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MAXIMAL SUBSEMIGROUPS CONTAINING A PARTICULAR SEMIGROUP

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ABSTRACT. We characterize maximal subsemigroups of the monoid $T(X)$ of all transformations on the set $X = \mathbb{N}$ of natural numbers containing a given subsemigroup W of $T(X)$ such that $T(X)$ is finitely generated over W. This paper gives a contribution to the characterization of maximal subsemigroups on the monoid of all transformations on an infinite set.

1. Introduction

In this paper, we want to continue the study of maximal subsemigroups of the semigroup $T(X)$ of all transformations on an infinite set, in particular, for the case X is countable. The maximal subsemigroups of $T(X)$ containing the symmetric group $\mathrm{Sym}(X)$ of all bijective mappings on an infinite set X are already known. They were determined by G. P. Gavrilov (X is countable) and by M. Pinsker (any infinite set X) characterizing maximal clones ([3],[6],[9]).

The setwise stabilizer of any finite set $Y \subseteq X$ under $\mathrm{Sym}(X)$ is a subgroup of $\mathrm{Sym}(X)$. In [2], the authors determine the maximal subsemigroups of $T(X)$ containing the setwise stabilizer of any finite set $Y \subseteq X$ under $\mathrm{Sym}(X)$. For a finite partition of X, one can also consider the (almost) stabilizer. They form subsemigroups of $\mathrm{Sym}(X)$ and in [2], the maximal subsemigroups of $T(X)$ containing such a subgroup are determined. Also in [2], the maximal subsemigroups containing the stabilizer of any uniform ultrafilter on X, which forms also a group, are determined.

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In the present paper, we consider a countable infinite set X and characterize for a given subsemigroup W such that there is a finite set U being a generator of $T(X)$ modulo W (see [7]), the maximal subsemigroups of $T(X)$ containing W. As a consequence of this result, we obtain a characterization of all maximal subsemigroups of $T(X)$ containing $T(X) \setminus S$, where S is a given maximal subsemigroup of $T(X)$ containing $\mathrm{Sym}(X)$.

If $\alpha \in T(X)$ and $A \subseteqq X$ such that the restriction of α to A is injective and have the same range as α, then we will refer A as transversal of α ($\ker\alpha$ denotes the kernel of α). We will also write $A\#\ker\alpha$ if A is a transversal of α.

Let $D(\alpha) := X \setminus \operatorname{im}\alpha$ ($\operatorname{im}\alpha$ denotes the range of α). The rank α, i.e. the cardinality of $\operatorname{im}\alpha$, is denoted by $\operatorname{rank}(\alpha) := |\operatorname{im}\alpha|$. Then $d(\alpha) := |D(\alpha)|$ is called defect of α and $c(\alpha) := \sum\limits_{y\in\operatorname{im}\alpha}(|y\alpha^{-1}|-1)$ is called collapse of α.

Moreover, we put $K(\alpha) := \{x \in \operatorname{im}\alpha \mid |x\alpha^{-1}| = \aleph_0\}$ and $k(\alpha) := |K(\alpha)|$ is called infinite contractive index. It is well known that $d(\beta) \leq d(\alpha\beta) \leq d(\alpha) + d(\beta)$, $k(\alpha\beta) \leq k(\alpha) + k(\beta)$ [3] and $c(\alpha) \leq c(\alpha\beta) \leq c(\alpha) + c(\beta)$ [1] for $\alpha, \beta \in T(X)$. For more background in the theory of transformation semigroups see [3] and [8].

2. The main result

We want to state the main theorem in a wider context, namely in the context of algebras of a given type $\tau = (n_i)_{i\in I}$ with integers $n_i \geq 1$ for $i \in I$. So, this section deals with concepts of Universal Algebra. We follow the usual notation in Universal algebra [5]. Later, we specify our considerations to transformation semigroups as algebras of type $\tau = (2)$.

Let $\underline{T} = \big(T; (f_i^A)_{i\in I}\big)$ be an algebra of type τ with the universe T and the n_i-ary operations f_i^A on A, for $i \in I$. Further, let $\underline{W}$ be a proper subalgebra of $\underline{T}$ ($\underline{W} < \underline{T}$). Then $\underline{T}$ is said to be finitely generated over $\underline{W}$ if there is a finite set $U \subseteq T$ such that $T = \langle W, U\rangle$ (we write $\langle W, U\rangle$ instead of $\langle W \cup U\rangle$). A proper subalgebra $\underline{W}$ of $\underline{T}$ is called maximal if $\underline{S} = \underline{W}$ for all $\underline{S} < \underline{T}$ containing W. Moreover, $H_T(W)$ denotes the collection of all finite sets $F \subseteq T \setminus W$ such that $\langle W, F\rangle = T$.

Let $\mathcal{F}$ be a collection of non-empty finite subsets of a set T. A choice-set for $\mathcal{F}$ is a subset H of T with $H \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. A choice set H for $\mathcal{F}$ is said to be minimal if no proper subset of H is a choice-set for $\mathcal{F}$. Let us mention two set-theoretical observations.

Lemma 2.1. *Let $\mathcal{F}$ be a collection of non-empty finite subsets of a set T and let K be a choice-set for $\mathcal{F}$. Then there is a minimal choice set H for $\mathcal{F}$ such that $H \subseteq K$.*

Proof. Consider the collection $\mathcal{H}$ of choice-sets H for $\mathcal{F}$ such that $H \subseteq K$. If $\mathcal{C} \subseteq \mathcal{H}$ is a chain under $\subseteq$ then $\bigcap \mathcal{C} \in \mathcal{H}$. Let $F \in \mathcal{F}$. One can write F in the form $F = \{x_1, x_2, \dots, x_n\}$ if $n \geq 1$ is the cardinality of F. Assume that $\bigcap \mathcal{C} \cap F = \emptyset$ then $x_i \notin C_i$ $(1 \leq i \leq n)$ for some $C_1, \dots, C_n \in \mathcal{C}$. But $\mathcal{C}$ is a chain, hence (up to renumbering) $C_1 \subseteq C_2 \subseteq \dots \subseteq C_n$, so $\{x_1, x_2, \dots, x_n\} \cap C_1 = \emptyset$, which is impossible for the choice-set C_1. Thus $\bigcap \mathcal{C}$ intersects $\mathcal{F}$. Now, Zorn's Lemma can be applied and yields the existence of a minimal element H within $\mathcal{H}$. This set H is a choice-set for $\mathcal{F}$ such that $H \subseteq K$. $\square$

