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A REPRESENTATION OF SKEW EFFECT ALGEBRAS

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ABSTRACT. It was shown by the author and R. Halaš that every effect algebra can be organized into a conditionally residuated structure. Skew effect algebras were introduced as a non-associative modification of effect algebras. Hence, there is natural question if a similar characterization by a certain residuated structure is possible. For this we use the so-called skew residuated structure introduced recently by the author and J. Krňávek. It is shown that this is really a suitable tool for the representation.

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Effect algebras, introduced by Foulis and Bennett [8], are equivalent with D-posets defined in [9], and they form a useful tool for description of the domain of probabilities and of observables in the logic of quantum mechanics. Effect algebras form an important tool also for investigations of time dimension of quantum events as it was described in [5]. A certain modification of an effect algebra which does not demand associativity of addition was introduced by the author and H. Länger in [7] under the name skew effect algebra. The motivation for these algebras was justified by the fact that skew effect algebras are closely related to a non-associative fuzzy logic introduced by M. Botur and R. Halaš in [3]. This non-associative fuzzy logic has its algebraic counterpart in the variety of commutative basic algebras, see [1], [2] for details.

Recently, it was shown by the author and R. Halaš in [4] that effect algebras can be represented by the so-called conditionally residuated structures. There is

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a natural question if something similar can be done also for skew effect algebras. A good candidate for this purpose could be the so-called skew residuated lattice introduced by the author and J. Krňávek in [6]. However, a certain modification is necessary. At first, our structures need not be lattices in general, and secondly, the operations should be defined only for suitable pairs of elements similarly as it is done in [4]. As a result, we can define the following concept.

Definition 1. A system $\mathcal{S} = (S; \leq, \cdot, \to, 0, 1)$ is called a *skew residuated structure* if

(i) $(S; \leq, 0, 1)$ is a poset where $0 \leq x \leq 1$ for each $x \in S$;

(ii) $(S; \cdot, 1)$ is a partial commutative groupoid satisfying
$x \cdot 1 = x$ for each $x \in S$;

(iii) $x \to y$ is defined iff $y \leq x$; in this case, also

$$x \to 0 \leq y \to 0 \quad \text{and} \quad (y \to 0) \to (x \to 0) = x \to y;$$

$x \cdot y$ is defined iff $x \to 0 \leq y$.

(iv) if $x \cdot y$ and $y \to z$ are defined then

$$x \cdot y = z \quad \text{if and only if} \quad x = y \to z;$$

(v) if $x \cdot y$ is defined then $x \cdot y \leq y$,
if $x \to y$ is defined then $y \leq x \to y$.

Remark 1. By a partial commutative groupoid is meant a set S with a partial binary operation $\cdot$ such that $x \cdot y$ is defined if and only if $y \cdot x$ is defined and then $x \cdot y = y \cdot x$.

From (ii) and (v) we get that also $x \cdot y \leq x$ and hence $x \cdot y \leq x \wedge y$ provided the infimum $x \wedge y$ exists in $(S; \leq)$.

Lemma 1. *Let $\mathcal{S} = (S; \leq, \cdot, \to, 0, 1)$ be a skew residuated structure. Then*

(a) $x \to x = 1$ *and* $1 \to x = x$*;*

(b) *if* $x \cdot a = x \cdot b$ *then* $a = b$*,*
if $x \to a = x \to b$ *then* $a = b$*,*
if $a \to x = b \to x$ *then* $a = b$*;*

(c) *if* $x \to y$ *is defined then* $x \cdot (x \to y)$ *is defined and* $x \cdot (x \to y) = y$*;*

(d) *if* $y \leq x$ *then* $(x \to y) \to y$ *is defined and* $(x \to y) \to y = x$*;*

(e) *for each* $x, y \in S$ *if* $x \cdot y$ *is defined then there exists a unique element* $b \in S$ *such that* $x = y \to b$ *and* $y = x \to b$*; this* $b = x \cdot y$*.*

P r o o f. Applying (v) and (iii) of Definition 1 for elements $x \to 0$ and $y \to 0$ we infer

$$x \to y = (y \to 0) \to (x \to 0) \geq x \to 0.$$

Hence, if $x \to y$ is defined then also $x \cdot (x \to y)$ is defined, and due to partial commutativity also $(x \to y) \cdot x$ is defined.

