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EXPLORING s-e-CONDITION AND APPLICATIONS TO SOME OSTROWSKI TYPE INEQUALITIES VIA HADAMARD FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, a new identity for Hadamard fractional integrals via differentiable mappings is established. A new concept named by s-e-condition is explored to overcome some essence difficulties from the singular kernels in Hadamard fractional integrals. With the help of the new concept s-e-condition and the obtained identity for Hadamard fractional integrals, some new Ostrowski type inequalities for Hadamard fractional integrals are obtained.

1. Introduction

Recently, Set [25] establish a new fractional integral identity via differentiable mappings and Riemann-Liouville fractional integrals.

Lemma 1.1. ([25: Lemma 2]) *Let $f\colon [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*

$$\begin{aligned}&\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{b-a}\left[{}_{RL}J^{\alpha}_{x^-}f(a)+{}_{RL}J^{\alpha}_{x^+}f(b)\right]\\&=\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)a)\,\mathrm{d}t-\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha}f'(tx+(1-t)b)\,\mathrm{d}t\end{aligned} \tag{1}$$

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where the symbol ${}_{RL}J^{\alpha}_{a+}f$ *and* ${}_{RL}J^{\alpha}_{b-}f$ *denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order* $\alpha \in R^+$ *are defined by*

$$({}_{RL}J^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \qquad (0 \le a < x \le b),$$

and

$$({}_{RL}J^{\alpha}_{b-}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,\mathrm{d}t, \qquad (0 \le a \le x < b),$$

respectively. Here $\Gamma(\cdot)$ *is the Gamma function.*

By using the above established integral identity (1) in Lemma 1.1 via s-convex mappings in the second sense, Set [25] establish many new Ostrowski type inequalities for Riemann-Liouville fractional integrals (see [25: Theorems 7–10]), which generalized the classical Ostrowski inequality (see [19]).

Another interesting identity similar to the identity (1) for Riemann-Liouville fractional integrals have also been established by Sarikaya et al. [24] in the following result.

Lemma 1.2. ([24: Lemma 2]) *Let* $f\colon [a,b] \to R$ *be a differentiable mapping on* (a,b) *with* $0 \le a < b$. *If* $f' \in L[a,b]$, *then the following equality for fractional integrals holds*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[{}_{RL}J^{\alpha}_{a+}f(b) + {}_{RL}J^{\alpha}_{b-}f(a)\big] \\ &= \frac{b-a}{2}\int_0^1 \big[(1-t)^{\alpha} - t^{\alpha}\big] f'(ta+(1-t)b)\,\mathrm{d}t. \end{aligned} \tag{2}$$

By using the above established integral identity (2) in Lemma 1.2 via convex mappings, Sarikaya et al. [24] establish many new Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals (see Theorem 3), which generalized the classical Hermite-Hadamard inequality (see Mitrinović and Lacković [15]).

In recent years, Ostrowski type inequalities and Hermite-Hadamard type inequalities were studied extensively by many researchers and numerous generalizations, extensions and variants of them appeared in a number of papers [1,2,4,5,7,8,16–18,23,27,28] and references therein.

It is remarkable that fractional calculus and its widely application have recently been paid more and more attentions. For more recent development on fractional calculus, one can see the monographs of Baleanu et al. [3], Diethelm [6], Kilbas et al. [10], Lakshmikantham et al. [12], Miller and Ross [14], Michalski [13], Podlubny [20] and Tarasov [26]; see also [9,11,22].

However, there are few work on Ostrowski type inequalities for Hadamard fractional integrals, even if Hadamard fractional integrals has been presented many years ago as Riemann-Liouville fractional integrals.

Definition 1.3. The left-sided and right-sided Hadamard fractional integrals of order $\alpha \in R^+$ of function $f(x)$ are defined by

$$({}_H J^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \Big(\ln \frac{x}{t}\Big)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}, \qquad (0 < a < x \leq b),$$

and

$$({}_H J^{\alpha}_{b^-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \Big(\ln \frac{t}{x}\Big)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}, \qquad (0 < a \leq x < b),$$

where $\Gamma(\cdot)$ is the Gamma function.