Lemma 2.2. *Let $\mathcal{F}$ be a collection of non-empty finite subsets of a set T and let $\mathcal{G}$ be a subcollection such that for each $F \in \mathcal{F}$ either $F \nsubseteq \bigcup \mathcal{G}$ or $G \subseteq F$ for some $G \in \mathcal{G}$. Then for each minimal choice-set H_0 for $\mathcal{G}$ there is a minimal choice-set H for $\mathcal{F}$ such that $H \cap \bigcup \mathcal{G} = H_0$.*

Proof. Let H_0 be minimal choice-set for $\mathcal{G}$. Then, we consider

$$K := H_0 \cap \bigcup \Big\{ F \setminus \bigcup \mathcal{G} \mid F \in \mathcal{F} \ \ \& \ \ (\forall G \in \mathcal{G})(G \nsubseteq F) \Big\}.$$

It is easy to verify that K is a choice-set for $\mathcal{F}$. Then by Lemma 2.1, there is a minimal choice-set $H \subseteq K$ for $\mathcal{F}$. Clearly, $H \cap \bigcup \mathcal{G} \subseteq K \cap \bigcup \mathcal{G}$ and $H_0 \subseteq \bigcup \mathcal{G}$ implies $K \cap \bigcup \mathcal{G} = H_0$, i.e. $H \cap \bigcup \mathcal{G} \subseteq H_0$. Moreover, $H \cap \bigcup \mathcal{G}$ is a choice-set for $\mathcal{G}$, so $H \cap \bigcup \mathcal{G} = H_0$, by the minimality. $\square$

Now we state the main theorem.

Theorem 2.1. *Let $\underline{T}$ be an algebra of type τ and let $\underline{W}$ be a proper subalgebra of $\underline{T}$ such that $\underline{T}$ is finitely generated over $\underline{W}$. Then a subalgebra $\underline{S}$ such that $\underline{W} \leq \underline{S} < \underline{T}$ is maximal if and only if $T \setminus S$ is a minimal choice-set for $H_T(W)$. For each minimal choice-set H for $H_T(W)$, the set $T \setminus H$ is the universe of a maximal subalgebra of $\underline{T}$.*

Proof. Suppose that $\underline{S}$ is maximal. Assume that $S \cap H \neq \emptyset$ for all minimal choice-sets H for $H_T(W)$. We are going to show that there is $F \in H_T(W)$ with $F \subseteq S$. Otherwise, $F \nsubseteq S$, i.e. $F \setminus S \neq \emptyset$, for all $F \in H_T(W)$. Then

$$K := \bigcup \{ F \setminus S \mid F \in H_T(W) \}$$

is a choice-set of $H_T(W)$. Then by Lemma 2.1, there exists a minimal choice-set H for $H_T(W)$ such that $H \subseteq K$. But $H \cap S = \emptyset$ is a contradiction. Hence, there is $F \in H_T(W)$ with $F \subseteq S$. Then $T = \langle W, F \rangle \subseteq S$, i.e. $S = T$, a contradiction. Hence there is a minimal choice-set H for $H_T(W)$ such that $S \cap H = \emptyset$, i.e. $S \subseteq T \setminus H$.

We want to show that $A := T \setminus H$ is closed under the operations of $\underline{T}$, i.e. $\underline{A}$ is a subalgebra of $\underline{T}$. For this, let $i \in I$ and $\alpha_1, \dots, \alpha_{n_i} \in A$. Assume that $s := f_i^A(\alpha_1, \dots, \alpha_{n_i}) \in H$. Then there is $F \in H_T(W)$ such that $F \cap H = \{s\}$ since H is a minimal choice-set for $H_T(W)$. Then $\big(F \setminus \{s\}\big) \cup \{\alpha_1, \dots, \alpha_{n_i}\} \in H_T(W)$.

But both $F \cap H = \{s\}$ and $(F \setminus \{s\}) \cup \{\alpha_1, \dots, \alpha_{n_i}\} \in H_T(W)$ implies $\alpha_j \in H$ for some $1 \leq j \leq n_i$, a contradiction. Hence $\underline{A}$ is a subalgebra of $\underline{T}$. Since S is maximal $S \subseteq A = T \setminus H$, this implies $S = T \setminus H$ and $H = T \setminus S$. This part of the proof shows in particular that $T \setminus H$ is the universe of a maximal subalgebra of $\underline{T}$ whenever H is a minimal choice-set H for $H_T(W)$.

Conversely, suppose that $T \setminus S$ is a minimal choice-set for $H_T(W)$. Then $S = T \setminus H$ such that $H := T \setminus S$ is a minimal choice-set for $H_T(W)$. It remains to show that $\underline{S}$ is maximal. Indeed, let $s \in H$. Then there is $F \in H_T(W)$ with $F \cap H = \{s\}$ because of the minimality of H. So, $T = \langle W, F\rangle \subseteq \langle T \setminus H, F\rangle = \langle T \setminus H, s\rangle$, i.e. $T = \langle T \setminus H, s\rangle$. This shows that $\underline{S}$ is maximal subalgebra of $\underline{T}$. $\square$

3. Maximal subsemigroups containing $\mathrm{Sym}(X)$

Let us introduce the following five sets:

- $\mathrm{Inj}(X) := \{\alpha \in T(X) \mid \mathrm{rank}(\alpha) = \aleph_0,\ c(\alpha) = 0 \text{ and } d(\alpha) \neq 0\}$ (the set of injective but not surjective mappings on X).
- $\mathrm{Sur}(X) := \{\alpha \in T(X) \mid \mathrm{rank}(\alpha) = \aleph_0,\ c(\alpha) \neq 0 \text{ and } d(\alpha) = 0\}$ (the set of surjective but not injective mappings on X).
- $C_p(X) := \{\alpha \in T(X) \mid \mathrm{rank}(\alpha) = \aleph_0,\ k(\alpha) = \aleph_0\}$.
- $\mathrm{IF}(X) := \{\alpha \in T(X) \mid \mathrm{rank}(\alpha) = \aleph_0,\ c(\alpha) = \aleph_0 \text{ and } d(\alpha) < \aleph_0\}$.
- $\mathrm{FI}(X) := \{\alpha \in T(X) \mid \mathrm{rank}(\alpha) = \aleph_0,\ d(\alpha) = \aleph_0 \text{ and } c(\alpha) < \aleph_0\}$.