(a) Since the operation $\cdot$ is commutative, (ii) gets $1 \cdot x = x$. By (iv) we obtain $1 = x \to x$. Since $x \cdot 1 = x$, we obtain $x = 1 \to x$.

(b) If $x \cdot a = y = x \cdot b$ then, by (v), $y \leq x$ and, by (iv), $a = x \to y = b$.

If $x \to a = c = x \to b$ then $a = c \cdot x = b$.

If $a \to x = b \to x = d$ then $d \cdot a = x = d \cdot b$ and, by the previous, we conclude $a = b$.

(c) Let $x \cdot (x \to y)$ be defined. Put $x \cdot (x \to y) = c$. Then $x \to y = x \to c$ and, by (b), $c = y$.

(d) If $y \leq x$ then $x \to y$ is defined. Put $z = x \to y$. Then $x \cdot z = y$ and, applying (v), $x \to y \geq y$ thus also $(x \to y) \to y$ is defined. Then $x \cdot z = y$ yields $x = z \to y = (x \to y) \to y$.

(e) Assume $x \cdot y$ is defined and $b = x \cdot y$. By (v) and (ii), $b \leq x \cdot y$ thus both $x \to b$ and $y \to b$ are defined. By (iv), $x = y \to b$ and, since also $y \cdot x = x \cdot y = b$, we get $y = x \to b$. By (b), this element is determined uniquely. □

In what follows, we recall several concepts on ordered sets taken mainly from [2]–[7].

Let $\mathcal{P} = (P; \leq, 0, 1)$ be a bounded poset. The interval $[a, 1]$ for $a \in P$ is called a *section*. Assume that there exists a mapping φ_a of $[a, 1]$ into itself satisfying

$$\varphi_a(\varphi_a(x)) = x \qquad \text{for} \quad x \in [a, 1] \text{ and } \varphi_a(a) = 1,\ \varphi_a(1) = a.$$

Then φ_a is called a *sectional switching involution on* $[a, 1]$.

A poset $\mathcal{P}$ is called a *poset with sectional switching involutions* if for each $a \in P$ there exists a sectional switching involution on $[a, 1]$ and, moreover, $x \leq y$ implies $\varphi_0(y) \leq \varphi_0(x)$. For the sake of brevity, we will write x^a instead of $\varphi_a(x)$ for $x \in [a, 1]$. The fact that $\mathcal{P}$ is a poset with sectional switching involutions will be expressed by the notation $\mathcal{P} = (P; \leq, 0, 1, (^a)_{a \in P})$.

Now, we can characterize skew residuated structures in these terms.

Theorem 1. *Let $\mathcal{S} = (S; \leq, \cdot, \to, 0, 1)$ be a skew residuated structure. For $y \leq x$ define $x^y = x \to y$. Then $\mathcal{P}(S) = (S; \leq, 0, 1, (^a)_{a \in S})$ is a poset with sectional switching involutions satisfying*

$$x^y = \left(y^0\right)^{\left(x^0\right)}.$$

P r o o f. By (d) of Lemma 1 we have

$$(x^y)^y = (x \to y) \to y = x$$

for $y \leq x$ thus the mapping $x \mapsto x^y$ is an involution on the interval $[y, 1]$. By (v) of Definition 1, $x^y = x \to y \geq y$ thus $x^y \in [y, 1]$, i.e. it is really a mapping of $[y, 1]$ into itself. Since $y^y = 1$ and $1^y = y$ as defined, it is a sectional switching involution on $[y, 1]$ for each $y \in S$. Finally, (iii) of Definition 1 gets: $x \leq y \implies y^0 \leq x^0$ thus the involution $x \mapsto x^0$ is really antitone as asked by definition. Moreover, (iii) yields $x^y = (y^0)^{(x^0)}$. □

This theorem gets a way how to represent skew effect algebras by means of skew residuated structures. Recall from [7] that every poset with sectional switching involutions satisfying the condition

$$x^y = \left(y^0\right)^{(x^0)}$$

can be converted into a skew effect algebra. For the reader's convenience, we recall the definition.

