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PRINCIPAL BUNDLE STRUCTURE ON JET PROLONGATIONS OF FRAME BUNDLES

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ABSTRACT. In this paper, we introduce the structure of a principal bundle on the r-jet prolongation J^rFX of the frame bundle FX over a manifold X. Our construction reduces the well-known principal prolongation W^rFX of FX with structure group G^r_n. For a structure group of J^rFX we find a suitable subgroup of G^r_n. We also discuss the structure of the associated bundles. We show that the associated action of the structure group of J^rFX corresponds with the standard actions of differential groups on tensor spaces.

1. Introduction

The concept of jet prolongations of principal and associated bundles is a fundamental tool in higher order differential geometry, the theory of differential invariants, and in applications (see, e.g., Brajerčík [2], Doupovec and Mikulski [3], Janyška [4], Kolář, Michor and Slovák [7], Kowalski and Sekizawa [9], Krupka [11], Kureš [13], Paták and Krupka [15]). This paper is a contribution to the structure theory of the prolongations.

One of the structure problems is the existence of a principal bundle structure on a jet prolongation of a principal bundle. This problem was originally studied by Kolář. His analysis was based on the works by Ehresmann and Libermann.

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For a principal G-bundle P over an n-dimensional manifold X he introduced a new principal bundle W^rP, where a structure group is the (r,n)-prolongation of G, denoted by G^r_n [5, 6]. Later, a modified prolongation formula was stated in [10], and the theory was explained in a more systematic way in [7] and [12].

The principal prolongation W^rP has found numerous applications in the geometry of differential invariants, calculus of variations, etc. In most of them, the underlying principal bundle was a bundle FX of frames over X. It was shown, in particular, that many applications utilize only the subgroup L^r_n, the differential group, and do not require properties of the whole structure group G^r_n. Geometrically it means, that in many applications only a reduction of G^r_n to a subgroup is needed (e.g., differential invariants, natural Lagrange structures, gauge theories). In the case of a frame bundle FX over a manifold X it is possible to use reduction of W^rFX to the bundle $F^{r+1}X$ of frames of order $r+1$, where the structure group G^r_n is reduced to the differential group L^{r+1}_n ([12]).

Jet prolongations of FX are also studied by Libermann in connection with prolongations of higher order connections [14]. She showed that the first jet prolongation J^1FX can be naturally identified with the principal bundle of second order *semi-holonomic* frames over X (see also [8]). Also a correspondence between the bundle of semi-holonomic r-jets of sections of FX and the bundle of semi-holonomic frames of order $r+1$ over X was described.

The aim of this paper is to study the existence of principal bundle structures on *holonomic* frame bundle of order r, J^rFX. Our main result consists in finding a Lie group which defines a principal bundle structure of J^rFX. The construction gives us a reduction of the principal prolongation W^rFX to J^rFX. We also study the associated actions of this newly introduced group on the corresponding associated fibre bundles. These actions generalize the standard tensor actions of differential groups to a broader class of type fibres.

Note that the prolongation procedure presented in this paper can be applied to an arbitrary principal L^1_n-bundle P. As an example we can take a principal L^1_n-bundle J^rP over the bundle C^rP of *r-th order connections* of P for any r (see, e.g., [1]).

2. Preliminaries

Let G be a Lie group, and denote by T^r_nG the set of all r-jets with source at the origin $0 \in \mathbb{R}^n$ and target in G. T^r_nG is a closed submanifold of $J^r(\mathbb{R}^n, G)$ of r-jets with source in $\mathbb{R}^n$ and target in G.

Let $\mathcal{S}, \mathcal{T} \in T^r_nG$, $\mathcal{S} = J^r_0 s$, $\mathcal{T} = J^r_0 t$, where $s, t\colon \mathbb{R}^n \to G$, be any elements. The rule

$$\mathcal{S} \cdot \mathcal{T} = J^r_0(s \cdot t),$$

where $(s \cdot t)(x) = s(x) \cdot t(x)$ is the group multiplication in G, defines a structure of Lie group on T^r_nG.

