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RECONCILIATION OF APPROACHES TO THE CONSTRUCTION OF CANONICAL EXTENSIONS OF BOUNDED LATTICES

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Dedicated to the 75th birthday of Professor Tibor Katriňák

(*Communicated by Miroslav Ploščica*)

ABSTRACT. We provide new insights into the relationship between different constructions of the canonical extension of a bounded lattice. This follows on from the recent construction of the canonical extension using Ploščica's maximal partial maps into the two-element set by Craig, Haviar and Priestley (2012). We show how this complete lattice of maps is isomorphic to the stable sets of Urquhart's representation and to the concept lattice of a specific context, and how to translate our construction to the original construction of Gehrke and Harding (2001). In addition, we identify the completely join- and completely meet-irreducible elements of the complete lattice of maximal partial maps.

1. Introduction

The theory of canonical extensions has proven to be a valuable tool for studying lattice-based algebras. A study of the symbiotic relationship between the theory of canonical extensions and the theory of natural dualities was started (in the present journal) by Haviar and Priestley [13]. For a recent survey of the theory of canonical extensions for lattice-based algebras, including a discussion

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of the important role of canonical extensions in the semantic modelling of logics, we refer the reader to Gehrke and Vosmaer [11]. The many applications of canonical extensions to non-classical logics motivate us to gain a fuller understanding of the constructions of canonical extensions for the most general lattice-based algebras: bounded lattices.

This paper was prompted by results obtained in [4], where we presented a new approach to the canonical extension of a bounded lattice, based on a lesser-known dual representation for the variety $\boldsymbol{\mathcal{L}}$ of bounded lattices due to Ploščica [15]. The canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ was introduced by Gehrke and Harding [8]; their $\mathbf{L}^\delta$ arises as the complete lattice of Galois-closed sets associated with the Galois connection between $\wp(\mathrm{Filt}(\mathbf{L}))$ and $\wp(\mathrm{Idl}(\mathbf{L}))$ coming from the relation $R \subseteq \mathrm{Filt}(\mathbf{L}) \times \mathrm{Idl}(\mathbf{L})$ given by $(F, I) \in R$ if and only if $F \cap I \neq \emptyset$. We briefly recall the Gehrke–Harding approach at the beginning of Section 4.

Canonical extensions of distributive lattices were introduced and investigated by Gehrke and Jónsson [9] by exploiting Priestley duality. The uniqueness of the canonical extension ensures that the later constructions in the whole variety $\boldsymbol{\mathcal{L}}$ given in [8] and [10] lead in the distributive case to the same completion as presented in [9]. However, even in the distributive case it can be unclear how to translate between the various concrete realisations of the canonical extension. And there have been significant obstacles to extending, in a fully satisfactory and transparent way, the duality approach beyond the distributive case. Therefore in [4] we focused on exploring deeply the interface between canonical extensions and duality theory for bounded lattices, with a particular emphasis on the categorical framework.

Different constructions of the canonical extension of a bounded lattice can be useful for different applications. The doubly-ordered set and resulting complex algebra (canonical extension) from Urquhart's representation [17] was used by Dzik, Orlowska and van Alten [6] to provide complete Kripke-style semantics for logics with negation. By contrast, the categorical setting of [4] was used to provide a functorial explanation for the fact that a lattice homomorphism between two lattices $\mathbf{L}$ and $\mathbf{K}$ is lifted, by the canonical extension construction, to a complete lattice homomorphism between their canonical extensions.

In this paper we focus on a reconciliation of different approaches to the construction of canonical extensions for the whole variety $\boldsymbol{\mathcal{L}}$ of bounded lattices while we further emphasize the role of Galois connections in the framework of the constructions. Of course, the variety $\boldsymbol{\mathcal{L}}$ is not finitely generated, thus no natural duality theory, as studied by Clark and Davey in [3], is available. However, the long-established representation for $\boldsymbol{\mathcal{L}}$ due to Urquhart [17] was usefully recast in the spirit of natural duality theory by Ploščica in [15]. Ploščica's topological representation of each $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ was our main tool for constructing the canonical extensions for members of $\boldsymbol{\mathcal{L}}$ in [4]. Roughly speaking, Ploščica's extension of the Priestley representation is accomplished by replacing total maps into $\{0, 1\}$, viewed either as a lattice or a partially ordered set, by appropriate maximally-defined partial maps of the same sort. We give a necessary recap of Ploščica's representation of bounded lattices early in Section 2 (see Proposition 2.4).

In Section 2 we also recall our construction from [4] of the canonical extension of $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ (see Theorem 2.6). Consider Ploščica's dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ of the lattice $\mathbf{L}$, where $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ is the set of maximal partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$ and $(f, g) \in E$ if and only if $f^{-1}(1) \cap g^{-1}(0) = \emptyset$. Then the canonical extension $\mathbf{L}^{\delta}$ of $\mathbf{L}$ is the lattice $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of all maximal partial E-preserving maps (MPE's, for short) from $\mathbf{X}$ to the two-element graph $\underset{\sim}{\mathbf{2}} = (\{0, 1\}; \leqslant)$.

Gehrke and Harding in [8: Remark 2.10] assert that the canonical extension of a bounded lattice $\mathbf{L}$ is isomorphic to the complete lattice of so-called ℓ-stable subsets of Urquhart's dual space of $\mathbf{L}$. In Section 2, our first new result shows directly that the ℓ-stable sets correspond to the maximal partial E-preserving maps (Theorem 2.9). We show that the complete lattice $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets of Ploščica's dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ of the lattice $\mathbf{L}$ (or, equivalently, the ℓ-stable subsets of Urquhart's dual space) ordered by inclusion, is order-isomorphic to the canonical extension $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of $\mathbf{L}$. For this, we define an order-isomorphism $\Phi_{CX}^{LS} \colon \mathrm{LS}(\mathbf{X}) \to \mathrm{C}(\mathbf{X})$ and we prove that this order-isomorphism fixes the elements of $\mathbf{L}$, thus $\mathrm{LS}(\mathbf{X})$ is the canonical extension of $\mathbf{L}$ (Corollary 2.10).

In Section 3 we turn to the framework of Formal Concept Analysis [7] and show that the canonical extension $\mathbf{L}^{\delta} = \mathrm{C}(\mathbf{X})$ of $\mathbf{L}$ is isomorphic to the concept lattice of a specific context associated to the complement $E^{\complement}$ of the graph relation E. More precisely, for a graph $\mathbf{X} = (X, E)$ we consider the context $\mathbb{K}(\mathbf{X}) := (X, X, E^{\complement})$ and the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of this context. We show that for an arbitrary graph $\mathbf{X} = (X, E)$ (not necessarily coming from a lattice $\mathbf{L}$) there is an order-isomorphism $\Phi_{CX}^{CL} \colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X})$ between the concept lattice and the lattice of MPE's (Proposition 3.1). When the graph is a dual of a lattice $\mathbf{L}$, that is $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, the order-isomorphism Φ_{CX}^{CL} fixes $\mathbf{L}$ and so the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X})$ is the canonical extension of the lattice $\mathbf{L}$ (Corollary 3.2). Theorem 3.3 shows a useful correspondence between the elements of the lattices $\mathrm{C}(\mathbf{X})$, $\mathrm{LS}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$.

In Section 4 we first reconcile the concept lattice studied in Section 3 with the canonical extension $\mathrm{GH}(\mathbf{L})$ of $\mathbf{L}$ introduced by Gehrke and Harding [8]. We further recall our construction of the canonical extension $\mathrm{C}(\mathbf{Y})$ of $\mathbf{L}$ via its bigger dual graph $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of 'special' partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$ due to Allwein and Hartonas [1,2]; this proved to be important for our categorical framework in [4]. The lattices $\mathrm{C}(\mathbf{X})$ and $\mathrm{C}(\mathbf{Y})$ are order-isomorphic via a specific (and quite complicated) map $\Phi^{CX}_{CY}\colon\ \mathrm{C}(\mathbf{X}) \to \mathrm{C}(\mathbf{Y})$ (see Proposition 4.1 and the formula (*) preceding it). By employing the bigger dual $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of a bounded lattice $\mathbf{L}$ due to Allwein and Hartonas [1,2], we identify the order-isomorphism $\Phi^{GH}_{CL}\colon\ \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ which fixes $\mathbf{L}$ (Proposition 4.2 and Proposition 4.3). Then we reconcile the canonical extensions $\mathrm{C}(\mathbf{X}) = \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ based on Ploščica's representation and $\mathrm{GH}(\mathbf{L})$ based on the Gehrke–Harding approach; we identify the order-isomorphism $\Phi^{CX}_{GH}\colon\ \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ which fixes $\mathbf{L}$.