In [6], the following proposition was proved. Note that we independently proved this proposition whilst of the work of G. P. Gavrilov and L. Heindorf, respectively. We thank Martin Goldstern for bringing these reference to our consideration at the AAA82 in Potsdam (June 2011) and for his set-theoretical observations which were essential for Section 2 in the present paper. For the sake of completeness, we include the proof of this proposition. (Another proof is also given in [2].)

Proposition 3.1. *The following semigroups of $T(X)$ are maximal:*

$$T(X) \setminus H$$

for $H \in \{\mathrm{Inj}(X), \mathrm{Sur}(X), C_p(X), \mathrm{IF}(X), \mathrm{FI}(X)\}$.

Proof.

1) Let $\alpha, \beta \in T(X) \setminus \mathrm{Inj}(X)$. Assume that $\alpha\beta \in \mathrm{Inj}(X)$. Then $c(\alpha) \leq c(\alpha\beta) = 0$, i.e. α is injective. Since $\alpha \notin \mathrm{Inj}(X)$, $\alpha \in \mathrm{Sym}(X)$. But $c(\alpha\beta) = 0$ and $\alpha \in \mathrm{Sym}(X)$ implies β is injective. Since $\beta \notin \mathrm{Inj}(X)$, $\beta \in \mathrm{Sym}(X)$. So $\alpha\beta \in \mathrm{Sym}(X)$, i.e. $\alpha\beta$ is surjective, contradicts $\alpha\beta \in \mathrm{Inj}(X)$. This shows that $T(X) \setminus \mathrm{Inj}(X)$ is a semigroup.

Let $\alpha \in \mathrm{Inj}(X)$. Then we will show that $\langle T(X) \setminus \mathrm{Inj}(X), \alpha \rangle = T(X)$. For this let $\beta \in \mathrm{Inj}(X)$. Let $a \in \operatorname{im}\beta$. Let $\gamma \in T(X)$ with $i\gamma = a$ for $i \in D(\alpha)$, and $i\gamma = i\alpha^{-1}\beta$ for $i \in \operatorname{im}\alpha$. Then $i\alpha\gamma = i\alpha\alpha^{-1}\beta = i\beta$ for all $i \in X$. This shows $\alpha\gamma = \beta$, where $\gamma \notin \mathrm{Inj}(X)$ since $D(\alpha) \neq \emptyset$. This shows that $T(X) \setminus \mathrm{Inj}(X)$ is maximal.

2) Let $\alpha, \beta \in T(X) \setminus \mathrm{Sur}(X)$. Assume that $\alpha\beta \in \mathrm{Sur}(X)$. Then $d(\beta) = 0$, i.e. β is surjective. Since $\beta \notin \mathrm{Sur}(X)$, $\beta \in \mathrm{Sym}(X)$. But $d(\alpha\beta) = 0$ and $\beta \in \mathrm{Sym}(X)$ implies α is surjective. Since $\alpha \notin \mathrm{Sur}(X)$, $\alpha \in \mathrm{Sym}(X)$. So $\alpha\beta \in \mathrm{Sym}(X)$, i.e. $\alpha\beta$ is injective, contradicts $\alpha\beta \in \mathrm{Sur}(X)$. This shows that $T(X) \setminus \mathrm{Sur}(X)$ is a semigroup.

Let $\alpha \in \mathrm{Sur}(X)$. Then we will show that $\langle T(X) \setminus \mathrm{Sur}(X), \alpha \rangle = T(X)$. For this let $\beta \in \mathrm{Sur}(X)$. For all $\overline{x} \in X/\ker\alpha$ we fix a $\overline{x}^* \in \overline{x}$. Then we consider the following $\delta \in T(X)$ with $i\delta = (i\beta\alpha^{-1})^*$ for all $i \in X$. Hence $i\delta\alpha = i\beta$ for all $i \in X$. This shows $\delta\alpha = \beta$. Since $\operatorname{im}\delta \subseteq \{\overline{x}^* \mid \overline{x} \in X/\ker\alpha\} \neq X$ (because α is not injective), $\delta \in T(X) \setminus \mathrm{Sur}(X)$. This shows $\beta = \delta\alpha \in \langle T(X) \setminus \mathrm{Sur}(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus \mathrm{Sur}(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus \mathrm{Sur}(X)$ is maximal.

3) Let $\alpha, \beta \in T(X) \setminus C_p(X)$. Then $k(\alpha) < \aleph_0$ and $k(\beta) < \aleph_0$. This shows $k(\alpha\beta) \leq k(\alpha) + k(\beta) < \aleph_0 + \aleph_0 = \aleph_0$. This shows that $\alpha\beta \in T(X) \setminus C_p(X)$.

Let $\alpha \in C_p(X)$. Then, we will show that $\langle T(X) \setminus C_p(X), \alpha \rangle = T(X)$. For this let $\beta \in C_p(X)$. Then, there is a bijection

$$f\colon X/\ker\beta \to \{x\alpha^{-1} \mid x \in K(\alpha)\}.$$

For each $\overline{x} \in X/\ker\beta$, there is an injective mapping

$$f_{\overline{x}}\colon \overline{x} \to f(\overline{x}).$$

We take the $\gamma \in T(X)$ with $i\gamma = f_{\overline{x}}(i)$ where $i \in \overline{x}$ for $\overline{x} \in X/\ker\beta$. Clearly, $\gamma \notin C_p(X)$. For $i, j \in X$, $i\beta = j\beta$ if and only if there is an $\overline{x} \in X/\ker\beta$ with $i, j \in \overline{x}$, i.e. $f_{\overline{x}}(i)\alpha = f_{\overline{x}}(j)\alpha$. But $f_{\overline{x}}(i)\alpha = f_{\overline{x}}(j)\alpha$ is equivalent to $i\gamma\alpha = j\gamma\alpha$, consequently, we have $i\gamma\alpha = j\gamma\alpha$ if and only if $i\beta = j\beta$. Further, let $\delta \in T(X)$ with $i\gamma\alpha\delta = i\beta$ for $i \in X$ and $i\delta = x_0$ (x_0 is any fixed element in X) for $i \in X \setminus \operatorname{im}\gamma\alpha$. Since $i\gamma\alpha = j\gamma\alpha$ if and only if $i\beta = j\beta$, δ is well defined and $K(\delta) \subseteq \{x_0\}$, i.e. $k(\delta) \leq 1$ and thus $\delta \in T(X) \setminus C_p(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus C_p(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus C_p(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus C_p(X)$ is maximal.