In this paper, we will establish a new identity for Hadamard fractional integrals, which is similar to the identities (1) and (2) for Riemann-Liouville fractional integrals. Since the form of Hadamard fractional integrals are more complex than Riemann-Liouville fractional integrals, we have to explore a new concept named by "s-e-condition" to overcome some essence difficulties from the special singular kernels $(\ln \frac{x}{t})^{\alpha-1}$and $(\ln \frac{t}{x})^{\alpha-1}$.

Definition 1.4. A function $f\colon I \subset (0,\infty) \to R$ is said to satisfy s-e-condition if

$$f(\mathrm{e}^{\lambda x+(1-\lambda)y}) \leq \lambda^s f(\mathrm{e}^x) + (1-\lambda)^s f(\mathrm{e}^y)$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

Remark 1.5. Let $f\colon I \subset (0,\infty) \to R$ be a nondecreasing and convex function. Then f satisfies the above s-e-condition.

In fact, for all $x, y \in I$, $\lambda \in [0,1]$ and some fixed $s \in (0,1]$, we have

$$\begin{aligned} f(\mathrm{e}^{\lambda x+(1-\lambda)y}) &\leq f(\lambda \mathrm{e}^x + (1-\lambda)\mathrm{e}^y) \\ &\leq \lambda f(\mathrm{e}^x) + (1-\lambda) f(\mathrm{e}^y) \leq \lambda^s f(\mathrm{e}^x) + (1-\lambda)^s f(\mathrm{e}^y). \end{aligned}$$

With the help of the new concept "s-e-condition" and an important identity for Hadamard fractional integrals (see Lemma 2.1), some new Ostrowski type inequalities for Hadamard fractional integrals are established.

2. Identity via Hadamard fractional integrals and differentiable mappings

In order to prove our main results we need the following identity via Hadamard fractional integrals and differentiable mappings.

Lemma 2.1. *Let $f\colon [a,b] \to R$ be a differentiable mapping on (a,b) with $0 < a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds*

$$\begin{aligned}
&\frac{(\ln x-\ln a)^\alpha+(\ln b-\ln x)^\alpha}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x^-}f(a)+{}_HJ^\alpha_{x^+}f(b)]\\
&=\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln a}f'(\mathrm{e}^{t\ln x+(1-t)\ln a})\,\mathrm{d}t\\
&\quad-\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln b}f'(\mathrm{e}^{t\ln x+(1-t)\ln b})\,\mathrm{d}t
\end{aligned}$$

for any $x \in (a,b)$.

P r o o f. Using integration by parts, we can state

$$\begin{aligned}
&\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln a}f'(\mathrm{e}^{t\ln x+(1-t)\ln a})\,\mathrm{d}t\\
&=t^\alpha\frac{f(\mathrm{e}^{t\ln x+(1-t)\ln a})}{\ln x-\ln a}\bigg|_0^1-\frac{\alpha}{\ln x-\ln a}\int_0^1 t^{\alpha-1}f(\mathrm{e}^{t\ln x+(1-t)\ln a})\,\mathrm{d}t\\
&=\frac{f(x)}{\ln x-\ln a}-\frac{\alpha}{(\ln x-\ln a)^{\alpha+1}}\int_a^x(\ln u-\ln a)^{\alpha-1}f(u)\frac{\mathrm{d}u}{u}\\
&=\frac{f(x)}{\ln x-\ln a}-\frac{\Gamma(\alpha+1)}{(\ln x-\ln a)^{\alpha+1}}\,{}_HJ^\alpha_{x^-}f(a),
\end{aligned}\tag{3}$$

and

$$\begin{aligned}
&\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln b}f'(\mathrm{e}^{t\ln x+(1-t)\ln b})\,\mathrm{d}t\\
&=t^\alpha\frac{f(\mathrm{e}^{t\ln x+(1-t)\ln b})}{\ln x-\ln b}\bigg|_0^1-\frac{\alpha}{\ln x-\ln b}\int_0^1 t^{\alpha-1}f(\mathrm{e}^{t\ln x+(1-t)\ln b})\,\mathrm{d}t\\
&=\frac{f(x)}{\ln x-\ln b}-\frac{\alpha}{(\ln b-\ln x)^{\alpha+1}}\int_b^x(\ln b-\ln u)^{\alpha-1}f(u)\frac{\mathrm{d}u}{u}\\
&=\frac{f(x)}{\ln x-\ln b}+\frac{\Gamma(\alpha+1)}{(\ln b-\ln x)^{\alpha+1}}\,{}_HJ^\alpha_{x^+}f(b).
\end{aligned}\tag{4}$$