In Section 5 we first show that in the case of an arbitrary graph $\mathbf{X} = (X, E)$, the sets $\{J_x \mid x \in X\}$ and $\{M_y \mid y \in X\}$ defined by (**) via the polars $E^{\mathsf{C}}_{\triangleleft}$ and $E^{\mathsf{C}}_{\triangleright}$ are join- and meet-dense in the complete lattice $\mathrm{C}(\mathbf{X})$ (Proposition 5.1). By combining it with a basic result of Formal Concept Analysis (Theorem 5.2), it gives an alternative argument to the reconciliation of the lattices $\mathrm{C}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ as proven in Proposition 3.1. Then in the special case $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ we describe the maps J_x and M_y by (***) via the quasi-orders $\leqslant_1$ and $\leqslant_2$. Using the description (***) we finally prove that the maps J_x $(x \in X)$ are exactly the completely join-irreducible elements and the maps M_y are exactly the completely meet-irreducible elements of the canonical extension $\mathrm{C}(\mathbf{X})$ of $\mathbf{L}$ (Proposition 5.5). This is an analogue of the result of Gehrke and Harding in [8: Lemma 3.4].

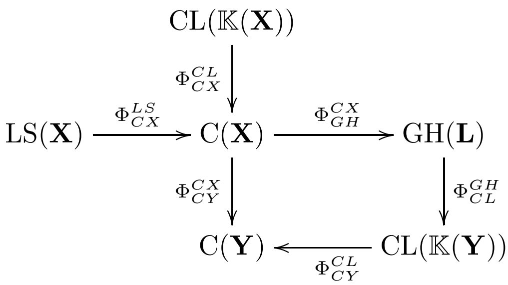

FIGURE 1. Reconciliation of the canonical extension constructions

In Figure 1 we provide a 'Reconciliation Diagram' which depicts the six different lattices studied in this paper which all serve as the canonical extension of an arbitrary bounded lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. Three of these lattices are defined via the

Ploščica dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ of $\mathbf{L}$: the lattices $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of MPE's, $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the concepts of the context $\mathbb{K}(\mathbf{X})$. Two of the lattices are defined via the Allwein–Hartonas dual graph $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of $\mathbf{L}$: the lattices $\mathrm{C}(\mathbf{Y}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ of MPE's and $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ of the concepts of the context $\mathbb{K}(\mathbf{Y})$. The lattice $\mathrm{GH}(\mathbf{L})$ comes from a Galois connection between $\wp(\mathrm{Filt}(\mathbf{L}))$ and $\wp(\mathrm{Idl}(\mathbf{L}))$. The diagram also depicts all the transition order-isomorphisms between the canonical extensions fixing the original lattice $\mathbf{L}$ that we mentioned above and whose establishment is the core of the present paper.

2. The canonical extension based on Ploščica's representation

A *completion* of a (bounded) lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ is defined to be a pair $(e, \mathbf{C})$ where $\mathbf{C}$ is a complete lattice and $e\colon \mathbf{L} \hookrightarrow \mathbf{C}$ is an embedding. An element of a completion $(e, \mathbf{C})$ of a (bounded) lattice $\mathbf{L}$ which is representable as a meet (join) of elements from $e(\mathbf{L})$ is called a *filter element* (*ideal element*); the sets of filter and ideal elements of $\mathbf{C}$ are denoted by $\mathbb{F}(\mathbf{C})$ and $\mathbb{I}(\mathbf{C})$, respectively. (We note that filter (ideal) elements are called closed (open) elements in the older literature.) A completion $(e, \mathbf{C})$ of $\mathbf{L}$ is said to be *dense* if every element of $\mathbf{C}$ is both a join of meets and a meet of joins of elements from $e(\mathbf{L})$; it is said to be *compact* if, for any sets $A \subseteq \mathbb{F}(\mathbf{C})$ and $B \subseteq \mathbb{I}(\mathbf{C})$ with $\bigwedge A \leqslant \bigvee B$, there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leqslant \bigvee B'$. (We note that, alternatively, the sets A, B in the definition of compactness above can be taken as arbitrary subsets of L.) A *canonical extension* of $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ has been defined as a dense and compact completion of $\mathbf{L}$ by Gehrke and Harding [8]. They showed that every bounded lattice $\mathbf{L}$ has a canonical extension and any two canonical extensions of $\mathbf{L}$ are isomorphic via an isomorphism that fixes the elements of $\mathbf{L}$.

The central idea in Ploščica's representation of bounded lattices [15] is the replacement of total maps by partial maps. Let $\boldsymbol{\mathcal{L}}$ be the category of all bounded lattices and bounded lattice homomorphisms. A partial map $f\colon \mathbf{L} \to \mathbf{K}$ between bounded lattices $\mathbf{L}$ and $\mathbf{K}$ is called a *partial homomorphism* if its domain is a 0,1-sublattice of $\mathbf{L}$ and the restriction $f{\restriction}_{\mathrm{dom}(f)}$ is an $\boldsymbol{\mathcal{L}}$-homomorphism. A partial homomorphism is said to be *maximal* if there is no partial homomorphism properly extending it; such a map is referred to as an MPH, for short. By Zorn's Lemma, every partial homomorphism can be extended to an MPH. For bounded lattices $\mathbf{L}$ and $\mathbf{K}$, we denote by $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \mathbf{K})$ the set of all MPH's from $\mathbf{L}$ to $\mathbf{K}$.

Let

$$\underline{\mathbf{2}} := \langle \{0,1\}; \vee, \wedge, 0, 1\rangle \quad \text{and} \quad \underset{\sim}{\mathbf{2}} := \langle \{0,1\}; \leqslant\rangle$$

denote, respectively, the two-element bounded lattice and the two-element ordered set with $0 < 1$. The topological structure $\underset{\sim}{\mathbf{2}}_{\mathcal{T}}$ is obtained by adding the discrete topology $\mathcal{T}$ to $\underset{\sim}{\mathbf{2}}$.

Following Ploščica [15], for any bounded lattice $\mathbf{L}$, the topological dual space of $\mathbf{L}$ is defined in the following way. We equip the set $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with the binary relation E defined by the rule

$$(f, g) \in E \quad \text{iff} \quad f(x) \leqslant g(x) \text{ for every } x \in \operatorname{dom}(f) \cap \operatorname{dom}(g);$$

we see that $(f, g) \in E$ if and only if $f^{-1}(1) \cap g^{-1}(0) = \emptyset$. When needed, we endow $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ with the topology $\mathcal{T}$ which has a subbasis of closed sets consisting of all the sets of the form $V_a = \{f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0\}$ and $W_a = \{f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1\}$, where $a \in L$. We let the dual of the lattice $\mathbf{L}$ be a graph with topology, $\mathrm{D}(\mathbf{L}) := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$. The topology of $\mathrm{D}(\mathbf{L})$ is T_1 and compact. If the lattice $\mathbf{L}$ is distributive, then $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) = \mathcal{L}(\mathbf{L}, \underline{\mathbf{2}})$, the relation E coincides with the pointwise partial order of maps and $\mathrm{D}(\mathbf{L})$ is the usual dual space of $\mathbf{L}$ in the Priestley duality [16].

In [4] we often deal with the dual of the lattice $\mathbf{L}$ without its topology, that is, considered as the graph $\mathrm{D}^{\flat}(\mathbf{L}) := (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. We first recall from [4] results concerning general graphs $\mathbf{X} = (X, E)$. By $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ we denote the set of all maximal partial E-preserving maps from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$.

Lemma 2.1. ([4: Lemma 2.1]) *Let $\mathbf{X} = (X, E)$ be a graph and φ a partial E-preserving map from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$. Then $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ if and only if*

(i) $\varphi^{-1}(0) = \{x \in X \mid \text{there is no } y \in \varphi^{-1}(1) \text{ with } (y, x) \in E\}$ *and*

(ii) $\varphi^{-1}(1) = \{x \in X \mid \text{there is no } y \in \varphi^{-1}(0) \text{ with } (x, y) \in E\}$.

The above lemma allows us to observe that for a graph $\mathbf{X} = (X, E)$ and $\varphi, \psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ we have

$$\varphi^{-1}(1) \subseteq \psi^{-1}(1) \iff \psi^{-1}(0) \subseteq \varphi^{-1}(0).$$

This observation is needed to motivate the definition of the right order on the set $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. The next result shows that the order yields a complete lattice.