Definition 2. A *skew effect algebra* is a partial algebra $\mathcal{S} = (S; +, ', 0, 1)$ with a total unary operation $'$ satisfying the following properties

(S1) If $x + y$ is defined then so is $y + x$ and $x + y = y + x$;

(S2) $x + y = 1$ if and only if $y = x'$;

(S3) if $x + 1$ is defined then $x = 0$;

(S4) if $x + y = z$ then $x' = z' + y$;

(S5) if $x' + (x + y)$ is defined then $y = 0$;

(S6) if $(x+y)+z$ is defined then there exists $u \in S$ such that $(x+y)+z = x+u$.

As pointed in [7], every effect algebra is a skew effect algebra but there is an example of a skew effect algebra which is not an effect algebra. A skew effect algebra is an effect algebra if and only if the partial operation $+$ is associative.

Given a skew effect algebra, one can introduce an order by the prescription

$$x \leq y \quad \text{if and only if} \quad \text{there exists } z \in S \text{ with } y = x + z.$$

We have $0 \leq x \leq 1$ for each $x \in S$ and for the unary operation $'$ we have $x'' = x$, $0' = 1$, $1' = 0$ and

$$x \leq y \quad \text{if and only if} \quad y' \leq x'.$$

Hence, applying Theorem 1, we can transform a skew residuated structure into a skew effect algebra as follows.

Corollary 1.1. *Let $\mathcal{S} = (S; \leq, \cdot, \to, 0, 1)$ be a skew residuated structure. Define*

$$x + y = \left(y^0\right)^x \qquad \text{for} \quad x \leq y^0$$

and $x' = x^0$ where $x^0 = x \to 0$ and $w^z = w \to z$. Then $\mathcal{E}(S) = (S; +, ', 0, 1)$ is a skew effect algebra.

P r o o f. By Theorem 1, $\mathcal{S}$ can be organized into a poset with sectional switching involutions satisfying

$$x^y = \left(y^0\right)^{\left(x^0\right)}.$$

By [7: Theorem 15], such a poset can be converted into a skew effect algebra by defining $x' = x^0$ and $x + y = \left(y^0\right)^x$. □

To prove the converse, we can state our next theorem.

Theorem 2. *Let $\mathcal{P} = (P; \leq, 0, 1, (^a)_{a \in P})$ be a poset with sectional switching involutions satisfying*

$$x^y = \left(y^0\right)^{\left(x^0\right)}.$$

Define $x \to y = x^y$ for $y \leq x$ and $x \cdot y = \left(x \to y^0\right)^0$ for $y^0 \leq x$. Then $\mathcal{P}(S) = (P; \leq, \cdot, \to, 0, 1)$ is a skew residuated structure.

P r o o f. We should verify it by definition. (i) holds trivially. By the assumption,

$$x \cdot y = \left(x \to y^0\right)^0 = \left(x^{(y^0)}\right)^0 = \left(y^{(x^0)}\right)^0 = \left(y \to x^0\right)^0 = y \cdot x$$

and

$$x \cdot 1 = (x \to 1^0)^0 = (x \to 0)^0 = (x^0)^0 = x$$

proving (ii). For (iii), $x \to y = x^y$ is defined if $y \leq x$. Since the involution $x \mapsto x^0$ is antitone, we have $x^0 \leq y^0$ and hence also $y^0 \to x^0$ is defined. Moreover,

$$(y \to 0) \to (x \to 0) = \left(y^0\right)^{(x^0)} = x^y = x \to y.$$

By definition, $x \cdot y = \left(x \to y^0\right)^0$ is defined if $y^0 \leq x$, i.e. if $x^0 \leq y$, thus if $x \to 0 \leq y$.