Multiplying both sides of (3) and (4) by $\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}$ and $\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}$, respectively, we have

$$\begin{aligned}&\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln a}f'(\mathrm{e}^{t\ln x+(1-t)\ln a})\,\mathrm{d}t\\&=\frac{(\ln x-\ln a)^{\alpha}f(x)}{\ln b-\ln a}-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}{}_HJ^{\alpha}_{x^-}f(a),\end{aligned}\tag{5}$$

and

$$\begin{aligned}&\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln b}f'(\mathrm{e}^{t\ln x+(1-t)\ln b})\,\mathrm{d}t\\&=-\frac{(\ln b-\ln x)^{\alpha}f(x)}{\ln b-\ln a}+\frac{\Gamma(\alpha+1)}{\ln b-\ln a}{}_HJ^{\alpha}_{x^+}f(b).\end{aligned}\tag{6}$$

From (5) and (6), we obtain the desired result. □

3. Ostrowski type inequalities via Hadamard fractional integrals and s-e-condition

Using Lemma 2.1, we can obtain the following fractional integral inequalities via Hadamard fractional integrals with the help of s-e-condition.

Theorem 3.1. *Let $f\colon [a,b]\subset(0,\infty)\to R$, be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L[a,b]$. If $|f'|$ satisfies s-e-condition on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\le M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned}&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\&\le\frac{Mb}{\ln b-\ln a}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\alpha+s+1}\end{aligned}$$

for any $x\in(a,b)$.

Proof. From Lemma 2.1 and since $|f'|$ satisfies s-e-condition on $[a,b]$, we have

$$\begin{aligned}&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\&\le\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln a}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln b}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t a^{(1-t)}(t^s|f^{'}(x)|+(1-t)^s|f^{'}(a)|)\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{(1-t)}(t^s|f^{'}(x)|+(1-t)^s|f^{'}(b)|)\,\mathrm{d}t\\
&\leq \frac{(\ln x-\ln a)^{\alpha+1}Mb}{\ln b-\ln a}\int_0^1 (t^{\alpha+s}+t^\alpha(1-t)^s)\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}Mb}{\ln b-\ln a}\int_0^1 (t^{\alpha+s}+t^\alpha(1-t)^s)\,\mathrm{d}t\\
&\leq \frac{Mb}{\ln b-\ln a}\left(\frac{1}{\alpha+s+1}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right)\\
&\qquad\times\left[(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}\right],
\end{aligned}
$$

where we use the fact that $x^t a^{(1-t)} \leq x \leq b$ and $x^t b^{(1-t)} \leq b$ via

$$
\int_0^1 t^{\alpha+s}\,\mathrm{d}t=\frac{1}{\alpha+s+1}\qquad\text{and}\qquad \int_0^1 t^\alpha(1-t)^s\,\mathrm{d}t=\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}.
$$

So using the reduction formula $\Gamma(n+1)=n\Gamma(n)(n>0)$ for Euler gamma function, the proof is completed. □

Theorem 3.2. *Let $f\colon [a,b]\subset(0,\infty)\to R$, be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ satisfies s-e-condition on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^\alpha+(\ln b-\ln x)^\alpha}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x-}f(a)+{}_HJ^\alpha_{x+}f(b)]\right|\\
&\leq \frac{Mb}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}
\end{aligned}
\tag{7}
$$

for any $x\in(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$.

Proof. From Lemma 2.1 and using the well known Höder inequality, we have

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\
&\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln a}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln b}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t.\\
&\leq \frac{(\ln x-\ln a)^{\alpha+1}b}{\ln b-\ln a}\int_0^1 t^{\alpha}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}b}{\ln b-\ln a}\int_0^1 t^{\alpha}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t.\\
&\leq \frac{(\ln x-\ln a)^{\alpha+1}b}{\ln b-\ln a}\left(\int_0^1 t^{p\alpha}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}b}{\ln b-\ln a}\left(\int_0^1 t^{p\alpha}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ satisfies s-e-condition on $[a,b]$ and $|f'(x)|\leq M$, we get

$$\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t\leq\frac{2M^q}{s+1},$$

and

$$\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|^q\,\mathrm{d}t\leq\frac{2M^q}{s+1}.$$