Theorem 2.2. ([4: Theorem 2.3]) *Let $\mathbf{X} = (X, E)$ be a graph with $x \in X$ and let $\{\varphi_i \mid i \in I\} \subseteq \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. Then the set $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ ordered by the rule*

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is a complete lattice where joins and meets are calculated by

$$\Big(\bigwedge_{i\in I}\varphi_i\Big)(x)=\begin{cases}1 & \textit{if}\ \ x\in\bigcap\limits_{i\in I}\varphi_i^{-1}(1),\\ 0 & \textit{if there is no } y\in\bigcap\limits_{i\in I}\varphi_i^{-1}(1)\ \textit{with}\ (y,x)\in E;\end{cases}$$

$$\Big(\bigvee_{i\in I}\varphi_i\Big)(x)=\begin{cases}1 & \textit{if there is no } y\in\bigcap\limits_{i\in I}\varphi_i^{-1}(0)\ \textit{with}\ (x,y)\in E,\\ 0 & \textit{if}\ \ x\in\bigcap\limits_{i\in I}\varphi_i^{-1}(0).\end{cases}$$

We make the family $\mathcal{G}_{\mathfrak{T}}$ of graphs with topology into a category in the following way. A map $\varphi\colon (X_1,E_1,\tau_1)\to(X_2,E_2,\tau_2)$ between graphs with topology is called a $\mathcal{G}_{\mathfrak{T}}$*-morphism* if it preserves the binary relation and is continuous as a map from (X_1,τ_1) to (X_2,τ_2). A partial map $\varphi\colon (X_1,E_1,\tau_1)\to(X_2,E_2,\tau_2)$ is called a *partial* $\mathcal{G}_{\mathfrak{T}}$*-morphism* if its domain is a τ_1-closed subset of X_1 and the restriction of φ to its domain is a morphism. A partial $\mathcal{G}_{\mathfrak{T}}$-morphism is called maximal, or an MPM, for short, if there is no partial $\mathcal{G}_{\mathfrak{T}}$-morphism properly extending it. For a graph with topology, $\mathbf{X}_{\mathfrak{T}}=(X,E,\mathfrak{T})$, we denote by $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ the set of MPM's from $\mathbf{X}_{\mathfrak{T}}$ to $\underset{\sim}{\mathbf{2}}_{\mathfrak{T}}$.

Proposition 2.3. ([4: Proposition 2.5]) *Let $\mathbf{X}_{\mathfrak{T}}=(X,E,\mathfrak{T})$ be a graph equipped with a T_1-topology and $\mathbf{X}=(X,E)$ be its untopologised counterpart. Then $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})\subseteq\mathcal{G}^{\mathrm{mp}}(\mathbf{X},\underset{\sim}{\mathbf{2}})$.*

In addition to the result above, the set $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ can in general be considered as a subposet of the poset $\mathcal{G}^{\mathrm{mp}}(\mathbf{X},\underset{\sim}{\mathbf{2}})$, with the partial order on $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ given, as in [15], by $\varphi\leqslant\psi$ iff $\varphi^{-1}(1)\subseteq\psi^{-1}(1)$. If in particular $\mathbf{X}_{\mathfrak{T}}=\mathrm{D}(\mathbf{L})$ then it was shown in [15] (see (iii) of Proposition 2.4 below) that the partial order $\leqslant$ on $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ is a lattice order, and the lattice $(\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathbf{X}_{\mathfrak{T}},\underset{\sim}{\mathbf{2}}_{\mathfrak{T}}),\leqslant)$ is then clearly a sublattice of the lattice $(\mathcal{G}^{\mathrm{mp}}(\mathbf{X},\underset{\sim}{\mathbf{2}}),\leqslant)$ (see Theorem 2.2).

Ploščica's representation of bounded lattices is summed up as follows.

Proposition 2.4 ([15: Lemmas 1.2, 1.5 and Theorem 1.7])**.** *Let $\mathbf{L}$ be a bounded lattice and let $\mathrm{D}(\mathbf{L})=(\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L},\underline{\mathbf{2}}),E,\mathfrak{T})$ be the dual of $\mathbf{L}$. For $a\in L$, let the evaluation map $e_a\colon \mathrm{D}(\mathbf{L})\to\underset{\sim}{\mathbf{2}}_{\mathfrak{T}}$ be defined by*

$$e_a(f)=\begin{cases}f(a) & a\in\mathrm{dom}(f),\\ - & \textit{undefined otherwise}.\end{cases}$$

Then the following hold:

(i) *The map e_a is an element of $\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathrm{D}(\mathbf{L}),\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ for each $a\in L$.*

(ii) *Every $\varphi\in\mathcal{G}^{\mathrm{mp}}_{\mathfrak{T}}(\mathrm{D}(\mathbf{L}),\underset{\sim}{\mathbf{2}}_{\mathfrak{T}})$ is of the form e_a for some $a\in L$.*

(iii) *The map* $e_{\mathbf{L}}\colon \mathbf{L} \to \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ *given by evaluation,* $a \mapsto e_a$ $(a \in L)$*, is an isomorphism of* $\mathbf{L}$ *onto the lattice* $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$*, ordered by* $\varphi \leqslant \psi$ *if and only if* $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$*.*

Theorem 2.2 tells us that, for a graph $\mathbf{X} = (X, E)$, the ordered set $\mathrm{C}(\mathbf{X})$ is a complete lattice. Now from Proposition 2.4, combined with the fact that every $\varphi \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, and in particular every evaluation map e_a, is an element of $\mathcal{G}^{\mathrm{mp}}(\mathrm{D}^{\flat}(\mathbf{L}), \underset{\sim}{\mathbf{2}})$, by Proposition 2.3, the following result can be obtained:

Proposition 2.5. ([4: Proposition 3.2]) *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L})$*. Then* $(e, \mathrm{C}(\mathbf{X}))$ *is a completion of* $\mathbf{L}$*, where* $e\colon a \mapsto e_a$ $(a \in L)$*.*

Proposition 2.5 identifies a completion for any bounded lattice. This is constructed from the dual space of the lattice. In the case that the lattice is distributive this certainly does give the canonical extension as introduced by Gehrke and Jónsson [9]. In [4] we proved that this completion, based on Ploščica's representation, supplies a canonical extension for an arbitrary bounded lattice.

Theorem 2.6. ([4: Theorem 3.11]) *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$*. The lattice* $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *ordered by*

$$\varphi \leqslant \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$$

is the canonical extension of $\mathbf{L}$*.*

In [15: Section 2], Ploščica demonstrates how his dual representation for lattices relates to the topological representation due to Urquhart [17]. At the level of the dual spaces, the passage back and forth between Urquhart's dual representation and Ploščica's is set up by a bijection between maximal disjoint filter-ideal pairs in L (as employed by Urquhart) and $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$. Instead of carrying a single binary relation E, Urquhart's dual spaces are equipped with a pair of quasi-orders, $\leqslant_1$ and $\leqslant_2$. Interpreted in terms of MPH's, these two relations are defined on the set $\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ as follows:

$$f \leqslant_1 g \iff f^{-1}(1) \subseteq g^{-1}(1) \qquad \text{and} \qquad f \leqslant_2 g \iff f^{-1}(0) \subseteq g^{-1}(0).$$

These quasi-orders $\leqslant_1$ and $\leqslant_2$ prove to be a valuable ancillary tool for working with graphs of the form $\mathrm{D}^{\flat}(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$.

Lemma 2.7 ([15: Theorem 2.1])**.** *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ *and let* $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$*. Then*

(i) $(f, g) \in E \iff \big(\exists\, h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})\big)(f \leqslant_1 h$ *and* $g \leqslant_2 h)$.

(ii) $f \leqslant_2 g \iff \big(\forall\, h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})\big)\big((h, g) \notin E$ *or* $(h, f) \in E\big)$.

(iii) $f \leqslant_1 g \iff \big(\forall\, h \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})\big)\big((g, h) \notin E$ *or* $(f, h) \in E\big)$.

It is a consequence of (i) above that the relation $\leqslant_1$ is contained in E, and $\geqslant_2$ is contained in E.

Lemma 2.8. ([4: Lemmas 3.4, 3.5]) *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, *let* $\mathbf{X} = \mathrm{D}^{\flat}(\mathbf{L}) = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. *Then for* $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$

(i) *if* $f \notin \varphi^{-1}(0)$ *there exists* $g \in \varphi^{-1}(1)$ *such that* $f \leqslant_2 g$;

(ii) *if* $f \notin \varphi^{-1}(1)$ *there exists* $g \in \varphi^{-1}(0)$ *such that* $f \leqslant_1 g$;

(iii) *if* $f \leqslant_2 g$ *and* $\varphi(f) = 0$, *then* $\varphi(g) = 0$;

(iv) *if* $f \leqslant_1 g$ *and* $\varphi(f) = 1$, *then* $\varphi(g) = 1$.

To explore the canonical extension in more depth, we now bring into play another aspect of Urquhart's duality. We recall that Urquhart constructs a lattice isomorphic to $\mathbf{L}$ by means of a Galois connection on the subsets of the dual space of $\mathbf{L}$. In terms of MPH's, this Galois connection is defined as follows: for a set $Y \subseteq \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, let

$$\ell(Y) := \big\{ f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid \text{ there is no } g \in Y \text{ with } f \leqslant_1 g \big\};$$
$$r(Y) := \big\{ f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid \text{ there is no } g \in Y \text{ with } f \leqslant_2 g \big\}.$$

The set Y is called *stable* if $Y = \ell r(Y)$ and Y is called *doubly closed* if both Y and $r(Y)$ are (topologically) closed. We note that Urquhart refers only to stable sets as defined above. However, we find it useful to follow the terminology of Allwein and Hartonas [2] and say that a set Y is *ℓ-stable* if $Y = \ell r(Y)$ and *r-stable* if $Y = r\ell(Y)$.