4) Let $\alpha, \beta \in T(X) \setminus \mathrm{IF}(X)$.

If $c(\alpha) < \aleph_0$ and $c(\beta) < \aleph_0$ then $c(\alpha\beta) \leq c(\alpha) + c(\beta) < \aleph_0$, i.e. $\alpha\beta \notin \mathrm{IF}(X)$.

If $d(\alpha) = \aleph_0$ and $c(\beta) < \aleph_0$ then $\left|\{\overline{x} \in X/\ker\beta \mid \overline{x} \cap \operatorname{im}\alpha = \emptyset\}\right| = \aleph_0$. This implies $\aleph_0 = d(\beta) \leq d(\alpha\beta) \leq d(\alpha) + d(\beta) = \aleph_0$, i.e. $\alpha\beta \notin \mathrm{IF}(X)$.

If $d(\beta) = \aleph_0$ then $d(\alpha\beta) \geq d(\beta) = \aleph_0$, i.e. $\alpha\beta \notin \mathrm{IF}(X)$.

Altogether, this shows that $\alpha\beta \in T(X) \setminus \mathrm{IF}(X)$.

Let $\alpha \in \mathrm{IF}(X)$. Then we will show that $\langle T(X) \setminus \mathrm{IF}(X), \alpha \rangle = T(X)$. For this let $\beta \in \mathrm{IF}(X)$. Let $\gamma \in T(X)$ with $\ker \gamma = \ker \beta$ and $\operatorname{im} \gamma \# \ker \alpha$. For each $\overline{x} \in X/\ker \beta$, we fix any $\overline{x}^* \in \overline{x}$. Since $c(\alpha) = \aleph_0$, $d(\gamma) = \aleph_0$, i.e. $\gamma \notin \mathrm{IF}(X)$. Further, let $\delta \in T(X)$ with $\operatorname{im} \alpha \# \ker \delta$ and $i\delta = (i(\gamma\alpha)^{-1})^* \beta$ for $i \in \operatorname{im} \alpha$. Since $\operatorname{im} \gamma \# \ker \alpha$, we have $\operatorname{im} \alpha = \operatorname{im} \gamma\alpha$ and δ is well defined. Because of $\operatorname{im} \gamma \# \ker \alpha$, $\ker \gamma\alpha = \ker \gamma = \ker \beta$, where $i\gamma\alpha\delta = (i\gamma\alpha(\gamma\alpha)^{-1})^* \beta = i\beta$ for $i \in X$. Note that from $\operatorname{im} \alpha \# \ker \delta$ and $d(\alpha) < \aleph_0$, it follows that $c(\delta) < \aleph_0$, i.e. $\delta \notin \mathrm{IF}(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus \mathrm{IF}(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus \mathrm{IF}(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus \mathrm{IF}(X)$ is maximal.

5) Let $\alpha, \beta \in T(X) \setminus \mathrm{FI}(X)$.

If $c(\alpha) = \aleph_0$ then $c(\alpha\beta) \geq c(\alpha) = \aleph_0$, i.e. $\alpha\beta \notin \mathrm{FI}(X)$.

If $d(\alpha) < \aleph_0$ and $c(\beta) = \aleph_0$ then $\left|\left\{i \in \operatorname{im} \alpha \mid (\exists j \in \operatorname{im} \alpha \setminus \{i\})(i\beta = j\beta)\right\}\right| = \aleph_0$. This implies $c(\alpha\beta) = \aleph_0$, i.e. $\alpha\beta \notin \mathrm{FI}(X)$.

If $d(\alpha) < \aleph_0$ and $d(\beta) < \aleph_0$ then $d(\alpha\beta) \leq d(\alpha) + d(\beta) < \aleph_0$, i.e. $\alpha\beta \notin \mathrm{FI}(X)$.

Altogether, this shows that $\alpha\beta \in T(X) \setminus \mathrm{FI}(X)$.

Let $\alpha \in \mathrm{FI}(X)$. Then, we will show that $\langle T(X) \setminus \mathrm{FI}(X), \alpha \rangle = T(X)$. For this, let $\beta \in \mathrm{FI}(X)$. Let $\gamma \in T(X)$ with $\ker \gamma = \ker \beta$ and $\operatorname{im} \gamma \# \ker \alpha$. For each $\overline{x} \in X/\ker \beta$, we fix any $\overline{x}^* \in \overline{x}$. Since $c(\alpha) < \aleph_0$, $d(\gamma) < \aleph_0$, i.e. $\gamma \notin \mathrm{FI}(X)$. Further, let $\delta \in T(X)$ with $\operatorname{im} \alpha \# \ker \delta$ and $i\delta = (i(\gamma\alpha)^{-1})^* \beta$ for $i \in \operatorname{im} \alpha$. Since $\operatorname{im} \gamma \# \ker \alpha$, $\operatorname{im} \alpha = \operatorname{im} \gamma\alpha$, $\ker \gamma\alpha = \ker \gamma = \ker \beta$ and δ is well defined, where $i\gamma\alpha\delta = (i\gamma\alpha(\gamma\alpha)^{-1})^* \beta = i\beta$ for $i \in X$. Note that from $\operatorname{im} \alpha \# \ker \delta$ and $d(\alpha) = \aleph_0$, it follows that $c(\delta) = \aleph_0$, i.e. $\delta \notin \mathrm{FI}(X)$. This shows $\beta = \gamma\alpha\delta \in \langle T(X) \setminus \mathrm{FI}(X), \alpha \rangle$. Consequently, $\langle T(X) \setminus \mathrm{FI}(X), \alpha \rangle = T(X)$. This shows that $T(X) \setminus \mathrm{FI}(X)$ is maximal. □

This proposition delivers the maximal subsemigroups of $T(X)$ containing $\mathrm{Sym}(X)$. In [2] (see also [4], [6]), the authors show the completeness of the list given in Proposition 3.1. We ask for the maximal subsemigroups of $T(X)$ containing the difference set between $T(X)$ and a maximal semigroup from this list.