On the other hand,

$$\int_0^1 t^{p\alpha}\,\mathrm{d}t=\frac{1}{p\alpha+1}.$$

Hence, we have

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\
&\leq\frac{Mb}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a},
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let $f\colon [a,b]\subset(0,\infty)\to R$, be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ satisfies s-e-condition on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\le M, x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned}
&\bigg|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\bigg|\\
&\le\ Mb\left(\frac{1}{1+\alpha q+s}\right)^{\frac{1}{q}}\left(1+\frac{\Gamma(\alpha q+1)\Gamma(s+1)}{\Gamma(\alpha q+s+1)}\right)^{\frac{1}{q}}\\
&\qquad\times\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a},
\end{aligned}$$

for any $x\in(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>0$.

P r o o f. From Lemma 2.1 and using the well known power mean inequality, we have

$$\begin{aligned}
&\bigg|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\bigg|\\
&\le\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln a}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln b}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t\\
&\le\frac{(\ln x-\ln a)^{\alpha+1}b}{\ln b-\ln a}\int_0^1 t^{\alpha}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}b}{\ln b-\ln a}\int_0^1 t^{\alpha}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t\\
&\le\frac{(\ln x-\ln a)^{\alpha+1}b}{\ln b-\ln a}\left(\int_0^1 t^{\alpha q}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}b}{\ln b-\ln a}\left(\int_0^1 t^{\alpha q}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ satisfies s-e-condition on $[a,b]$ and $|f'(x)| \le M$, we get

$$\begin{aligned}\int_0^1 t^{\alpha q}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t &\le \int_0^1 t^{\alpha q+s}|f'(x)|^q\,\mathrm{d}t+\int_0^1 t^{\alpha q}(1-t)^s|f'(a)|^q\,\mathrm{d}t\\ &= \frac{|f'(x)|^q}{\alpha q+s+1}+|f'(a)|^q\int_0^1 t^{\alpha q}(1-t)^s\,\mathrm{d}t\\ &= \frac{|f'(x)|^q}{\alpha q+s+1}+|f'(a)|^q\beta(\alpha q+1,s+1)\\ &= \frac{|f'(x)|^q}{\alpha q+s+1}+|f'(a)|^q\frac{\Gamma(\alpha q+1)\Gamma(s+1)}{\Gamma(\alpha q+s+2)}\\ &\le \frac{M^q}{\alpha q+s+1}\left(1+\frac{\Gamma(\alpha q+1)\Gamma(s+1)}{\Gamma(\alpha q+s+1)}\right),\end{aligned}$$

and

$$\begin{aligned}\int_0^1 t^{\alpha q}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t &\le \int_0^1 t^{\alpha q+s}|f'(x)|^q\,\mathrm{d}t+\int_0^1 t^{\alpha q}(1-t)^s|f'(b)|^q\,\mathrm{d}t\\ &\le \frac{M^q}{\alpha q+s+1}\left(1+\frac{\Gamma(\alpha q+1)\Gamma(s+1)}{\Gamma(\alpha q+s+1)}\right)\end{aligned}$$

where β is Euler beta function and we used the fact that

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \text{and} \qquad \Gamma(n+1)=n\Gamma(n) \quad (n>0).$$

Hence, we have

$$\begin{aligned}&\left|\frac{(\ln x-\ln a)^\alpha+(\ln b-\ln x)^\alpha}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x-}f(a)+{}_HJ^\alpha_{x+}f(b)]\right|\\ &\le Mb\left(\frac{1}{1+\alpha q+s}\right)^{\frac1q}\left(1+\frac{\Gamma(\alpha q+1)\Gamma(s+1)}{\Gamma(\alpha q+s+1)}\right)^{\frac1q}\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a},\end{aligned}$$

which completes the proofs. □

Note Remark 1.5, Theorem 3.1 and Theorem 3.2, one can obtain the following results immediately.

Corollary 3.4. *Let $f\colon [a,b]\subset(0,\infty)\to R$, be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L[a,b]$. If $|f'|$ is convex and nondecreasing on $[a,b]$ and $|f'(x)|\le M, x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\
&\leq\frac{Mb}{\ln b-\ln a}\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\alpha+1},
\end{aligned}
$$

for any $x\in(a,b)$.