Ploščica [15: Theorem 2.2] shows that, for a lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and $Y \subseteq \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, the subset Y is a doubly closed ℓ-stable set if and only if $Y = \varphi^{-1}(1)$ for some $\varphi \in \mathfrak{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathrm{D}(\mathbf{L}), \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$. Gehrke and Harding assert without proof that the canonical extension of a bounded lattice is isomorphic to the complete lattice of ℓ-stable subsets of Urquhart's dual space [8: Remark 2.10].

The first new result in this paper shows directly that the ℓ-stable sets above do correspond to the maximal partial E-preserving maps. (We remark that the proof is similar to that used to prove [15: Theorem 2.2].)

Theorem 2.9. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *For any* $Y \subseteq X$ *the following are equivalent:*

(1) Y *is an ℓ-stable set;*

(2) $Y = \varphi^{-1}(1)$ *for some* $\varphi \in \mathfrak{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$.

P r o o f. Consider an ℓ-stable subset $Y \subseteq X$, and define φ_Y as follows:

$$\varphi_Y(f) = \begin{cases} 1 & \text{if } f \in Y, \\ 0 & \text{if } f \in r(Y). \end{cases}$$

To show that φ_Y preserves E, let $f, g \in \operatorname{dom}(\varphi_Y)$ and suppose $(f, g) \in E$ but $\varphi_Y(f) = 1$ and $\varphi_Y(g) = 0$. From $\varphi_Y(f) = 1$ we get $f \in Y = \ell r(Y)$ and $\varphi_Y(g) = 0$ gives $g \in r(Y)$. Lemma 2.7(i) gives us that there exists an MPH h such that $f \leqslant_1 h$ and $g \leqslant_2 h$. Now $f \in \ell r(Y)$ means that for all $j \in r(Y)$ we have $f \nleqslant_1 j$ and hence $h \notin r(Y)$. This implies that there exists $k \in Y$ such that $h \leqslant_2 k$. The transitivity of $\leqslant_2$ then implies that $g \leqslant_2 k$ and hence $g \notin r(Y)$, a contradiction. In order to see that φ_Y is maximal, suppose that $\operatorname{dom} \varphi_Y \subseteq \operatorname{dom} \psi$ for some $\psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. It is clear that $Y \subseteq \psi^{-1}(1)$ and $r(Y) \subseteq \psi^{-1}(0)$. If $f \in \psi^{-1}(1)$ and $g \in \psi^{-1}(0)$, then $(f, g) \notin E$ and hence $f \nleqslant_1 g$ and $g \nleqslant_2 f$. Thus any $f \in \psi^{-1}(1)$ must be in $\ell(\psi^{-1}(0))$ and similarly $\psi^{-1}(0) \subseteq r(\psi^{-1}(1))$. We then get $\psi^{-1}(1) \subseteq \ell(\psi^{-1}(0)) \subseteq \ell r(Y) = Y$ and hence $Y = \psi^{-1}(1)$. We also have $r(Y) = \psi^{-1}(0)$ and so $\psi = \varphi_Y$, showing that φ_Y is maximal.

Given $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$, define $Y = \varphi^{-1}(1)$. We will show that $r(Y) = \varphi^{-1}(0)$ and hence that Y is an ℓ-stable set. If $f \in \varphi^{-1}(0)$ then, by Lemma 2.1, for all $g \in \varphi^{-1}(1)$ we have that $(g, f) \notin E$. Since $f \leqslant_2 g$ implies that $(g, f) \in E$, we have that $f \nleqslant_2 g$ and hence $f \in r(Y) = \{h \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2}) \mid (\forall k \in \varphi^{-1}(1))(h \nleqslant_2 k)\}$. Thus $\varphi^{-1}(0) \subseteq r(Y)$. If we suppose that $f \notin \varphi^{-1}(0)$, then by Lemma 2.8(i) we have that there exists $g \in \varphi^{-1}(1)$ such that $f \leqslant_2 g$. Thus $f \notin r(\varphi^{-1}(1)) = r(Y)$ and hence $r(Y) = \varphi^{-1}(0)$. We can similarly show that $\ell(\varphi^{-1}(0)) = \varphi^{-1}(1)$ and so conclude that $Y = \ell r(Y)$. □

Let $\mathrm{LS}(\mathbf{X})$ denote the complete lattice of ℓ-stable subsets of $\mathbf{X}$, ordered by inclusion. We define a map $\Phi^{LS}_{CX} \colon \mathrm{LS}(\mathbf{X}) \to \mathrm{C}(\mathbf{X})$ by $Y \mapsto \varphi_Y$. The previous theorem shows that it is a well-defined bijection. The next result which is an easy consequence of the previous theorem shows that Φ^{LS}_{CX} is an order-isomorphism fixing $\mathbf{L}$, and thus the lattice $\mathrm{LS}(\mathbf{X})$ is the canonical extension of $\mathbf{L}$ via the embedding $a \mapsto e_a^{-1}(1)$ $(a \in L)$.

Corollary 2.10. *Let $\mathbf{L}$ be a bounded lattice and $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2}), E)$.*

(i) *The lattice $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets of $\mathbf{X}$ is order-isomorphic to the canonical extension $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ of $\mathbf{L}$ via the map Φ^{LS}_{CX}.*

(ii) *The lattice $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets of $\mathbf{X}$ is the canonical extension of $\mathbf{L}$.*

Proof. Part (i) follows from Theorem 2.9 and the fact that for any ℓ-stable subsets Y, Z of $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{2})$, clearly $Y \subseteq Z$ if and only if $\varphi_Y^{-1}(1) \subseteq \varphi_Z^{-1}(1)$.

For part (ii) it remains to show that Φ^{LS}_{CX} preserves the embedded copies of $\mathbf{L}$ in each of the complete lattices $\mathrm{LS}(\mathbf{X})$ and $\mathrm{C}(\mathbf{X})$.

From Proposition 2.5 we know that **L** is embedded in C(**X**) via the embedding $e\colon a \mapsto e_a$ $(a \in L)$, while from Ploščica's result [15: Theorem 2.2] mentioned above it follows that $a \mapsto e_a^{-1}(1)$ $(a \in L)$ is an embedding of **L** into LS(**X**). Now we see that $\Phi_{CX}^{LS}(e_a^{-1}(1)) = e_a$. □

3. The canonical extension as a concept lattice

In this section we first show that the complete lattice C(**X**), as constructed for the graph $\mathbf{X} = (X, E)$ in Theorem 2.2, is isomorphic to the concept lattice, in the framework of Formal Concept Analysis (FCA, for short), of a specific context associated to the complement of the relation E. A similar representation of lattices, though only in the finite case, was used by Wille in [18]. The addition of topology to a generalization of Wille's contexts was used later to obtain a representation theorem for arbitrary lattices by Hartung [12].

We begin with some basics from Formal Concept Analysis (the main source is [7], but for our purposes, [5: Chapter 3] is sufficient). A *context* **K** is a triple (O, P, I) such that O and P are sets and $I \subseteq O \times P$. The elements of O are called *objects* and the elements of P are called *attributes* meaning properties of the objects. Then so-called *polar maps* (*polars*, for short) of the relation I set up a Galois connection

$$I_{\triangleright}\colon (\wp(O), \subseteq) \to (\wp(P), \supseteq) \quad \text{and} \quad I_{\triangleleft}\colon (\wp(P), \supseteq) \to (\wp(O), \subseteq)$$

defined by

$$I_{\triangleright}(U) = \big\{a \in P \mid (\forall o \in U)\big((o,a) \in I\big)\big\} \qquad \text{and}$$
$$I_{\triangleleft}(V) = \big\{o \in O \mid (\forall a \in V)\big((o,a) \in I\big)\big\}.$$

Hence one can associate to the context **K** a complete lattice

$$\mathrm{CL}(\mathbf{K}) = \big\{A \subseteq O \mid (I_{\triangleleft} \circ I_{\triangleright})(A) = A\big\}$$

of Galois-closed sets. In the FCA literature the complete lattice CL(**K**) is called the *concept lattice of the context* **K**. Its elements are usually referred to as pairs (A, B) where $I_{\triangleright}(A) = B$ and $I_{\triangleleft}(B) = A$. Such a pair is called a *concept* where A is referred to as the *extent* and B as the *intent* of the concept.

For our graphs $\mathbf{X} = (X, E)$ we shall consider the context

$$\mathbb{K}(\mathbf{X}) := (X, X, E^{\complement})$$

where the base set X of the graph **X** stands for both objects O and attributes P and the relation I is the complement of the graph relation: $E^{\complement} = (X \times X)\backslash E$.