Let us consider the r-th differential group of $\mathbb{R}^n$, denoted L^r_n, which consists of all invertible r-jets of mappings $\alpha\colon \mathbb{R}^n \to \mathbb{R}^n$ with source and target at the origin $0 \in \mathbb{R}^n$, and multiplication is given by the composition of jets. Each element $A = J^r_0\alpha \in L^r_n$ defines a mapping $\varphi(A)\colon T^r_nG \to T^r_nG$ by the formula $\varphi(A)(\mathcal{S}) = \mathcal{S} \circ A^{-1}$. $\varphi(A)$ is an automorphism of the Lie group T^r_nG, and the mapping $A \mapsto \varphi(A)$ is a homomorphism of the Lie group L^r_n into the group $\mathrm{Aut}\, T^r_nG$ of automorphisms of T^r_nG. The exterior semi-direct product $L^r_n \times_\varphi T^r_nG$, associated with the homomorphism φ (see [12]), is a Lie group with the multiplication

$$(A, \mathcal{S}) \cdot (B, \mathcal{T}) = (A \cdot B, \mathcal{S} \cdot (\mathcal{T} \circ A^{-1})),$$

where $A, B \in L^r_n$, $\mathcal{S}, \mathcal{T} \in T^r_nG$. This Lie group is called the *(r, n)-prolongation* of G and is denoted by G^r_n.

Let F^rX denote the set of all r-frames, i.e., invertible r-jets $Z = J^r_0\zeta$ with source at the origin $0 \in \mathbb{R}^n$ and target in the n-dimensional manifold X. F^rX is endowed with a natural structure of principal bundle with the structure group L^r_n and is called the bundle of r-frames over X.

Let P be a principal bundle over an n-dimensional manifold X, let π be its projection. Let J^rP denote the r-jet prolongation of P. The r-jet of a section γ of P at a point $x \in X$ is denoted by $\Upsilon = J^r_x\gamma$.

Consider the fibre product $W^rP = F^rX \oplus J^rP$, i.e., the submanifold in $F^rX \times J^rP$ of pairs (Z, Υ) such that Z and Υ belong to the fibre over the same point of X. For every $(Z, \Upsilon) \in W^rP$, $Z = J^r_0\zeta$, $\Upsilon = J^r_x\gamma$, where $x = \zeta(0)$, and every $(A, \mathcal{S}) \in G^r_n$, $A = J^r_0\alpha$, $\mathcal{S} = J^r_0 s$, we put

$$(Z, \Upsilon) \cdot (A, \mathcal{S}) = (Z \cdot A, \Upsilon \cdot (\mathcal{S} \circ Z^{-1})) = (J^r_0(\zeta \circ \alpha), J^r_x(\gamma \cdot (s \circ \zeta^{-1}))), \tag{1}$$

where $(\gamma \cdot (s \circ \zeta^{-1}))(x) = \gamma(x) \cdot s(\zeta^{-1}(x))$ is the right action of G on P. Then (1) is a right action of the group G^r_n on W^rP which defines a structure of principal G^r_n-bundle on W^rP. W^rP is called the *(principal) r-prolongation* of P.

In this paper we consider a frame bundle over an n-dimensional manifold X instead of P. A *frame* at a point $x \in X$ is a pair $\Xi = (x, \xi)$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is an ordered basis of the tangent space T_xX. The set of all frames in all points of X is denoted by FX and the structure $\mu\colon FX \to X$ is a principal bundle with the structure group $Gl_n(R)$ (the general linear group of $\mathbb{R}^n$); the dimension of a fibre is n^2.

For every chart (U, φ), $\varphi = (x^i)$, on X, $x \in U$, the *associated chart* (V, ψ), $\psi = (x^i, x^i_j)$, on FX, is defined by $V = \mu^{-1}(U)$, and

$$x^i(\Xi) = x^i(\mu(\Xi)), \quad \xi_j = x^i_j(\Xi) \left(\frac{\partial}{\partial x^i}\right)_x,$$

where $\Xi \in V$.