Corollary 3.5. *Let $f\colon [a,b]\subset(0,\infty)\to R$, be a differentiable mapping on (a,b) with $0<a<b$ such that $f'\in L[a,b]$. If $|f'|^q$ is convex and nondecreasing on $[a,b]$, $p,q>1$ and $|f'(x)|\leq M$, $x\in[a,b]$, then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\
&\leq\frac{Mb}{(1+p\alpha)^{\frac{1}{p}}}\frac{(\ln x-\ln a)^{\alpha+1}+(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a},
\end{aligned}
$$

for any $x\in(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$.

4. Further results

Lemma 4.1. *For $\alpha>0$ and $k>0$, we have*

$$
I(\alpha):=\int_0^1 t^{\alpha-1}k^t\,\mathrm{d}t=k\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(\ln k)^{i-1}}{(\alpha)_i}<+\infty,
$$

where

$$
(\alpha)_i=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+i-1).
$$

Moreover, it holds

$$
\left|I(\alpha)-k\sum_{i=1}^{m}\frac{(-\ln k)^{i-1}}{(\alpha)_i}\right|\leq\frac{|\ln k|}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln k|\mathrm{e}}{m-1}\right)^{m-1}.
$$

P r o o f. Using integration by parts, one can obtain

$$
I(\alpha)=\frac{k}{\alpha}-\frac{\ln k}{\alpha}I(\alpha+1).
$$

Further, we find

$$
I(\alpha+1)=\frac{k}{\alpha+1}-\frac{\ln k}{\alpha+1}I(\alpha+2).
$$

As a result,

$$
I(\alpha)=\frac{k}{\alpha}-\frac{k(\ln k)}{\alpha(\alpha+1)}+\frac{\ln^2 k}{\alpha(\alpha+1)}I(\alpha+2).
$$

Repeating the same steps as above, we can get

$$I(\alpha) = k\sum_{i=1}^{m}\frac{(-\ln k)^{i-1}}{(\alpha)_i} + \frac{(-\ln k)^m}{(\alpha)_m}I(\alpha+m).$$

Let $m \to \infty$, then the proof is completed due to

$$\sum_{i=1}^{m}\frac{(-\ln k)^{i-1}}{(\alpha)_i}$$

is the standard Leibnitz series, which is convergence [21]. As a matter of fact, this series absolutely converges.

Moreover, since $I(\alpha+m) \le \max\{1,k\}$ and by Stirling's formula

$$\left|\frac{(-\ln k)^m}{(\alpha)_m}\right| \le \frac{|\ln k|^m}{\alpha(m-1)!} = \frac{|\ln k|}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln k|\mathrm{e}}{m-1}\right)^{m-1},$$

we get

$$\left|I(\alpha) - k\sum_{i=1}^{m}\frac{(-\ln k)^{i-1}}{(\alpha)_i}\right| \le \frac{|\ln k|^m}{(\alpha)_m}I(\alpha+m) \le \frac{|\ln k|\max\{1,k\}}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln k|\mathrm{e}}{m-1}\right)^{m-1}.$$

Clearly, the convergence is rapid as $m \to \infty$. □

Theorem 4.2. *Under the assumptions of* Theorem 3.1*, the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a}f(x) - \frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x-}f(a) + {}_HJ^\alpha_{x+}f(b)]\right| \\
&\le \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}xM\sum_{i=1}^{\infty}\Big(-\ln\frac{x}{a}\Big)^{i-1}\left[\frac{1}{(\alpha+s+1)_i} + \frac{1}{(\alpha+1)_i}\right] \\
&\quad + \frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}xM\sum_{i=1}^{\infty}\Big(-\ln\frac{x}{b}\Big)^{i-1}\left[\frac{1}{(\alpha+s+1)_i} + \frac{1}{(\alpha+1)_i}\right],
\end{aligned}$$

for any $x \in (a,b)$.