4. Maximal subsemigroups containing $\mathrm{Inj}(X)$, $\mathrm{Sur}(X)$, $C_p(X)$, $\mathrm{IF}(X)$, $\mathrm{FI}(X)$

Now we want to determine the maximal subsemigroups of $T(X)$ containing $T(X) \setminus S$, where S is one of the five maximal subsemigroups of $T(X)$ containing $\mathrm{Sym}(X)$. First, we characterize the maximal subsemigroups of $T(X)$ containing $\mathrm{Inj}(X)$ and $\mathrm{Sur}(X)$, respectively. Note that we do not need Theorem 2.1 here. It is well known that the set $F(X)$ of all transformations of finite rank forms an ideal of $T(X)$ and $\mathrm{Inf}(X) := T(X) \setminus F(X)$ generates $T(X)$. The next lemma shows that any maximal subsemigroup S of $T(X)$ has the form $S = F(X) \cup T$ for some $T \subset \mathrm{Inf}(X)$.

Lemma 4.1. *Let S be a maximal subsemigroup of $T(X)$. Then $F(X) \subset S$.*

Proof. We have $\mathrm{Inf}(X) \nsubseteq S$ (since $S \neq T(X)$). Since $F(X)$ forms an ideal of $T(X)$ both $\mathrm{Inf}(X) \cap F(X) = \emptyset$ and $\mathrm{Inf}(X) \nsubseteq S$ implies $S \subseteq S \cup F(X) \neq T(X)$. Because of the maximality of S, we have $S = S \cup F(X)$, i.e. $F(X) \subset S$. □

Lemma 4.2. *Let* $\mathrm{Sur}(X) \subset S \leq T(X)$ *with* $\mathrm{Inj}(X) \cap S \neq \emptyset$ *and* $\mathrm{FI}(X) \cap S \neq \emptyset$. *Then*

$$S = T(X).$$

Proof. We have $F(X) \subset S$ by Lemma 4.1. Hence, we have to consider only the elements of $\mathrm{Inf}(X)$. Let $\alpha \in \mathrm{Sym}(X)$. Then there is a $\beta \in \mathrm{Inj}(X) \cap S$ and we take the $\gamma \in T(X)$ with $i\gamma = i$ for $i \in D(\beta)$ and $i\beta\gamma = i\alpha$ for $i \in X$. Clearly, γ is well defined (since β is injective) and $\gamma \in \mathrm{Sur}(X)$. Since $\operatorname{im}\beta = X \setminus D(\beta)$, this shows that $\beta\gamma = \alpha$, and consequently, $\mathrm{Sym}(X) \subset S$. Let us put

$$A := \bigl\{\alpha \in \mathrm{Inf}(X) \mid d(\alpha) < \aleph_0\bigr\}$$
$$B := \bigl\{\alpha \in \mathrm{Inf}(X) \mid d(\alpha) = \aleph_0\bigr\}.$$

Clearly, $\mathrm{Inf}(X) = A \cup B$. Let now $\alpha \in A$. If $d(\beta) < \aleph_0$ then for each natural number $k \geq 1$, there is a natural number $r \geq 1$ such that $d(\beta^r) \geq k$. Since $\beta^r \in \mathrm{Inj}(X) \cap S$, we can assume that $d(\beta) \geq d(\alpha)$. Since $d(\beta) \geq d(\alpha)$, there is a $\gamma_1 \in \mathrm{Sur}(X)$ such that γ_1 restricted to $\operatorname{im}\beta$ is bijective with $\operatorname{im}\alpha$ as range and $D(\beta)\gamma_1 = D(\alpha)$. We take the $\gamma_2 \in \mathrm{Inf}(X)$ with $i\gamma_2$ is the unique element in $i\alpha\gamma_1^{-1}\beta^{-1}$ for $i \in X$. Since β is injective, we have $\gamma_2 \in \mathrm{Sur}(X) \cup \mathrm{Sym}(X)$. Then we have $i\gamma_2\beta\gamma_1 = i\alpha\gamma_1^{-1}\beta^{-1}\beta\gamma_1 = i\alpha$ for $i \in X$. This shows $\alpha = \gamma_2\beta\gamma_1 \in S$, and consequently, $A \subset S$.

Let now $\alpha \in B$. Moreover, there is a $\delta \in \mathrm{FI}(X) \cap S$. Then there is a $\eta \in A$ with $\ker\alpha = \ker\eta$ and $\operatorname{im}\eta \# \ker\delta$. Since $d(\alpha) = d(\delta) = \aleph_0$, there is a bijection $f\colon D(\delta) \to D(\alpha)$. Because of $\ker\alpha = \ker\eta$ and $\operatorname{im}\eta \# \ker\delta$, we have $i\delta^{-1}\eta^{-1} \in X/\ker\alpha$ for $i \in \operatorname{im}\delta$. We define $\gamma_3 \in T(X)$ setting $i\gamma_3 = f(i)$ for $i \in D(\delta)$ and $i\gamma_3$ is a unique element in $i\delta^{-1}\eta^{-1}\alpha$ for $i \in \operatorname{im}\delta$. It is easy to verify that γ_3 is well defined. Then for $i \in X$, $i\eta\delta\gamma_3 = i\alpha$. This shows $\alpha = \eta\delta\gamma_3$ and consequently, $B \subset S$. Altogether, $\mathrm{Inf}(X) = A \cup B \subseteq S$ and thus $S = T(X)$. □

Lemma 4.3. *Let* $\mathrm{Inj}(X) \subset S \leq T(X)$ *with* $H \cap S \neq \emptyset$ *for all* $H \in \bigl\{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\bigr\}$. *Then*

$$S = T(X).$$

Proof. We show that then $\mathrm{Sur}(X) \subset S$. If we have $\mathrm{Sur}(X) \subset S$ then from $\mathrm{Inj}(X) \cap S \neq \emptyset$ and $\mathrm{FI}(X) \cap S \neq \emptyset$ (because of $\mathrm{Inj}(X) \subset S$) it follow $S = T(X)$ by Lemma 4.2.