FX can be identified with the bundle F^1X of all invertible 1-jets with source at the origin $0 \in \mathbb{R}^n$ and target in X. To every $\Xi = (x, \xi) \in FX$, $x \in U$, we assign a 1-jet $Z = J^1_0\zeta$ such that $\zeta(0) = x$ and $D_j(x^i \circ \zeta)(0) = x^i_j(\Xi)$. This defines a bijection between the bundles FX and F^1X. Due to this identification, in what follows, we will use the notation FX also for F^1X.

Let us consider a local trivialization of the principal bundle FX. For every $x \in X$ there exists an open subset $U \subset X$ and a diffeomorphism ϕ, such that the diagram

$$\begin{array}{ccc} \mu^{-1}(U) & \xrightarrow{\ \phi\ } & U \times L^1_n \\ \mu \downarrow & & \downarrow p_1 \\ U & \xrightarrow{\ \mathrm{id}\ } & U \end{array} \tag{2}$$

commutes. By p_1 (p_2) we denote the projection onto the first (second) component. Let $J^1_0\zeta \in \mu^{-1}(U)$, $\zeta(0) = x$, and let $t_z\colon \mathbb{R}^n \to \mathbb{R}^n$, $z \in \mathbb{R}^n$, be a translation of $\mathbb{R}^n$ given by $t_z(w) = w - z$. Then an identification $\phi\colon \mu^{-1}(U) \to U \times L^1_n$, associated with a chart (U, φ), is defined by $J^1_0\zeta \mapsto (\zeta(0), J^1_0\tilde{\zeta})$, where $\tilde{\zeta}\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\tilde{\zeta} = t_{\varphi(x)} \circ \varphi \circ \zeta. \tag{3}$$

If $\Xi = J^1_0\zeta \in \mu^{-1}(U)$, and (a^i_j) are the (first) canonical coordinates on L^1_n, then $\phi(\Xi) = (\mu(\Xi), A)$, where $a^i_j(A) = x^i_j(\Xi)$.

Let $FX \times L^1_n \ni (\Xi, A) \mapsto \Xi \cdot A \equiv R_A(\Xi) \in FX$ be a *right action* of the structure group L^1_n on FX, in the corresponding coordinates given by

$$\bar{x}^i = x^i \circ R_A = x^i, \quad \bar{x}^i_j = x^i_j \circ R_A = x^i_k \cdot a^k_j(A).$$

The mapping $\chi = p_2 \circ \phi\colon \mu^{-1}(U) \to L^1_n$ satisfies

$$\chi(R_A(\Xi)) = \chi(\Xi) \cdot A \tag{4}$$

for every $\Xi \in \mu^{-1}(U)$ and every $A \in L^1_n$.

By J^rFX we denote the *r-jet prolongation* of FX. The *r-jet* of a section γ of FX, at a point $x \in X$, is denoted by $J^r_x\gamma$, and the assignment $x \mapsto J^r\gamma(x) = J^r_x\gamma$ is the *r-jet prolongation* of γ. The *canonical jet projections* $\mu^{r,0}\colon J^rFX \to FX$ (the target projection), and $\mu^r\colon J^rFX \to X$ (the source projection), are defined by $\mu^{r,0}(J^r_x\gamma) = \gamma(x)$, and $\mu^r(J^r_x\gamma) = x$, respectively. With a chart (U, φ) we also associate a chart (V^r, ψ^r) on J^rFX, where $V^r = (\mu^r)^{-1}(U)$, and $\psi^r = (x^i, x^i_j, x^i_{j,k_1}, x^i_{j,k_1k_2}, \dots, x^i_{j,k_1k_2\dots k_r})$.