Proof. Note Lemma 4.1 and using s-e-conditions, we have

$$\begin{aligned}
&\left|\frac{(\ln x-\ln a)^\alpha + (\ln b - \ln x)^\alpha}{\ln b - \ln a}f(x) - \frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x-}f(a) + {}_HJ^\alpha_{x+}f(b)]\right| \\
&\le \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln a}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t \\
&\quad + \frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha \mathrm{e}^{t\ln x+(1-t)\ln b}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t a^{(1-t)}(t^s|f'(x)|+(1-t)^s|f'(a)|)\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha x^t b^{(1-t)}(t^s|f'(x)|+(1-t)^s|f'(b)|)\,\mathrm{d}t\\
&\leq \frac{aM(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left[\frac{x}{a}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}}{(\alpha+s+1)_i}+\frac{x}{a}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}}{(\alpha+1)_i}\right]\\
&\quad+\frac{bM(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left[\frac{x}{b}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}}{(\alpha+s+1)_i}+\frac{x}{b}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}}{(\alpha+1)_i}\right]\\
&\leq \frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}xM\sum_{i=1}^{\infty}\Big(-\ln\frac{x}{a}\Big)^{i-1}\left[\frac{1}{(\alpha+s+1)_i}+\frac{1}{(\alpha+1)_i}\right]\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}xM\sum_{i=1}^{\infty}\Big(-\ln\frac{x}{b}\Big)^{i-1}\left[\frac{1}{(\alpha+s+1)_i}+\frac{1}{(\alpha+1)_i}\right],
\end{aligned}
$$

which completes the proof. □

Theorem 4.3. *Under the assumptions of* Theorem 3.2*, the following inequality for fractional integrals with $\alpha>0$ holds:*

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^\alpha+(\ln b-\ln x)^\alpha}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^\alpha_{x^-}f(a)+{}_HJ^\alpha_{x^+}f(b)]\right|\\
&\leq \frac{xM(\ln x-\ln a)^{\alpha+1}}{(\ln b-\ln a)p^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}p^i}{(\alpha p+1)_i}\right]^{\frac{1}{p}}\\
&\quad+\frac{xM(\ln b-\ln x)^{\alpha+1}}{(\ln b-\ln a)p^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}p^i}{(\alpha p+1)_i}\right]^{\frac{1}{p}},
\end{aligned}
$$

for any $x\in(a,b)$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Note Lemma 4.1 and using Hölder inequality and s-e-conditions, we have

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^\alpha+(\ln b-\ln x)^\alpha}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^\alpha_{x^-}f(a)+{}_HJ^\alpha_{x^+}f(b)\right]\right|\\
&\leq \frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^\alpha\Big(\frac{x}{a}\Big)^t|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\left(\frac{x}{b}\right)^{t}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t\\
&\le \frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}\left(\frac{x}{a}\right)^{pt}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}\left(\frac{x}{b}\right)^{pt}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 |f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le \frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}\left(\frac{x}{a}\right)^{pt}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 t^s|f'(x)|^q+(1-t)^s|f'(a)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left(\int_0^1 t^{\alpha p}\left(\frac{x}{b}\right)^{pt}\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1 t^s|f'(x)|^q+(1-t)^s|f'(b)|^q\,\mathrm{d}t\right)^{\frac{1}{q}}\\
&\le \frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\frac{1}{p^{\alpha+\frac{1}{p}}}\left[\left(\frac{x}{a}\right)^{p}\sum_{i=1}^{\infty}\frac{\left(-\ln\frac{x}{a}\right)^{i-1}p^{\alpha p+i}}{(\alpha p+1)_i}\right]^{\frac{1}{p}}\left(\frac{2M^q}{s+1}\right)^{\frac{1}{q}}\\
&\quad+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\frac{1}{p^{\alpha+\frac{1}{p}}}\left[\left(\frac{x}{b}\right)^{p}\sum_{i=1}^{\infty}\frac{\left(-\ln\frac{x}{b}\right)^{i-1}p^{\alpha p+i}}{(\alpha p+1)_i}\right]^{\frac{1}{p}}\left(\frac{2M^q}{s+1}\right)^{\frac{1}{q}}\\
&\le \frac{xM(\ln x-\ln a)^{\alpha+1}}{(\ln b-\ln a)p^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}p^i}{(\alpha p+1)_i}\right]^{\frac{1}{p}}\\
&\quad+\frac{xM(\ln b-\ln x)^{\alpha+1}}{(\ln b-\ln a)p^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}p^i}{(\alpha p+1)_i}\right]^{\frac{1}{p}},
\end{aligned}
$$

which completes the proof. □

Theorem 4.4. *Under the assumptions of* Theorem 3.3, *the following inequality for fractional integrals with $\alpha>0$ holds for any $x\in(a,b)$ and some $q>0$:*