Let $\alpha \in \mathrm{Sur}(X)$. Moreover, there is a $\beta \in C_p(X) \cap S$. Then there is a bijection

$$f\colon X/\ker\alpha \to \bigl\{x\beta^{-1} \mid x \in K(\beta)\bigr\}.$$

For each $\overline{x} \in X/\ker\alpha$, there is an injective but not surjective mapping

$$f_{\overline{x}}\colon \overline{x} \to f(\overline{x}).$$

We take the $\gamma \in T(X)$ with $i\gamma = f_{\overline{x}}(i)$ where $i \in \overline{x}$ for $\overline{x} \in X/\ker\alpha$. It is easy to verify that $\gamma \in \mathrm{Inj}(X)$. There are $\delta \in \mathrm{Sur}(X) \cap S$ and $\eta \in \mathrm{IF}(X) \cap S$. If $c(\delta) < \aleph_0$ then from $c(\delta) > 0$, it follows that $c(\delta^r) > d(\eta)$ for some $r \in \mathbb{N}$, where $\delta^r \in \mathrm{Sur}(X) \cap S$. Hence, we can assume that $c(\delta) > d(\eta)$ and there is a set $A \subseteq X$ with $A \# \ker\delta$ and a bijection

$$h_1\colon \operatorname{im}\eta \to A$$

and an injective but not surjective mapping

$$h_2\colon D(\eta) \to X \setminus A.$$

We take the $\gamma_1 \in T(X)$ with $i\gamma_1 = h_1(i)$ for $i \in \operatorname{im}\eta$ and $i\gamma_1 = h_2(i)$ for $i \in D(\eta)$. It is easy to verify that $\gamma_1 \in \mathrm{Inj}(X)$. In fact, $\eta\gamma_1\delta \in \mathrm{Sur}(X) \cap S$ with $c(\eta\gamma_1\delta) = \aleph_0$ since $\eta \in \mathrm{IF}(X)$. So, we can assume that $c(\delta) = \aleph_0$. For $i, j \in X$, $i\alpha = j\alpha$ if and only if there is an $\overline{x} \in X/\ker\alpha$ with $i, j \in \overline{x}$, i.e. $f_{\overline{x}}(i)\beta = f_{\overline{x}}(j)\beta$. But $f_{\overline{x}}(i)\beta = f_{\overline{x}}(j)\beta$ is equivalent to $i\gamma\beta = j\gamma\beta$, consequently, we have $i\gamma\beta = j\gamma\beta$ if and only if $i\alpha = j\alpha$. Further, let $B \subseteq X$ with $B \# \ker\delta$ and

$$\varphi\colon D(\gamma\beta) \to X \setminus B$$

be an injective but not surjective transformation (such one exists since $c(\delta) = \aleph_0$ implies $|X \setminus B| = \aleph_0$). Then the transformation γ_2 on X with $i\gamma\beta\gamma_2$ is the unique element in $i\alpha\delta^{-1} \cap B$ for $i \in X$ and $i\gamma_2 = \varphi(i)$ for $i \in D(\gamma\beta)$ belongs to $\mathrm{Inj}(X)$ since $B \# \ker\delta$. So, we have $i\gamma\beta\gamma_2\delta = i\alpha$ for $i \in X$. This shows that $\gamma\beta\gamma_2\delta = \alpha$, and consequently, $\mathrm{Sur}(X) \subset S$. □

Now we are able to characterize the maximal subsemigroups of $T(X)$ containing $\mathrm{Inj}(X)$ and $\mathrm{Sur}(X)$, respectively.

Theorem 4.1. *Let* $\mathrm{Sur}(X) \subset S \leq T(X)$. *Then* S *is maximal if and only if* $S = T(X) \setminus \mathrm{Inj}(X)$ *or* $S = T(X) \setminus \mathrm{FI}(X)$.

Proof. By Proposition 3.1, both $T(X) \setminus \mathrm{Inj}(X)$ and $T(X) \setminus \mathrm{FI}(X)$ are maximal subsemigroups of $T(X)$. Suppose that S is a maximal subsemigroup of $T(X)$. Then $\mathrm{Inj}(X) \cap S = \emptyset$ or $\mathrm{FI}(X) \cap S = \emptyset$ by Lemma 4.2, i.e. $S \subseteq T(X) \setminus \mathrm{Inj}(X)$ or $S \subseteq T(X) \setminus \mathrm{FI}(X)$ and thus $S = T(X) \setminus \mathrm{Inj}(X)$ or $S = T(X) \setminus \mathrm{FI}(X)$ because of the maximality of S. □

Theorem 4.2. *Let* $\mathrm{Inj}(X) \subset S \leq T(X)$. *Then* S *is maximal if and only if* $S = T(X) \setminus H$ *for some* $H \in \{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\}$.

Proof. By Proposition 3.1, $T(X) \setminus H$ ($H \in \{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\}$) are maximal subsemigroups of $T(X)$. If S is a maximal subsemigroup of $T(X)$ then $H \cap S = \emptyset$ for some $H \in \{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\}$ by Lemma 4.3, i.e. $S \subseteq T(X) \setminus H$ for some $H \in \{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\}$. The maximality of S provides the assertion. □

Finally, we want to determine the maximal subsemigroups of $T(X)$ containing H for $H \in \{C_p(X), \mathrm{IF}(X), \mathrm{FI}(X)\}$ using Theorem 2.1. First, we state that $\mathrm{FI}(X)$ as well as $\mathrm{IF}(X)$ are subsemigroups of $T(X)$.

Lemma 4.4. $\mathrm{FI}(X)$ *is a subsemigroup of* $T(X)$.

Proof. Let $\alpha, \beta \in \mathrm{FI}(X)$. Then we have $c(\alpha\beta) \leq c(\alpha) + c(\beta) < \aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 = d(\beta) \leq d(\alpha\beta)$. This shows that $\alpha\beta \in \mathrm{FI}(X)$. □

Lemma 4.5. $\mathrm{IF}(X)$ *is a subsemigroup of* $T(X)$.

Proof. Let $\alpha, \beta \in \mathrm{IF}(X)$. Then we have $d(\alpha\beta) \leq d(\alpha) + d(\beta) < \aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 = c(\alpha) \leq c(\alpha\beta)$. This shows that $\alpha\beta \in \mathrm{IF}(X)$. □

Let us consider the set $C_p(X) \cap \mathrm{Sur}(X)$. Then we have:

Lemma 4.6. *We have* $\langle \mathrm{FI}(X), \alpha \rangle = T(X)$ *for all* $\alpha \in C_p(X) \cap \mathrm{Sur}(X)$.