It is well known that J^rFX has a structure of fibre bundle with the standard fibre $T^r_nL^1_n = J^r_0(\mathbb{R}^n, L^1_n)$ (the manifold of all jets of mappings from $\mathbb{R}^n \to L^1_n$ with source in $0 \in \mathbb{R}^n$ and target in L^1_n). To describe a fibre bundle structure on J^rFX, we take a local trivialization of FX, consisting of pairs (U, ϕ) (see (2)), and we introduce the corresponding local trivialization of J^rFX. Let (U, φ) be a coordinate chart on X. Over U we have the mapping

$$\Phi\colon J^rFX|_U = (\mu^r)^{-1}(U) \to U \times T^r_nL^1_n, \tag{5}$$

in the form $J^rFX|_U \ni J^r_x\gamma \mapsto (x, J^r_0\bar{\gamma})$, where $\bar{\gamma}\colon \mathbb{R}^n \to L^1_n$ is defined by

$$\bar{\gamma} = \chi \circ \gamma \circ \varphi^{-1} \circ t_{-\varphi(x)}. \tag{6}$$

Obviously, Φ is smooth. Its inverse $\Phi^{-1}\colon U \times T^r_nL^1_n \to J^rFX|_U$; $(x, J^r_0 s) \mapsto J^r_x\delta$, with $\delta\colon U \to FX$, $\delta(y) = \phi^{-1}(y, (s \circ t_{\varphi(x)} \circ \varphi)(y))$, $y \in U$, is also smooth.

Thus, the collection of pairs (U, Φ) represent the local trivializations of J^rFX such that the diagram

$$\begin{array}{ccc} J^rFX|_U & \xrightarrow{\ \Phi\ } & U \times T^r_nL^1_n \\ \mu^{r,0}\big\downarrow & & \big\downarrow \mathrm{id}\times\rho \\ FX|_U & \xrightarrow{\ \phi\ } & U \times L^1_n \\ \mu\big\downarrow & & \big\downarrow p_1 \\ U & \xrightarrow{\ \mathrm{id}\ } & U \end{array} \tag{7}$$

commutes. In (7), $\rho\colon T^r_nL^1_n \to L^1_n$ is the projection (of the fibred manifold) defined by $T^r_nL^1_n \ni J^r_0 s \mapsto s(0) \in L^1_n$.

3. Principal bundle structure on J^rFX

In contrast with [7,12], in this section we introduce a modified group operation on $T^r_nL^1_n$, denoted $*$, and a structure of principal bundle on J^rFX with the structure group $(T^r_nL^1_n, *)$.

Let us recall that $T^r_nL^1_n$ consists of all jets of smooth mappings from $\mathbb{R}^n \to L^1_n$ with source in $0 \in \mathbb{R}^n$ and target in L^1_n. The identification $T^r_nL^1_n = J^r_0(\mathbb{R}^n, L^1_n)$ (closed submanifold of $J^r(\mathbb{R}^n, L^1_n)$) induces a structure of smooth manifold (also a structure of fibred manifold $\rho\colon T^r_nL^1_n \to L^1_n$). Moreover, $T^r_nL^1_n$ is endowed with a Lie group structure, where multiplication is given as follows (see, e.g., [12]). Let $\mathcal{S}, \mathcal{T} \in T^r_nL^1_n$, $\mathcal{S} = J^r_0 s$, $\mathcal{T} = J^r_0 t$, where $s, t\colon \mathbb{R}^n \to L^1_n$, then

$$\mathcal{S} \cdot \mathcal{T} = J^r_0(s \cdot t),$$

where $(s \cdot t)(x) = s(x) \cdot t(x)$ (the group multiplication in L^1_n). In what follows we wish to introduce another Lie group structure on $T^r_nL^1_n$. Under the correspondence between the general linear group $Gl_n(\mathbb{R})$ and the differential group L^1_n we have the following

Lemma 1. *For every $S \in L^1_n$ there exists a unique linear automorphism s_0 of $\mathbb{R}^n$ such that $S = J^1_0 s_0$.*

As a direct consequence of Lemma 1 we have:

Corollary 1. *For every $\mathcal{S} = J^r_0 s \in T^r_n L^1_n$ there exists a unique linear automorphism $s_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $s(0) = J^1_0 s_0$.*

The mapping s_0 is said to be *associated* with $\mathcal{S}$.