We define a Galois connection via polars

$$E^{\mathsf{C}}_{\triangleright}\colon (\wp(X),\subseteq) \to (\wp(X),\supseteq) \qquad \text{and} \qquad E^{\mathsf{C}}_{\triangleleft}\colon (\wp(X),\supseteq) \to (\wp(X),\subseteq)$$

as above by

$$E^{\mathsf{C}}_{\triangleright}(Y) = \big\{x \in X \mid (\forall\, y \in Y)\big((y,x) \notin E\big)\big\},$$
$$E^{\mathsf{C}}_{\triangleleft}(Y) = \big\{z \in X \mid (\forall\, y \in Y)\big((z,y) \notin E\big)\big\}.$$

The concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X}) = (X, X, E^{\mathsf{C}})$, given by

$$\mathrm{CL}(\mathbb{K}(\mathbf{X})) = \big\{Y \subseteq X \mid (E^{\mathsf{C}}_{\triangleleft} \circ E^{\mathsf{C}}_{\triangleright})(Y) = Y\big\},$$

ordered by inclusion, will now be shown to be order-isomorphic to the lattice $\mathrm{C}(\mathbf{X})$ coming from the graph $\mathbf{X}$. Let $Y \subseteq X$ be an element of $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$. We define a map $\Phi^{CL}_{CX}\colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X}), Y \mapsto \varphi_Y$ where $\varphi_Y\colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ is defined by

$$\varphi_Y(x) = \begin{cases} 1 & \text{if } x \in Y,\\ 0 & \text{if } x \in E^{\mathsf{C}}_{\triangleright}(Y). \end{cases}$$

We emphasize that the first result below does not rely on the graph coming from a lattice. Rather, it establishes in the general case the relationship between MPE's and Galois-closed sets under the E^{C} relation. The subsequent corollary highlights the fact that when the graph $\mathbf{X}$ does come from some lattice $\mathbf{L} \in \boldsymbol{\mathcal{L}}$, then the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ described above is the canonical extension of the lattice $\mathbf{L}$.

Proposition 3.1. *Let $\mathbf{X} = (X, E)$ be a graph. The concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the context $\mathbb{K}(\mathbf{X})$ is order-isomorphic to the lattice $\mathrm{C}(\mathbf{X})$ via the map Φ^{CL}_{CX}.*

Proof. We start by showing that the map Φ^{CL}_{CX} is well-defined. To this end, let $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and consider $y \in \varphi_Y^{-1}(1)$ and $x \in \varphi_Y^{-1}(0)$. Since $y \in Y$, and $x \in E^{\mathsf{C}}_{\triangleright}(Y)$, we have $(y,x) \notin E$ and hence φ_Y is E-preserving. If $\operatorname{dom}\varphi_Y \subsetneq \operatorname{dom}\psi$ and ψ extends φ_Y for some $\psi \in \mathrm{C}(\mathbf{X})$, then there exists $x \in \operatorname{dom}(\psi) \setminus \operatorname{dom}(\varphi_Y)$. Since $x \notin E^{\mathsf{C}}_{\triangleright}(Y)$, there exists $y \in Y = \varphi_Y^{-1}(1)$ such that $(y,x) \in E$. Thus $x \notin \psi^{-1}(0)$ and so $\psi(x) = 1$. Now $x \notin \varphi_Y^{-1}(1)$ means that $x \notin Y = (E^{\mathsf{C}}_{\triangleleft} \circ E^{\mathsf{C}}_{\triangleright})(Y)$. This implies that there exists $z \in E^{\mathsf{C}}_{\triangleright}(Y)$ such that $(x,z) \in E$. But, $z \in \psi^{-1}(0)$, which contradicts ψ being E-preserving and hence φ_Y is maximal.

For $\psi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, let $Y = \psi^{-1}(1)$ and by part (i) of Lemma 2.1 we have $\psi^{-1}(0) = E^{\mathsf{C}}_{\triangleright}(\psi^{-1}(1)) = E^{\mathsf{C}}_{\triangleright}(Y)$. Applying part (ii) of Lemma 2.1, and using the definition of $E^{\mathsf{C}}_{\triangleright}$ gives us $Y = \psi^{-1}(1) = (E^{\mathsf{C}}_{\triangleleft} \circ E^{\mathsf{C}}_{\triangleright})(Y)$ and hence $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$.

Thus we have a bijective correspondence between elements of $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and MPE's from $\mathbf{X}$ into $\underset{\sim}{\mathbf{2}}$. We can further conclude that these two complete lattices are order-isomorphic. For $Y, Z \subseteq X$, clearly $Y \subseteq Z$ in $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ if and only if $\varphi_Y \leqslant \varphi_Z$ in $\mathrm{C}(\mathbf{X})$. $\square$

Corollary 3.2. *Let* $\mathbf{L}$ *be a bounded lattice. Let* $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\mathbb{K}(\mathbf{X}) = (X, X, E^{\complement})$ *be the context associated to* $\mathrm{E}^{\complement}$. *Then the concept lattice* $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ *of the context* $\mathbb{K}(\mathbf{X})$ *is the canonical extension of the lattice* $\mathbf{L}$ *via the embedding* $a \mapsto e_a^{-1}(1)$ $(a \in L)$.

P r o o f. We shall show that the order-isomorphism $\Phi^{CL}_{CX} \colon \mathrm{CL}(\mathbb{K}(\mathbf{X})) \to \mathrm{C}(\mathbf{X})$ preserves the embedded copies of $\mathbf{L}$ in each of the complete lattices $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ and $\mathrm{C}(\mathbf{X})$.

From Proposition 2.5 we know that $\mathbf{L}$ is embedded in $\mathrm{C}(\mathbf{X})$ via the embedding $e \colon a \mapsto e_a$ $(a \in L)$, while from Hartung's work [12: Theorem 2.1.8] it follows that $a \mapsto e_a^{-1}(1) = \{ f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid a \in f^{-1}(1) \}$ $(a \in L)$ is an embedding of $\mathbf{L}$ into $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$. It is clear that the order-isomorphism Φ^{CL}_{CX} maps $e_a^{-1}(1)$ into e_a $(a \in L)$. □

So far we have presented three different approaches to the construction of the canonical extension of a bounded lattice $\mathbf{L}$: via its dual graph $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ and the lattices $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ of MPE's, $\mathrm{LS}(\mathbf{X})$ of ℓ-stable subsets and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ of the concepts of the context $\mathbb{K}(\mathbf{X})$. The next result summarizes the equivalence of these approaches.

Theorem 3.3. *Let* $\mathbf{L} \in \mathcal{L}$. *Let* $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and let* $\mathbb{K}(\mathbf{X}) = (X, X, E^{\complement})$ *be the context associated to* $\mathbf{X}$. *Then for* $Y \subseteq X$, *the statements in* (i) *are equivalent.*

(i) (1) $Y = \varphi^{-1}(1)$ *for some* $\varphi \in \mathrm{C}(\mathbf{X})$;
(2) $Y \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$;
(3) $Y = E^{\complement}_{\triangleleft}(B)$ *for some* $B \subseteq X$;
(4) $Y \in \mathrm{LS}(\mathbf{X})$, *that is,* Y *is an* ℓ*-stable set;*
(5) $Y = \ell(A)$ *for some* $A \subseteq X$.

Dually, for $Z \subseteq X$, *the statements in* (ii) *are equivalent.*

(ii) (1) $Z = \varphi^{-1}(0)$ *for some* $\varphi \in \mathrm{C}(\mathbf{X})$;
(2) $E^{\complement}_{\triangleright}(Z) \in \mathrm{CL}(\mathbb{K}(\mathbf{X}))$;
(3) $Z = E^{\complement}_{\triangleright}(B)$ *for some* $B \subseteq X$;
(4) Z *is an* r*-stable set;*
(5) $Z = r(A)$ *for some* $A \subseteq X$.

P r o o f. For part (i), the equivalence of (1) and (2) is the result of Proposition 3.1 above. The equivalence of (1) and (4) is the result of Theorem 2.9. The equivalences of (2) and (3), and of (4) and (5), are well known from the theory of Galois connections. □

4. The canonical extension based on the Gehrke–Harding approach

The original construction of a canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L}$ by Gehrke and Harding [8: Proposition 2.6] uses a Galois connection between $\wp(\mathrm{Filt}(\mathbf{L}))$ and $\wp(\mathrm{Idl}(\mathbf{L}))$. The relation $(F, I) \in R$ iff $F \cap I \neq \emptyset$ gives rise to the Galois connection $R_\triangleright \colon \wp(\mathrm{Filt}(\mathbf{L})) \to \wp(\mathrm{Idl}(\mathbf{L}))$ and $R_\triangleleft \colon \wp(\mathrm{Idl}(\mathbf{L})) \to \wp(\mathrm{Filt}(\mathbf{L}))$ where the polars are given for $A \subseteq \mathrm{Filt}(\mathbf{L})$ and $B \subseteq \mathrm{Idl}(\mathbf{L})$ by

$$R_\triangleright(A) = \bigl\{ I \in \mathrm{Idl}(\mathbf{L}) \mid (\forall\, F \in A)\bigl((F, I) \in R\bigr)\bigr\}$$

and

$$R_\triangleleft(B) = \bigl\{ F \in \mathrm{Filt}(\mathbf{L}) \mid (\forall\, I \in B)\bigl((F, I) \in R\bigr)\bigr\}.$$

The Galois-closed subsets of $\mathrm{Filt}(\mathbf{L})$, $\bigl\{A \subseteq \mathrm{Filt}(\mathbf{L}) \mid A = (R_\triangleleft \circ R_\triangleright)(A)\bigr\}$, ordered by inclusion, form the canonical extension of $\mathbf{L}$ [8]. To distinguish it from the previous constructions, we will denote it using the first letters of the surnames of its inventors by $\mathrm{GH}(\mathbf{L})$. The lattice $\mathbf{L}$ is embedded in $\mathrm{GH}(\mathbf{L})$ via the embedding $a \mapsto A_a := \bigl\{F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F\bigr\}$.