Proof. Let $\alpha \in C_p(X) \cap \mathrm{Sur}(X)$, $\beta \in \mathrm{Inj}(X)$, and $A \subseteq X$ be a transversal of α. We put $\gamma \in T(X)$ setting $x\gamma$ is the unique element in $x\beta\alpha^{-1} \cap A$ for all $x \in X$. It is easy to verify that $\operatorname{im}\gamma \subseteq A$ and $d(\gamma) = |X \setminus \operatorname{im}\gamma| \geq |X \setminus A| = \aleph_0$ since $\alpha \in C_p(X)$ implies $c(\alpha) = \aleph_0$ and any transversal of α miss infinite many elements of X. Let $i, j \in X$ with $i\gamma = j\gamma$. This implies $(i\beta\alpha^{-1} \cap A)\alpha = (j\beta\alpha^{-1} \cap A)\alpha$, $i\beta = j\beta$, and $i = j$ since $\beta \in \mathrm{Inj}(X)$. Thus $\gamma \in \mathrm{Inj}(X)$ and $c(\gamma) = 0 < \aleph_0$. Consequently, $\gamma \in \mathrm{FI}(X)$. Because of $x\gamma\alpha = x\beta$ for all $x \in X$, we have $\beta = \gamma\alpha \in \langle \mathrm{FI}(X), \alpha \rangle$. This shows that $\mathrm{Inj}(X) \subseteq \langle \mathrm{FI}(X), \alpha \rangle$. Moreover, $\langle \mathrm{FI}(X), \alpha \rangle \cap H \neq \emptyset$ for $H \in \{\mathrm{Sur}(X), C_p(X), \mathrm{IF}(X)\}$. By Lemma 4.3, we have $\langle \mathrm{FI}(X), \alpha \rangle = T(X)$. □

Lemma 4.6 shows that $T(X)$ is finitely generated over $\mathrm{FI}(X)$. Then Lemma 4.4 and Theorem 2.1 imply:

Proposition 4.3. *Let* $S \leq T(X)$ *with* $\mathrm{FI}(X) \subset S$. *Then the following statements are equivalent:*

(i) *S is maximal.*

(ii) *$T(X) \setminus S$ is a minimal choice-set for* $H_{T(X)}(\mathrm{FI}(X))$.
$T(X) \setminus H$ is a maximal subsemigroup containing $\mathrm{FI}(X)$ *whenever H is a minimal choice-set for* $H_{T(X)}(\mathrm{FI}(X))$.

It will turn out that there are 2^c many maximal subsemigroups of $T(X)$ containing $\mathrm{FI}(X)$, where c denotes the cardinality corresponding to the continuum.

Lemma 4.7. *There holds $\left|H_{T(X)}(\mathrm{FI}(X))\right| = 2^c$, i.e. there are 2^c many maximal subsemigroups of $T(X)$ containing $\mathrm{FI}(X)$.*

P r o o f. Let X be the set of all positive integer and let $\mathcal{A}$ be any collection of continuum many almost disjoint subsets of $X \setminus \{1, 2\}$. Further, let $\beta \in C_p(X) \cap \mathrm{Sur}(X)$. For $Y \in \mathcal{A}$, we fix a transformation β^Y with the same kernel as β but the range $Y \cup \{1\}$ and a surjective transformation β_Y with the kernel $\left\{(y, y) \mid y \in Y\right\} \cup \left\{(a, b) \mid a, b \in X \setminus Y\right\}$. Then it is easy to check that $\beta^Y \beta_Y \in C_p(X) \cap \mathrm{Sur}(X)$, but $\beta^Y, \beta_Y \notin C_p(X) \cap \mathrm{Sur}(X)$. Thus $A_Y := \left\{\beta^Y, \beta_Y\right\} \in H_{T(X)}(\mathrm{FI}(X))$ by Lemma 4.6. It is also easy to verify that

(1): $A_{Y_1} \neq A_{Y_2} = \emptyset$ whenever $Y_1 \neq Y_2$ and

(2): $\beta^{Y_1}\beta^{Y_2}, \beta_{Y_1}\beta_{Y_2}, \beta^{Y_1}\beta_{Y_2}, \beta_{Y_2}\beta^{Y_1} \notin \mathrm{Sur}(X)$ whenever $Y_1 \neq Y_2$.

Because of (1), the collection

$$\mathcal{G} := \{A_Y \mid Y \in \mathcal{A}\}$$

consists of continuum many elements. Let $F \in H_{T(X)}(\mathrm{FI}(X))$. If $A_Y \subsetneq F$ for all $Y \in \mathcal{A}$, then $F \subsetneq \bigcup \mathcal{G}$. Otherwise, $F \subseteq \bigcup \mathcal{G}$ and F contains exactly one element from each A_Y, $Y \in \mathcal{A}$. Then by (2), it is not hard to see that $\left\langle \mathrm{FI}(X), F \right\rangle \cap \mathrm{Sur}(X) \cap C_p(X) = \emptyset$, a contradiction. Then, by Lemma 2.2 for each minimal choice-set H_0 for $\mathcal{G}$, there is a minimal choice-set H for $\mathcal{F}$ such that $H \cap \bigcup \mathcal{G} = H_0$. It follows that there are at least as many minimal choice-sets for $\mathcal{F}$ as there are for $\mathcal{G}$. Since $\mathcal{G}$ consists of pairwise disjoint two-element sets, there are at least 2^c many minimal choice-sets for $\mathcal{F}$. $\square$

Now, we consider the set $\mathrm{FI}(X) \cap \mathrm{Inj}(X)$. Here, we get:

Lemma 4.8. *We have $\left\langle \mathrm{IF}(X), \alpha \right\rangle = T(X)$ for all $\alpha \in \mathrm{FI}(X) \cap \mathrm{Inj}(X)$.*