For every $\mathcal{S}, \mathcal{T} \in T^r_n L^1_n$, we define a multiplication on $T^r_n L^1_n$, denoted $*$, by

$$\mathcal{S} * \mathcal{T} = J^r_0 (s \cdot (t \circ s_0^{-1})), \tag{8}$$

where $\cdot$ denotes the group multiplication in L^1_n, and $s_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ is the mapping associated with $\mathcal{S}$.

Lemma 2. *$(T^r_n L^1_n, *)$ is a Lie group.*

Proof. $T^r_n L^1_n$ has a structure of smooth manifold. Obviously, $\mathcal{S} * \mathcal{T} \in T^r_n L^1_n$ for every $\mathcal{S}, \mathcal{T} \in T^r_n L^1_n$.

Further, we show that $(\mathcal{S} * \mathcal{T}) * \mathcal{U} = \mathcal{S} * (\mathcal{T} * \mathcal{U})$. Let $\mathcal{S}, \mathcal{T}, \mathcal{U} \in T^r_n L^1_n$, $\mathcal{S} = J^r_0 s$, $\mathcal{T} = J^r_0 t$, $\mathcal{U} = J^r_0 u$, $s, t, u \colon \mathbb{R}^n \to L^1_n$. By Lemma 1, there exist mappings $s_0, t_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $s(0) = S = J^1_0 s_0$, $t(0) = T = J^1_0 t_0$. If we denote $\mathcal{S} * \mathcal{T} = J^r_0 v$, then by (8), $v(0) = (s \cdot (t \circ s_0^{-1}))(0) = s(0) \cdot t(0) = S \cdot T$, and the mapping $v_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ associated with $\mathcal{S} * \mathcal{T}$ is of the form $v_0 = s_0 \circ t_0$. Indeed, $J^1_0 (s_0 \circ t_0) = J^1_0 s_0 \cdot J^1_0 t_0 = S \cdot T = v(0)$, and $s_0 \circ t_0$ is linear, thus, by Corollary 1, we get $v_0 = s_0 \circ t_0$. Further,

$$\begin{aligned}(\mathcal{S} * \mathcal{T}) * \mathcal{U} &= J^r_0 (s \cdot (t \circ s_0^{-1})) * J^r_0 u = J^r_0 (s \cdot (t \circ s_0^{-1}) \cdot (u \circ (s_0 \circ t_0)^{-1})) \\ &= J^r_0 (s \cdot (t \circ s_0^{-1}) \cdot (u \circ t_0^{-1} \circ s_0^{-1})) = J^r_0 (s \cdot (t \cdot (u \circ t_0^{-1}) \circ s_0^{-1})) \\ &= J^r_0 s * J^r_0 (t \cdot (u \circ t_0^{-1})) = \mathcal{S} * (\mathcal{T} * \mathcal{U}).\end{aligned}$$

The identity element of $*$ defined on $T^r_n L^1_n$ is $\mathcal{E} = J^r_0 e$, where $e \colon \mathbb{R}^n \to L^1_n$ is the constant mapping assigning the identity matrix $E \in L^1_n$ to every $z \in \mathbb{R}^n$. The corresponding mapping $e_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ with the property $J^1_0 e_0 = e(0)$ is $\mathrm{id}_{\mathbb{R}^n}$. Note, that $\mathcal{E}$ coincides with the identity element of $(T^r_n L^1_n, \cdot)$.

The inverse $\mathcal{S}^{-1}$ of an element $\mathcal{S} = J^r_0 s \in T^r_n L^1_n$ is given by $\mathcal{S}^{-1} = J^r_0 s^{-1}$, where $s^{-1}(z) = ((s \circ s_0)(z))^{-1}$. The mapping associated with $\mathcal{S}^{-1}$ is s_0^{-1}.