In [4] we applied a duality theorem of Allwein and Hartonas [1, 2] in which they used a bigger dual space of a bounded lattice described in terms of disjoint filter-ideal pairs. This set of disjoint pairs of subsets of $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ is in bijective correspondence with the set of partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$ where the preimage of $\{1\}$ is a filter and the preimage of $\{0\}$ is an ideal. As mentioned in Section 1, we introduced these in [4] as *special* partial homomorphisms, or SPH's for short, and denoted their set by $\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$. Also, we equipped this set with the same binary relation E, that is, for $f, g \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$, we say $(f, g) \in E$ if and only if $f^{-1}(1) \cap g^{-1}(0) = \emptyset$.

The following proposition from [4] shows that the canonical extension constructed from the graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ of maximal partial homomorphisms is order-isomorphic to the complete lattice of MPE's from the graph $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of special partial homomorphisms into $\underset{\sim}{\mathbf{2}}$. To present it, we recall that a map $\Phi^{CX}_{CY} \colon \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \to \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ is defined for $\varphi \in \boldsymbol{\mathcal{G}}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ and for $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$ by

$$(\Phi^{CX}_{CY}(\varphi))(f) = \begin{cases} 1 & \text{if } \bigl(\forall\, g \in \varphi^{-1}(0)\bigr)\bigl((f, g) \notin E_Y\bigr), \\ 0 & \text{if } (\forall\, h)\bigl[\bigl(\forall\, g \in \varphi^{-1}(0)\bigr)\bigl((h, g) \notin E_Y \implies (h, f) \notin E_Y\bigr)\bigr], \\ - & \text{otherwise.} \end{cases} \tag{$*$}$$

Proposition 4.1. ([4: Theorem 4.7]) *Let* $\mathbf{L}$ *be a bounded lattice. Further, consider the graphs* $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E_X)$ *and* $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$. *Then* $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ *is order-isomorphic to* $\mathcal{G}^{\mathrm{mp}}(\mathbf{Y}, \underset{\sim}{\mathbf{2}})$ *via the isomorphism* Φ^{CX}_{CY} *defined in* (*). *Moreover,* Φ^{CX}_{CY} *restricts to a bijection between the maps defined by evaluation.*

Given a bounded lattice $\mathbf{L}$, the canonical extension $\mathbf{L}^\delta$ is unique up to an isomorphism which fixes $\mathbf{L}$ [8: Proposition 2.7]. Thus one can conclude from Proposition 4.1 above that the complete lattice of MPE's from $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ into $\underset{\sim}{\mathbf{2}}$ is also a canonical extension of the bounded lattice $\mathbf{L}$.

We now use the bigger dual $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ of a bounded lattice $\mathbf{L}$. For the complement of this graph relation, $E_Y^{\complement} = (Y \times Y)\backslash E_Y$, on the bigger dual graph $\mathbf{Y}$, we shall consider the context $\mathbb{K}(\mathbf{Y}) := (Y, Y, E_Y^{\complement})$ and the concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ of this context as defined in the previous section. Our first result in this section reconciles this concept lattice $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ with the canonical extension $\mathrm{GH}(\mathbf{L})$ of $\mathbf{L}$ by Gehrke–Harding.

Proposition 4.2. *Let* $\mathbf{L}$ *be a bounded lattice and let* $\mathbf{Y} = (\mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$. *Then the complete lattice* $\mathrm{GH}(\mathbf{L}) = \{A \subseteq \mathrm{Filt}(\mathbf{L}) \mid A = (R_\triangleleft \circ R_\triangleright)(A)\}$ *is order-isomorphic to the concept lattice* $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ *via the following isomorphism*

$$\Phi^{GH}_{CL} \colon \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y})) : \quad A \mapsto \{f \in \mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f^{-1}(1) \in A\}.$$

Proof. In the proof we denote Φ^{GH}_{CL} simply by Φ and use $E_\triangleright^{\complement}$ and $E_\triangleleft^{\complement}$ for the polars coming from $E_Y^{\complement}$. First we observe that for any $g \in \mathcal{L}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$,

$$\begin{aligned} g \in E_\triangleright^{\complement}(\Phi(A)) &\iff \big(\forall h \in \Phi(A)\big)\big((h,g) \notin E_Y\big) \\ &\iff \big(\forall h \in \Phi(A)\big)\big(h^{-1}(1) \cap g^{-1}(0) \neq \emptyset\big) \\ &\iff (\forall F \in A)\big(F \cap g^{-1}(0) \neq \emptyset\big) \\ &\iff g^{-1}(0) \in R_\triangleright(A). \end{aligned}$$

We have $\Phi(A) \in \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ since

$$\begin{aligned} f \in (E_\triangleleft^{\complement} \circ E_\triangleright^{\complement})(\Phi(A)) &\iff \big(\forall g \in E_\triangleright^{\complement}(\Phi(A))\big)\big((f,g) \notin E_Y\big) \\ &\iff \big(\forall g \in E_\triangleright^{\complement}(\Phi(A))\big)\big(f^{-1}(1) \cap g^{-1}(0) \neq \emptyset\big) \\ &\iff \big(\forall I \in R_\triangleright(A)\big)\big(f^{-1}(1) \cap I \neq \emptyset\big) \\ &\iff f^{-1}(1) \in (R_\triangleleft \circ R_\triangleright)(A) = A \\ &\iff f \in \Phi(A). \end{aligned}$$

It is clear that Φ is 1-1. To see that Φ is onto, for any $Z \in \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$, we consider the set $A = \{F \in \mathrm{Filt}(\mathbf{L}) \mid F = f^{-1}(1) \text{ for some } f \in Z\}$. It is easy to see that $\Phi(A) = Z$. Finally we show that $A \in \mathrm{GH}(\mathbf{L})$ by using similar arguments as above: for any $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$,

$$\begin{aligned} f^{-1}(1) \in (R_{\triangleleft} \circ R_{\triangleright})(A) &\iff \big(\forall I \in R_{\triangleright}(A)\big)\big(f^{-1}(1) \cap I \neq \emptyset\big) \\ &\iff \big(\forall g \in E_{\triangleright}^{\complement}(\Phi(A))\big)\big(f^{-1}(1) \cap g^{-1}(0) \neq \emptyset\big) \\ &\iff \big(\forall g \in E_{\triangleright}^{\complement}(Z)\big)\big(f^{-1}(1) \cap g^{-1}(0) \neq \emptyset\big) \\ &\iff \big(\forall g \in E_{\triangleright}^{\complement}(Z)\big)\big((f,g) \notin E_Y\big) \\ &\iff f \in (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(Z) = Z \\ &\iff f^{-1}(1) \in A. \end{aligned}$$

For $A, A' \subseteq \mathrm{Filt}(\mathbf{L})$ we clearly have $A \subseteq A'$ if and only if $\Phi(A) \subseteq \Phi(A')$, so Φ is an order-isomorphism. □

As before, we need to confirm that this isomorphism restricts to the representation of the lattice.

Proposition 4.3. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{Y} = (\boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}), E_Y)$ *and* $\mathbb{K}(\mathbf{Y}) := (Y, Y, E_Y^{\complement})$. *Then the order-isomorphism* $\Phi_{CL}^{GH} \colon \mathrm{GH}(\mathbf{L}) \to \mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ *preserves the embedded copies of* $\mathbf{L}$ *in each of the complete lattices.*

P r o o f. Consider the embeddings $a \mapsto A_a := \big\{F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F\big\}$ of $\mathbf{L}$ into $\mathrm{GH}(\mathbf{L})$ and $a \mapsto \bar{e}_a^{-1}(1)$ of $\mathbf{L}$ into $\mathrm{CL}(\mathbb{K}(\mathbf{Y}))$ $(a \in L)$. Now for $\Phi := \Phi_{CL}^{GH}$ we have

$$\begin{aligned} \Phi(A_a) &= \big\{f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f^{-1}(1) \in A_a\big\} \\ &= \big\{f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid a \in f^{-1}(1)\big\} \\ &= \big\{f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1\big\} \\ &= \bar{e}_a^{-1}(1). \end{aligned}$$

□

We recall that for $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and its Ploščica dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, we have $a \in L$ embedded as e_a in $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ while by the Gehrke–Harding approach $a \in L$ is embedded as A_a in $\mathrm{GH}(\mathbf{L})$. We have

$$\begin{aligned} e_a^{-1}(1) &= \big\{f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1\big\}, \\ e_a^{-1}(0) &= \big\{f \in \boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0\big\}, \\ A_a &= \big\{F \in \mathrm{Filt}(\mathbf{L}) \mid a \in F\big\}. \end{aligned}$$

We remember that elements of $\mathrm{C}(\mathbf{X})$ are MPE's from $\mathbf{X}$ to $\underset{\sim}{\mathbf{2}}$. Now, we want to assign to each MPE a set of filters of $\mathbf{L}$. In the search for a mapping

$\Phi^{CX}_{GH}\colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ such that $\Phi^{CX}_{GH}(e_a) = A_a$, we require that every filter F in A_a must have the property that $a \in F$. We now extend this requirement to arbitrary elements of $\mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. For $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ we define a subset $\Phi(\varphi) \subseteq \mathrm{Filt}(\mathbf{L})$ by

$$\Phi(\varphi) := \big\{F \in \mathrm{Filt}(\mathbf{L}) \mid \big(\forall f \in \varphi^{-1}(0)\big)\big(F \cap f^{-1}(0) \neq \emptyset\big)\big\}.$$

The proof that the map $\Phi^{CX}_{GH}\colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ is well-defined follows easily from the definition. To show that Φ^{CX}_{GH} is an order isomorphism from $\mathrm{C}(\mathbf{X})$ to $\mathrm{GH}(\mathbf{L})$, we will make use of the existing isomorphisms.