P r o o f. Let $\alpha \in \mathrm{FI}(X) \cap \mathrm{Inj}(X)$ and $\beta \in \mathrm{Inj}(X)$. We put $\gamma \in T(X)$ setting

$$\begin{aligned} i\alpha\gamma &:= i\beta \qquad \text{for} \quad i \in X \\ i\gamma &:= f(i) \qquad \text{for} \quad i \in D(\alpha) \end{aligned}$$

where

$$f\colon D(\alpha) \to D(\beta) \cup x_0\alpha$$

is a surjective but not injective transformation such that $|D(\alpha) \setminus \Sigma| = \aleph_0$ for some transversal Σ of f and any fixed $x_0 \in X$ (we consider f as transformation in $D(\alpha)$). Since $\alpha \in \mathrm{Inj}(X)$, the transformation γ is well defined. Such a mapping exists because of $d(\alpha) = \aleph_0$. Since $|D(\alpha) \setminus \Sigma| = \aleph_0$ for some transversal Σ of f, we have $c(\gamma) = \aleph_0$. Moreover, $\mathrm{im}\,\gamma = \{x\gamma \mid x \in X\} = \{x\gamma \mid x \in \mathrm{im}\,\alpha\} \cup \{x\gamma \mid x \in X \setminus \mathrm{im}\,\alpha\} = \mathrm{im}\,\beta \cup (X \setminus \mathrm{im}\,\beta) \cup \{x_0\alpha\} = X$. Hence $d(\gamma) = 0$.

This shows that $\gamma \in \mathrm{IF}(X)$. By definition, we have $\beta = \alpha\gamma \in \big\langle \mathrm{IF}(X), \alpha \big\rangle$. This shows that $\mathrm{Inj}(X) \subseteq \big\langle \mathrm{IF}(X), \alpha \big\rangle$. Moreover, $\big\langle \mathrm{IF}(X), \alpha \big\rangle \cap H \neq \emptyset$ for $H \in \big\{ \mathrm{Sur}(X), C_p(X), \mathrm{IF}(X) \big\}$. By Lemma 4.3, we have $\big\langle \mathrm{IF}(X), \alpha \big\rangle = T(X)$. $\square$

Lemma 4.8 shows that $T(X)$ is finitely generated over $\mathrm{IF}(X)$. Then Lemma 4.5 and Theorem 2.1 imply:

Proposition 4.4. *Let $S \leq T(X)$ with $\mathrm{IF}(X) \leq S$. Then the following statements are equivalent:*

(i) *S is maximal.*

(ii) *$T(X) \setminus S$ is a minimal choice-set for $H_{T(X)}(\mathrm{IF}(X))$.*
$T(X) \setminus H$ is a maximal subsemigroup containing $\mathrm{IF}(X)$, whenever H is a minimal choice-set for $H_{T(X)}(\mathrm{IF}(X))$.

Lemma 4.9. $\big\langle C_p(X) \big\rangle \cap (\mathrm{Inj}(X) \cap \mathrm{FI}(X)) = \emptyset$.

P r o o f. Let $\alpha, \beta \in T(X)$ with $c(\alpha) = c(\beta) = \aleph_0$. Then $\aleph_0 = c(\alpha) \leq c(\alpha\beta)$, i.e. $c(\alpha\beta) = \aleph_0$. Since $c(\alpha) = \aleph_0$ for all $\alpha \in C_p(X)$, this shows that $\big\langle C_p(X) \big\rangle \cap \mathrm{FI}(X) = \emptyset$. $\square$

Since we can decompose a countable set into countable many countable sets, it is routine that each transformation α with $\big(\exists \overline{x} \in X/\ker\alpha\big)\big(|\overline{x}| = \aleph_0\big)$ can be written as product $\beta\gamma$ of appropriate transformations $\beta, \gamma \in C_p(X)$. Moreover, it is clear that $\big\{ \alpha \in T(X) \mid \big(\exists \overline{x} \in X/\ker\alpha\big)\big(|\overline{x}| = \aleph_0\big) \big\}$ is subsemigroup of $T(X)$. Hence $\langle C_p(X) \rangle = \big\{ \alpha \in T(X) \mid \big(\exists \overline{x} \in X/\ker\alpha\big)\big(|\overline{x}| = \aleph_0\big) \big\}$.

In order to guaranty that $T(X)$ is finitely generated over $\big\langle C_p(X) \big\rangle$, we prove:

Lemma 4.10. *We have $\big\langle C_p(X), \alpha \big\rangle = T(X)$ for all $\alpha \in \mathrm{FI}(X) \cap \mathrm{Inj}(X)$.*

P r o o f. We show that $\mathrm{Inj}(X) \subset \big\langle C_p(X), \alpha \big\rangle$. If we have it then from $\big\langle C_p(X), \alpha \big\rangle \cap H \neq \emptyset$ for $H \in \big\{ \mathrm{Sur}(X), C_p(X), \mathrm{IF}(X) \big\}$ it follows that $\big\langle C_p(X), \alpha \big\rangle = T(X)$ by Lemma 4.3. For this let $\beta \in \mathrm{Inj}(X)$. Let $\alpha \in S \cap (\mathrm{FI}(X) \cap \mathrm{Inj}(X))$, i.e. $d(\alpha) = \aleph_0$. Further let $\{I_k \mid k \in X\}$ be a decomposition of $D(\alpha)$ in infinitely many infinite subsets. Then we take $\gamma \in T(X)$ with $X/\ker\gamma = \{I_k \cup \{k\alpha\} \mid k \in X\}$ and $i\gamma = k\beta$ for $i \in I_k \cup \{k\alpha\}$ and $k \in X$. It is easy to verify that $\gamma \in C_p(X) \subset \langle C_p(X), \alpha \rangle$. This provides $k\alpha\gamma = k\beta$ for $k \in X$. This shows $\beta = \alpha\gamma \in S$. Consequently, $\mathrm{Inj}(X) \subset \big\langle C_p(X), \alpha \big\rangle$. $\square$

Proposition 4.5. *Let $S \leq T(X)$ with $C_p(X) \subseteq S$. Then the following statements are equivalent:*

(i) *S is maximal.*

(ii) *$T(X) \setminus S$ is a minimal choice-set for $H_{T(X)}(\big\langle C_p(X) \big\rangle)$.*
$T(X) \setminus H$ is a maximal subsemigroup containing $C_p(X)$ whenever H is a minimal choice-set for $H_{T(X)}(\big\langle C_p(X) \big\rangle)$.

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