Denoting for a moment the group multiplication in L^1_n by Ψ, we obtain (8) in the form

$$\mathcal{S} * \mathcal{T} = J^r_{(S,T)} \Psi \circ J^r_0 (s \times (t \circ s_0^{-1})) = J^r_{(S,T)} \Psi \circ (\mathcal{S}, \mathcal{T}) \circ (J^r_0 \, \mathrm{id}_{\mathbb{R}^n}, J^r_0 s_0^{-1}).$$

Since the composition of jets is smooth, the product $\mathcal{S} * \mathcal{T}$ depends smoothly on $\mathcal{S}$ and $\mathcal{T}$, and we see that the group structure in $(T^r_n L^1_n, *)$ is compatible with its smooth structure. Thus $(T^r_n L^1_n, *)$ is a Lie group. □

Let $(a^i_j, a^i_{j,k}, a^i_{j,kl})$ denote the coordinates on $T^2_n L^1_n$, let (b^i_j) be the second canonical coordinates on L^1_n, i.e., $a^i_k \cdot b^k_j = \delta^i_j$. Then, for any $\mathcal{S}, \mathcal{T} \in T^2_n L^1_n$, the coordinate expressions of $*$ on $T^2_n L^1_n$ are

$$\begin{aligned} a^i_j(\mathcal{S} * \mathcal{T}) &= a^i_k(\mathcal{S}) a^k_j(\mathcal{T}), \quad b^i_j(\mathcal{S} * \mathcal{T}) = b^i_k(\mathcal{T}) b^k_j(\mathcal{S}), \\ a^i_{j,k}(\mathcal{S} * \mathcal{T}) &= a^i_{l,k}(\mathcal{S}) a^l_j(\mathcal{T}) + a^i_l(\mathcal{S}) a^l_{j,m}(\mathcal{T}) b^m_k(\mathcal{S}), \\ a^i_{j,kl}(\mathcal{S} * \mathcal{T}) &= a^i_{m,kl}(\mathcal{S}) a^m_j(\mathcal{T}) + a^i_{m,k}(\mathcal{S}) a^m_{j,p}(\mathcal{T}) b^p_l(\mathcal{S}) \\ &\quad + a^i_{m,l}(\mathcal{S}) a^m_{j,p}(\mathcal{T}) b^p_k(\mathcal{S}) + a^i_m(\mathcal{S}) a^m_{j,pq}(\mathcal{T}) b^q_l(\mathcal{S}) b^p_k(\mathcal{S}), \end{aligned}$$

and the coordinates of the identity element $\mathcal{E}$ of $T^r_n L^1_n$ are

$$a^i_j(\mathcal{E}) = \delta^i_j, \qquad a^i_{j,k_1 k_2 \dots k_m}(\mathcal{E}) = 0, \quad 1 \le m \le r. \tag{9}$$

Unless otherwise stated, from now on by the Lie group $T^r_n L^1_n$ we mean $(T^r_n L^1_n, *)$. Now we are in a position to define an action of $T^r_n L^1_n$ on $J^r FX$. Let $\Upsilon = J^r_x \gamma \in J^r FX$, where $\gamma \colon U \to FX$ is a smooth section, $x \in U \subset X$. Using the local trivialization of FX, mentioned in Section 2, and Lemma 1, we have

Lemma 3. *For every $\Upsilon = J^r_x \gamma \in J^r FX$ there exists a unique invertible smooth mapping $\gamma_0 \colon \mathbb{R}^n \to X$ such that $\gamma_0(0) = x$, $J^1_0 \gamma_0 = \gamma(x)$, and $\tilde{\gamma}_0 \colon \mathbb{R}^n \to \mathbb{R}^n$, defined by* (3)*, is a linear mapping.*

The mapping γ_0 is said to be *associated* with Υ. Let $\mathcal{S} \in T^r_n L^1_n$, $\mathcal{S} = J^r_0 s$. We define a mapping $J^r FX \times T^r_n L^1_n \ni (\Upsilon, \mathcal{S}) \mapsto \Upsilon * \mathcal{S} \in J^r FX$ by

$$\Upsilon * \mathcal{S} = J^r_x(\gamma \cdot (s \circ \gamma_0{}^{-1})). \tag{10}$$