Theorem 4.4. *Let* $\mathbf{L} \in \boldsymbol{\mathcal{L}}$. *Let* $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ *and* $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$. *Then the map* $\Phi^{CX}_{GH}\colon \mathrm{C}(\mathbf{X}) \to \mathrm{GH}(\mathbf{L})$ *given for* $\varphi \in \mathrm{C}(\mathbf{X})$ *by*

$$\Phi(\varphi) = \big\{F \in \mathrm{Filt}(\mathbf{L}) \mid \big(\forall f \in \varphi^{-1}(0)\big)\big(F \cap f^{-1}(0) \neq \emptyset\big)\big\}$$

is an order-isomorphism. Further, it preserves the embedded copies of $\mathbf{L}$ *in the lattices* $\mathrm{C}(\mathbf{X})$ *and* $\mathrm{GH}(\mathbf{L})$.

Proof. We will show that $\Phi^{CX}_{GH} = \big(\Phi^{GH}_{CL}\big)^{-1} \circ \big(\Phi^{CL}_{CY}\big)^{-1} \circ \Phi^{CX}_{CY}$. To do this we will show that for every $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ we have

$$\Phi^{CX}_{CY}(\varphi) = \Phi^{CL}_{CY}\big(\Phi^{GH}_{CL}(A_\varphi)\big)$$

where $A_\varphi = \big\{F \in \mathrm{Filt}(\mathbf{L}) \mid \big(\forall f \in \varphi^{-1}(0)\big)\big(F \cap f^{-1}(0) \neq \emptyset\big)\big\}$. For $f \in \boldsymbol{\mathcal{L}}^{\mathrm{sp}}(\mathbf{L}, \underline{\mathbf{2}})$,

$$\begin{aligned}
\Big(\Phi^{CL}_{CY}\big(\Phi^{GH}_{CL}(A_\varphi)\big)\Big)(f) = 1 &\iff f \in \big(\Phi^{GH}_{CL}\big)(A_\varphi)\\
&\iff f^{-1}(1) \in A_\varphi\\
&\iff \big(\forall g \in \varphi^{-1}(0)\big)\big(f^{-1}(1) \cap g^{-1}(0) \neq \emptyset\big)\\
&\iff \big(\Phi^{CX}_{CY}\big)(\varphi)(f) = 1.
\end{aligned}$$

□

5. Completely join-irreducible and completely meet-irreducible elements of $\mathrm{C}(\mathbf{X})$

Completely join-irreducible and completely meet-irreducible elements play an important role in the theory of canonical extensions. When the lattice-based algebra represents the formula algebra of a logic, these elements of the canonical extension are used as the elements of the frames which provide relational semantics for the logic. We begin by proving a result regarding join- and meet-dense subsets of $\mathrm{C}(\mathbf{X})$ which applies to the general case of arbitrary graphs.

Proposition 5.1. *Let* $\mathbf{X} = (X, E)$ *be any graph and define maps* $J_x \colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ *and* $M_y \colon \mathbf{X} \to \underset{\sim}{\mathbf{2}}$ *by*

$$J_x(z) = \begin{cases} 1 & \text{if } z \in (E_{\triangleleft}^{\complement} \circ E_{\triangleright}^{\complement})(\{x\}) \\ 0 & \text{if } z \in (E_{\triangleright}^{\complement})(\{x\}), \end{cases} \qquad M_y(z) = \begin{cases} 1 & \text{if } z \in (E_{\triangleleft}^{\complement})(\{y\}), \\ 0 & \text{if } z \in (E_{\triangleright}^{\complement} \circ E_{\triangleleft}^{\complement})(\{y\}). \end{cases} \tag{$**$}$$

for $z \in X$. *Then the set* $\{J_x \mid x \in X\}$ *is join-dense in* $\mathrm{C}(\mathbf{X})$ *and the set* $\{M_y \mid y \in X\}$ *is meet-dense in* $\mathrm{C}(\mathbf{X})$.

P r o o f. The fact that $J_x, M_y \in \mathrm{C}(\mathbf{X})$ for any $x, y \in X$ follows from Lemma 2.1. We will show that for any $\varphi \in \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ we have $\varphi = \bigvee\{J_x \mid J_x \leqslant \varphi\}$. We note that by definition $\bigvee\{J_x \mid J_x \leqslant \varphi\} \leqslant \varphi$. For $x \in X$ we have

$$\begin{aligned} J_x \leqslant \varphi &\iff J_x^{-1}(1) \subseteq \varphi^{-1}(1) \\ &\iff \varphi^{-1}(0) \subseteq J_x^{-1}(0) = E_{\triangleright}^{\complement}(\{x\}) \\ &\iff \big(\forall z \in \varphi^{-1}(0)\big)\big((x, z) \notin E\big) \\ &\iff x \in \varphi^{-1}(1). \end{aligned}$$

Further, we see that

$$\begin{aligned} \varphi \leqslant \bigvee\{J_x \mid J_x \leqslant \varphi\} &\iff \varphi^{-1}(1) \subseteq \Big(\bigvee\{J_x \mid J_x \leqslant \varphi\}\Big)^{-1}(1) \\ &\iff \big(\bigvee\{J_x \mid J_x \leqslant \varphi\}\big)^{-1}(0) \subseteq \varphi^{-1}(0) \\ &\iff \bigcap\{J_x^{-1}(0) \mid J_x \leqslant \varphi\} \subseteq \varphi^{-1}(0) \\ &\iff \bigcap\{J_x^{-1}(0) \mid x \in \varphi^{-1}(1)\} \subseteq \varphi^{-1}(0). \end{aligned}$$

Now, if $y \in \bigcap\{J_x^{-1}(0) \mid x \in \varphi^{-1}(1)\}$ we have that $(x, y) \notin E$ for all $x \in \varphi^{-1}(1)$ and hence, by Lemma 2.1, $y \in \varphi^{-1}(0)$. This shows $\varphi = \bigvee\{J_x \mid J_x \leqslant \varphi\}$. The proof that $\bigwedge\{M_y \mid \varphi \leqslant M_y\}$ is similar. □

Now we can set up maps $\mathbb{J} \colon \mathbf{X} \to \mathrm{C}(\mathbf{X})$ and $\mathbb{M} \colon \mathbf{X} \to \mathrm{C}(\mathbf{X})$ defined by

$$\mathbb{J} \colon x \mapsto J_x \quad \text{and} \quad \mathbb{M} \colon y \mapsto M_y.$$

These satisfy the properties of the maps γ and μ in the following basic result of FCA (noting that $\mathbb{J}(x) \leqslant \mathbb{M}(y)$ if and only if $(x, y) \notin E$):

Theorem 5.2. ([7: Theorem 1.3]) *Let* $\mathbf{K} = (O, P, I)$ *be a context and consider the complete lattice* $\mathrm{CL}(\mathbf{K})$. *A lattice* $\mathbf{L}$ *is isomorphic to* $\mathrm{CL}(\mathbf{K})$ *if and only if there are mappings* $\gamma \colon O \to \mathbf{L}$ *and* $\mu \colon P \to \mathbf{L}$ *such that* $\gamma(O)$ *is join dense in* $\mathbf{L}$, $\mu(P)$ *is meet dense in* $\mathbf{L}$, *for all* $o \in O, a \in P$, $(o, a) \in I$ *if and only if* $\gamma(o) \leqslant \mu(a)$.

Now by combining Proposition 5.1 and Theorem 5.2 we are able to conclude an alternative argument to our result proved in Proposition 3.1 stating that the lattices $\mathrm{C}(\mathbf{X})$ and $\mathrm{CL}(\mathbb{K}(\mathbf{X}))$ are isomorphic.

We now turn our attention back to the case of $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$.

