_j v_j \frac{\partial}{\partial v_j} \right).$$

Hence, the partial differential equation (4.1) becomes

$$\sum_{i=1}^r \left(x_i \frac{\partial \Phi_n}{\partial x_i} + m_i y_i \frac{\partial \Phi_n}{\partial y_i} \right) + \sum_{j=1}^s \left(u_j \frac{\partial \Phi_n}{\partial u_j} + m_j v_j \frac{\partial \Phi_n}{\partial v_j} \right) = n \Phi_n.$$

(ii) Let $\{R_n(\mathbf{x}, \mathbf{y})\}_{n=0}^{\infty}$ be the multivariable Humbert polynomials and $\{Q_n(\mathbf{u}, \mathbf{v})\}_{n=0}^{\infty}$ be the Chan-Chyan-Srivastava multivariable polynomials so that

$$\{\Phi_n\}_{n=0}^{\infty} = \left\{ P_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) g_k^{(\beta_1, \dots, \beta_s)}(u_1, \dots, u_s) \right\}_{k=0, n=0}^{n, \infty}.$$

In this case, from (1.5) and (1.6)

$$L = \sum_{i=1}^r \left(x_i \frac{\partial}{\partial x_i} + m_i y_i \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad N = \sum_{j=1}^s u_j \frac{\partial}{\partial u_j}.$$

Thus, for this product polynomial set, we obtain the equation:

$$\sum_{i=1}^r \left(x_i \frac{\partial \Phi_n}{\partial x_i} + m_i y_i \frac{\partial \Phi_n}{\partial y_i} \right) + \sum_{j=1}^s u_j \frac{\partial \Phi_n}{\partial u_j} = n \Phi_n.$$

(iii) Let $\{R_n(\mathbf{x}, \mathbf{y})\}_{n=0}^{\infty}$ be the multivariable Humbert polynomials and $\{Q_n(\mathbf{u}, \mathbf{v})\}_{n=0}^{\infty}$ be the Lagrange-Hermite multivariable polynomials so that

$$\{\Phi_n\}_{n=0}^{\infty} = \left\{ P_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) h_k^{(\beta_1, \dots, \beta_s)}(u_1, \dots, u_s) \right\}_{k=0, n=0}^{n, \infty}.$$

Then, from (1.5) and (1.7),

$$L = \sum_{i=1}^r \left(x_i \frac{\partial}{\partial x_i} + m_i y_i \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad N = \sum_{j=1}^s j u_j \frac{\partial}{\partial u_j}.$$

Hence, the equation (4.1) becomes

$$\sum_{i=1}^r \left(x_i \frac{\partial \Phi_n}{\partial x_i} + m_i y_i \frac{\partial \Phi_n}{\partial y_i} \right) + \sum_{j=1}^s j u_j \frac{\partial \Phi_n}{\partial u_j} = n \Phi_n.$$

(iv) Let $\{R_n(\mathbf{x}, \mathbf{y})\}_{n=0}^{\infty}$ be the multivariable Humbert polynomials and $\{Q_n(\mathbf{u}, \mathbf{v})\}_{n=0}^{\infty}$ be Erkus-Srivastava multivariable polynomials. Then

$$\{\Phi_n\}_{n=0}^{\infty} = \left\{ P_{n-k}^{(\alpha_1, \dots, \alpha_r)}(\mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{C}) u_k^{(\beta_1, \dots, \beta_s)}(u_1, \dots, u_s) \right\}_{k=0, n=0}^{n, \infty}$$

satisfies

$$\sum_{i=1}^r \left(x_i \frac{\partial \Phi_n}{\partial x_i} + m_i y_i \frac{\partial \Phi_n}{\partial y_i} \right) + \sum_{j=1}^s m_j u_j \frac{\partial \Phi_n}{\partial u_j} = n \Phi_n.$$

Some special cases of this Remark give the partial differential equations, which were presented in [8], satisfied by the products of any two of the Chan-Chyan-Srivastava, Lagrange-Hermite and Erkus-Srivastava multivariable polynomials.

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