Lemma 4. (10) *defines a right action of $T^r_n L^1_n$ on $J^r FX$.*

Proof. Let $\Upsilon = J^r_x \gamma \in J^r FX$, $\gamma \colon U \to FX$ and $\mathcal{S}, \mathcal{T} \in T^r_n L^1_n$, $\mathcal{S} = J^r_0 s$, $\mathcal{T} = J^r_0 t$. Let us denote $\Upsilon * \mathcal{S} = J^r_x(\gamma \cdot (s \circ \gamma_0{}^{-1})) = J^r_x \delta$. First, we notice that the corresponding mapping δ_0 associated with $J^r_x \delta$ is equal to $\gamma_0 \circ s_0$. Indeed,

$$(\gamma_0 \circ s_0)(0) = \gamma_0(s_0(0)) = \gamma_0(0) = x = \delta_0(0).$$

Further,

$$\begin{aligned} x^i_j(J^1_0(\gamma_0 \circ s_0)) &= D_j(x^i(\gamma_0 \circ s_0))(0) = D_k(x^i \circ \gamma_0)(s_0(0)) \cdot D_j(a^k \circ s_0)(0) \\ &= D_k(x^i \circ \gamma_0)(0) \cdot D_j(a^k \circ s_0)(0) = x^i_k(J^1_0 \gamma_0) \cdot a^k_j(J^1_0 s_0) \\ &= x^i_k(\gamma(x)) \cdot a^k_j(s(0)), \end{aligned}$$

and

$$x^i_j(\delta(x)) = x^i_j((\gamma \cdot (s \circ \gamma_0^{-1}))(x)) = x^i_k(\gamma(x)) \cdot a^k_j(s(0)).$$

Thus $\delta(x) = (\gamma \cdot (s \circ \gamma_0^{-1}))(x) = J^1_0(\gamma_0 \circ s_0)$. Moreover, denoting $\omega_0 = \gamma_0 \circ s_0$, by (3) we have that

$$\tilde{\omega}_0 = t_{\varphi(x)} \circ \varphi \circ \gamma_0 \circ s_0 = \tilde{\gamma}_0 \circ s_0$$

is linear, and therefore, using Lemma 3, we can conclude that

$$\delta_0 = \gamma_0 \circ s_0. \tag{11}$$

Now, by (10) and (11) we can write

$$\begin{aligned}(\Upsilon * \mathcal{S}) * \mathcal{T} &= J^r_x((\gamma \cdot (s \circ \gamma_0^{-1})) \cdot (t \circ (\gamma_0 \circ s_0)^{-1}))\\ &= J^r_x((\gamma \cdot (s \circ \gamma_0^{-1})) \cdot (t \circ s_0^{-1} \circ \gamma_0^{-1}))\\ &= J^r_x(\gamma \cdot ((s \cdot (t \circ s_0^{-1})) \circ \gamma_0^{-1})) = \Upsilon * (\mathcal{S} * \mathcal{T}).\end{aligned}$$

Finally, it is obvious that $\Upsilon * \mathcal{E} = J^r_x(\gamma \cdot (e \circ \gamma_0^{-1})) = J^r_x \gamma = \Upsilon$, because $e(z) = E$ for all $z \in \mathbb{R}^n$. □