Lemma 5.3. *Let $\mathbf{L} \in \boldsymbol{\mathcal{L}}$ and let $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. For $x \in X$ we have $\uparrow_1 x = (E_{\triangleleft}^{\mathsf{C}} \circ E_{\triangleright}^{\mathsf{C}})(\{x\})$. Dually, for $y \in X$ we have $\uparrow_2 y = (E_{\triangleright}^{\mathsf{C}} \circ E_{\triangleleft}^{\mathsf{C}})(\{y\})$.*

Proof. Let $x \leqslant_1 z$. To prove that $z \in (E_{\triangleleft}^{\mathsf{C}} \circ E_{\triangleright}^{\mathsf{C}})(\{x\})$ we need to show that for every $y \in E_{\triangleright}^{\mathsf{C}}(\{x\})$ we have $(z, y) \notin E$. Suppose for a contradiction that there exists $y \in E_{\triangleright}^{\mathsf{C}}(\{x\})$ such that $(z, y) \in E$. By Lemma 2.7(i) there exists w such that $z \leqslant_1 w$ and $y \leqslant_2 w$. Now by transitivity of $\leqslant_1$ we have that $x \leqslant_1 w$ and hence $(x, w) \in E$. Combining this with $y^{-1}(0) \subseteq w^{-1}(0)$ we get that $(x, y) \in E$. This contradicts the fact that $y \in E_{\triangleright}^{\mathsf{C}}(\{x\})$ and hence $\uparrow_1 x \subseteq (E_{\triangleleft}^{\mathsf{C}} \circ E_{\triangleright}^{\mathsf{C}})(\{x\})$.

Now consider $z \in X$ such that $x \not\leqslant_1 z$. By Lemma 2.7(iii) we have that there must exist y such that $(z, y) \in E$ and $(x, y) \notin E$. Now clearly $y \in E_{\triangleright}^{\mathsf{C}}(\{x\})$ and so $z \notin (E_{\triangleleft}^{\mathsf{C}} \circ E_{\triangleright}^{\mathsf{C}})(\{x\})$. □

The following result is a straightforward consequence of Proposition 5.1 specifying the maps J_x and M_y via the quasi-orders $\leqslant_1$ and $\leqslant_2$ in the special case $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. Note that the above lemma can be combined with Theorem 3.3 to see that the maps J_x and M_y below are indeed members of $\mathrm{C}(\mathbf{X})$.

Corollary 5.4. *Let $\mathbf{L}$ be a bounded lattice. Let $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ and let $\varphi \in \mathrm{C}(\mathbf{X})$. Then*

$$\varphi = \bigvee \big\{ J_x \mid x \in \varphi^{-1}(1) \big\} \ \text{ and } \ \varphi = \bigwedge \big\{ M_y \mid y \in \varphi^{-1}(0) \big\}$$

where

$$J_x(z) = \begin{cases} 1 & \text{if } x \leqslant_1 z, \\ 0 & \text{if } z \in r(\uparrow_1 x) \end{cases} \qquad \text{and} \qquad M_y(z) = \begin{cases} 1 & \text{if } z \in l(\uparrow_2 y), \\ 0 & \text{if } y \leqslant_2 z \end{cases} \qquad (***)$$

for $z \in X$.

For a bounded lattice $\mathbf{L}$ and its dual graph $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$, we are now ready to reveal which elements of the canonical extension $\mathrm{C}(\mathbf{X})$ of $\mathbf{L}$ are the completely join-irreducibles and the completely meet-irreducibles; let us denote them by $J^{\infty}(\mathrm{C}(\mathbf{X}))$ and $M^{\infty}(\mathrm{C}(\mathbf{X}))$, respectively. Our proof is similar to the proof of the equivalent result due to Gehrke and Harding [8: Lemma 3.4].

Proposition 5.5. *Let $\mathbf{L}$ be a bounded lattice and $\mathbf{X} = (\boldsymbol{\mathcal{L}}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. Then*

$$J^{\infty}(\mathrm{C}(\mathbf{X})) = \big\{ J_x \mid x \in X \big\} \quad \text{and} \quad M^{\infty}(\mathrm{C}(\mathbf{X})) = \big\{ M_y \mid y \in X \big\}$$

where J_x and M_y are defined by $(***)$.

P r o o f. We first show that J_x is such that $J_x = \bigwedge F$ where F is a member of a maximally-disjoint filter-ideal pair on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{2}_{\mathcal{T}})$. Since $x \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$, we have that $x^{-1}(1)$ is such a filter on L, and so consider the set of MPM's $F = \{e_a \in \mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{2}_{\mathcal{T}}) \mid a \in x^{-1}(1)\}$. Since $\mathbf{L}$ is isomorphic to $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{2}_{\mathcal{T}})$ via the map $a \mapsto e_a$ as indicated by Proposition 2.4, we have that F is a member of a maximally-disjoint filter-ideal pair on $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{2}_{\mathcal{T}})$. Similarly we can consider the corresponding ideal $J = \{e_a \mid a \in x^{-1}(0)\}$.

Since J_x and $\bigwedge F$ are elements of $\mathrm{C}(\mathbf{X}) = \mathcal{G}^{\mathrm{mp}}(\mathbf{X}, \underset{\sim}{2})$ in order to show that they are equal we need only to prove that $J_x^{-1}(1) = (\bigwedge F)^{-1}(1)$. By definition of the meet in $\mathrm{C}(\mathbf{X})$,

$$\Big(\bigwedge F\Big)^{-1}(1) = \bigcap\{e_a^{-1}(1) \mid a \in x^{-1}(1)\}.$$

Thus $z \in (\bigwedge F)^{-1}(1)$ if and only if for all $a \in x^{-1}(1)$, $e_a(z) = z(a) = 1$. That is, $z \in (\bigwedge F)^{-1}(1)$ if and only if $x^{-1}(1) \subseteq z^{-1}(1)$. This gives us the final equivalence, that $z \in (\bigwedge F)^{-1}$ if and only if $z \in {\uparrow_1} x = J_x^{-1}(1)$.

Now assume that $J_x = \bigvee_{i\in I} \varphi_i$. Since the filter elements are join dense in $\mathrm{C}(\mathbf{X})$, we can consider the case where φ_i is a filter element for each $i \in I$. That is, $\varphi_i = \bigwedge_{e_a\in F_i} e_a$ with F_i a filter of $\mathcal{G}^{\mathrm{mp}}_{\mathcal{T}}(\mathbf{X}_{\mathcal{T}}, \underset{\sim}{2}_{\mathcal{T}})$. If J_x is not completely join-irreducible, then for every $i \in I$, $\varphi_i^{-1}(1) \subsetneq J_x^{-1}(1)$. That is, $\bigwedge F_i < \bigwedge F$ and hence $F \subsetneq F_i$ for each $i \in I$. Since F is maximally disjoint from J, there exists $b_i \in F_i \cap J$ for each $i \in I$. In $\mathrm{C}(\mathbf{X})$ we have

$$\bigwedge F = \bigvee_{i\in I}\Big(\bigwedge_{e_a\in F_i} e_a\Big) \leqslant \bigvee_{i\in I} e_{b_i} \leqslant \bigvee J.$$

By compactness, there must exist $e_a \in F \cap J$, a contradiction as they are disjoint. Hence there must exist $j \in I$ such that $J_x = \varphi_j = \bigwedge F_j$.

We have shown that each of the maps J_x is completely join-irreducible in $\mathrm{C}(\mathbf{X})$. For $\varphi \in J^\infty(\mathrm{C}(\mathbf{X}))$, Corollary 5.4 gives us $\varphi = \bigvee\{J_x \mid x \in \varphi^{-1}(1)\}$. Since φ is completely join-irreducible we have that $\varphi \in \{J_x \mid x \in X\}$. The equality $M^\infty(\mathrm{C}(\mathbf{X})) = \{M_y \mid y \in X\}$ can be proven analogously. □

After using the description of the completely join-irreducibles and the completely meet-irreducibles of the canonical extension $\mathrm{C}(\mathbf{X})$ in the form (***), via the quasi-orders $\leqslant_1$ and $\leqslant_2$, we can conclude this section with their description in the form (**) used in the more general case in Proposition 5.1.

Corollary 5.6. *Let* $\mathbf{L}$ *be a bounded lattice and* $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. *Then*

$$J^\infty(\mathrm{C}(\mathbf{X})) = \{J_x \mid x \in X\} \quad \textit{and} \quad M^\infty(\mathrm{C}(\mathbf{X})) = \{M_y \mid y \in X\}$$

where for $z \in X$

$$J_x(z) = \begin{cases} 1 & \textit{if } z \in (E^{\mathsf{C}}_{\triangleleft} \circ E^{\mathsf{C}}_{\triangleright})(\{x\}), \\ 0 & \textit{if } z \in (E^{\mathsf{C}}_{\triangleright})(\{x\}) \end{cases} \qquad M_y(z) = \begin{cases} 1 & \textit{if } z \in (E^{\mathsf{C}}_{\triangleleft})(\{y\}), \\ 0 & \textit{if } z \in (E^{\mathsf{C}}_{\triangleright} \circ E^{\mathsf{C}}_{\triangleleft})(\{y\}). \end{cases}$$

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