Prove (iv). Assume $x \cdot y = z$. Then $z = \left(x \to y^0\right)^0$ and hence

$$z^0 = x \to y^0 = x^{(y^0)}.$$

This yields

$$y^z = \left(z^0\right)^{(y^0)} = \left(x^{(y^0)}\right)^{(y^0)} = x$$

thus $x = y \to z$.

Conversely, let $x = y \to z = y^z = \big(z^0\big)^{(y^0)}$. Then

$$x^{(y^0)} = \Big(\big(z^0\big)^{(y^0)}\Big)^{(y^0)} = z^0$$

whence $z = \big(x \to y^0\big)^0 = x \cdot y$.

It remains to prove (v). Assume $x \cdot y$ is defined, i.e. $y^0 \leq x$. Then $x \in \big[y^0, 1\big]$ and

$$x \to y^0 = x^{(y^0)} \in \big[y^0, 1\big]\,,$$

i.e. $x \to y^0 \geq y^0$. Hence

$$x \cdot y = \big(x \to y^0\big)^0 \leq \big(y^0\big)^0 = y.$$

Further, $x \to y = x^y$ and hence $y \leq x \to y$. □

The foregoing theorem enables us to complete our representation of skew effect algebras. By [7: Theorem 13], every skew effect algebra can be recognized as a poset with sectional switching involutions satisfying

$$x^y = \big(y^0\big)^{(x^0)}\,,$$

where $x^y = x' + y$ for $y \leq x$. Hence, by Theorem 2, if we define

$$x \cdot y = \big(x \to y^0\big)^0 = (x' + y')' \quad \text{and} \quad x \to y = x^y,$$

we obtain a skew residuated structure. Thus we have the following.

Corollary 2.1. *Let $\mathcal{E} = (E; +, ', 0, 1)$ be a skew effect algebra. Define*

$$x \cdot y = (x' + y')' \quad \text{and} \quad x \to y = x' + y.$$

Let $\leq$ be defined as follows

$$x \leq y \quad \text{if and only if} \quad y = x + z \text{ for some } z \in E.$$

Then $\mathcal{S}(E) = (E; \leq, \cdot, \to, 0, 1)$ is a skew residuated structure.

Let us note that to every skew effect algebra $\mathcal{E}$ the structure $\mathcal{S}(E)$ is assigned in a unique way. If we assign to $\mathcal{S}(E)$ a skew effect algebra $\mathcal{E}(\mathcal{S}(E))$ as done by Corollary 1.1 then $\mathcal{E} = \mathcal{E}(\mathcal{S}(E))$.

Also conversely, starting with a skew residuated structure $\mathcal{S}$ and applying Corollaries 1.1 and 2.1, we can show that $\mathcal{S}(\mathcal{E}(S)) = \mathcal{S}$. Namely, the ordering $\leq$ and the constants 0, 1 coincide in $\mathcal{S}$ and in $\mathcal{S}(\mathcal{E}(S))$. Denote by $\circ$ and $\Longrightarrow$ the binary operations of $\mathcal{S}(\mathcal{E}(S))$. Then

$$x \Longrightarrow y = x' + y = x^0 + y = \big(y^0\big)^{(x^0)} = x^y = x \to y$$

by Theorem 1 and, moreover, both $x \implies y$ and $x \to y$ are defined if $y \le x$.

For the operation $\circ$, we have $x \circ y = (x' + y')' = (x \to y')'$ thus $x \circ y$ is defined if $x \to y'$ is defined which is the case when $y \to 0 \le x$, i.e. if $x \cdot y$ is defined. Further,

$$x \circ y = (x' + y')' = \left(x^0 + y^0\right)^0 = \left(y^{(x^0)}\right)^0 = (y \to (x \to 0)) \to 0.$$

To complete the proof, we mention that $y = (y \to (x \to 0)) \to (x \to 0)$ by (d) of Lemma 1. Applying (iii) of Definition 1 we get $x \to y = (y \to 0) \to (x \to 0)$ and hence $y = x \to (((y \to (x \to 0)) \to 0)$. By (iv) we obtain

$$y \cdot x = (y \to (x \to 0)) \to 0 = (x \to (y \to 0)) \to 0 = x \cdot y.$$

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