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)]\right|\\
&\le \frac{xM(\ln x-\ln a)^{\alpha+1}}{(\ln b-\ln a)q^{\frac{1}{q}}}\sum_{i=1}^{\infty}\left(-\ln\frac{x}{a}\right)^{i-1}q^i\left[\frac{1}{(\alpha q+s+1)_i}+\frac{1}{(\alpha q+1)_i}\right]\\
&\quad+\frac{xM(\ln b-\ln x)^{\alpha+1}}{(\ln b-\ln a)q^{\frac{1}{q}}}\sum_{i=1}^{\infty}\left(-\ln\frac{x}{b}\right)^{i-1}q^i\left[\frac{1}{(\alpha q+s+1)_i}+\frac{1}{(\alpha q+1)_i}\right].
\end{aligned}
$$

Proof. Note Lemma 4.1 and using Hölder inequality and s-e-conditions, we have

$$
\begin{aligned}
&\left|\frac{(\ln x-\ln a)^{\alpha}+(\ln b-\ln x)^{\alpha}}{\ln b-\ln a}f(x)-\frac{\Gamma(\alpha+1)}{\ln b-\ln a}\left[{}_HJ^{\alpha}_{x^-}f(a)+{}_HJ^{\alpha}_{x^+}f(b)\right]\right|\\
&\leq\frac{(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln a}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|\,\mathrm{d}t\\
&\quad+\frac{(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\int_0^1 t^{\alpha}\mathrm{e}^{t\ln x+(1-t)\ln b}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|\,\mathrm{d}t\\
&\leq\frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left[\int_0^1 t^{\alpha q}\left(\frac{x}{a}\right)^{tq}|f'(\mathrm{e}^{t\ln x+(1-t)\ln a})|^q\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&\quad+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left[\int_0^1 t^{\alpha q}\left(\frac{x}{b}\right)^{tq}|f'(\mathrm{e}^{t\ln x+(1-t)\ln b})|^q\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&\leq\frac{a(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\left[\int_0^1 t^{\alpha q}\left(\frac{x}{a}\right)^{tq}(t^s|f'(x)|^q+(1-t)^s|f'(a)|^q)\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&\quad+\frac{b(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\left[\int_0^1 t^{\alpha q}\left(\frac{x}{b}\right)^{tq}(t^s|f'(x)|^q+(1-t)^s|f'(b)|^q)\,\mathrm{d}t\right]^{\frac{1}{q}}\\
&\leq\frac{aM(\ln x-\ln a)^{\alpha+1}}{\ln b-\ln a}\\
&\quad\times\left[\frac{(\frac{x}{a})^q}{q^{\alpha q+s+1}}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}}{(\alpha q+s+1)_i}q^{\alpha q+s+i}+\frac{(\frac{x}{a})^q}{q^{\alpha q+1}}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{a})^{i-1}}{(\alpha q+1)_i}q^{\alpha q+i}\right]^{\frac{1}{q}}\\
&\quad+\frac{bM(\ln b-\ln x)^{\alpha+1}}{\ln b-\ln a}\\
&\quad\times\left[\frac{(\frac{x}{b})^q}{q^{\alpha q+s+1}}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}}{(\alpha q+s+1)_i}q^{\alpha q+s+i}+\frac{(\frac{x}{b})^q}{q^{\alpha q+1}}\sum_{i=1}^{\infty}\frac{(-\ln\frac{x}{b})^{i-1}}{(\alpha q+1)_i}q^{\alpha q+i}\right]^{\frac{1}{q}}\\
&\leq\frac{xM(\ln x-\ln a)^{\alpha+1}}{(\ln b-\ln a)q^{\frac{1}{q}}}\sum_{i=1}^{\infty}\left(-\ln\frac{x}{a}\right)^{i-1}q^i\left[\frac{1}{(\alpha q+s+1)_i}+\frac{1}{(\alpha q+1)_i}\right]\\
&\quad+\frac{xM(\ln b-\ln x)^{\alpha+1}}{(\ln b-\ln a)q^{\frac{1}{q}}}\sum_{i=1}^{\infty}\left(-\ln\frac{x}{b}\right)^{i-1}q^i\left[\frac{1}{(\alpha q+s+1)_i}+\frac{1}{(\alpha q+1)_i}\right],
\end{aligned}
$$

which completes the proof. □

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