Let $(x^i, x^i_j, x^i_{j,k}, x^i_{j,kl})$ denote the fibred coordinates on J^2FX, and let y^j_k be the inverse matrix of x^i_j. For any $\Upsilon \in J^2FX$, $\mathcal{S} \in T^2_n L^1_n$, the coordinate expressions of (10) on J^2FX are given by

$$\begin{aligned}x^i(\Upsilon * \mathcal{S}) &= x^i(\Upsilon),\\ x^i_j(\Upsilon * \mathcal{S}) &= x^i_k(\Upsilon) a^k_j(\mathcal{S}), \quad y^i_j(\Upsilon * \mathcal{S}) = b^i_k(\mathcal{S}) y^k_j(\Upsilon),\\ x^i_{j,k}(\Upsilon * \mathcal{S}) &= x^i_{l,k}(\Upsilon) a^l_j(\mathcal{S}) + x^i_l(\Upsilon) a^l_{j,m}(\mathcal{S}) y^m_k(\Upsilon),\\ x^i_{j,kl}(\Upsilon * \mathcal{S}) &= x^i_{m,kl}(\Upsilon) a^m_j(\mathcal{S}) + x^i_{m,k}(\Upsilon) a^m_{j,p}(\mathcal{S}) y^p_l(\Upsilon)\\ &\quad + x^i_{m,l}(\Upsilon) a^m_{j,p}(\mathcal{S}) y^p_k(\Upsilon) + x^i_m(\Upsilon) a^m_{j,pq}(\mathcal{S}) y^q_l(\Upsilon) y^p_k(\Upsilon).\end{aligned} \tag{12}$$

Theorem 1. *J^rFX with the right action (10) becomes a principal $T^r_n L^1_n$-bundle.*

P r o o f. J^rFX has a structure of fibre bundle with the standard fibre $T^r_n L^1_n$; its local trivialization is described in Section 2. The action (10) of $T^r_n L^1_n$ on J^rFX is free. Indeed, if we suppose that for some $\Upsilon = J^r_x \gamma \in J^rFX$ and $\mathcal{S} = J^r_0 s \in T^r_n L^1_n$, we have $\Upsilon * \mathcal{S} = \Upsilon$, then by (10),

$$J^r_x(\gamma \cdot (s \circ {\gamma_0}^{-1})) = J^r_x \gamma. \tag{13}$$

This implies $(\gamma \cdot (s \circ {\gamma_0}^{-1}))(x) = \gamma(x) \cdot s(0) = \gamma(x)$, which gives us $s(0) = E$ (identity element of L^1_n), i.e., $a^i_j(s(0)) = a^i_j(\mathcal{S}) = \delta^i_j$, because the action of L^1_n on FX is free. Further, from (13), $x^i_{j,k}(\Upsilon * \mathcal{S}) = x^i_{j,k}(\Upsilon)$, and using (12), we get $a^i_{j,k}(\mathcal{S}) = 0$. Since relations similar to equations (12) hold for any r, continuing analogously, we finally get that $a^i_{j,k_1 k_2 \dots k_m}(\mathcal{S}) = 0$ for all $1 \le m \le r$, which by (9) means that $\mathcal{S} = \mathcal{E}$ and (10) is free.

Finally, using the local trivialization of J^rFX, consisting of the collection of pairs (U, Φ), where the diffeomorphism Φ (5) is defined by (6), we shall show that Φ is equivariant with respect to the right action (10) of $T^r_n L^1_n$ on J^rFX and the group operation (8) on $T^r_n L^1_n$. Let $\Upsilon = J^r_x \gamma \in J^rFX|_U$ and $\mathcal{S} = J^r_0 s \in T^r_n L^1_n$. Let us denote $\tau = p_2 \circ \Phi$. We wish to show that $\tau(\Upsilon * \mathcal{S}) = \tau(\Upsilon) * \mathcal{S}$. We have $\tau(\Upsilon) = J^r_0 \bar{\gamma}$ with $\bar{\gamma} \colon \mathbb{R}^n \to L^1_n$ given by (6), and by (8) we get

$$\tau(\Upsilon) * \mathcal{S} = J^r_0(\bar{\gamma} \cdot (s \circ \bar{\gamma}_0^{-1})) = J^r_0((\chi \circ \gamma \circ \varphi^{-1} \circ t_{-\varphi(x)}) \cdot (s \circ \bar{\gamma}_0^{-1})), \tag{14}$$

where $\bar{\gamma}_0 \colon \mathbb{R}^n \to \mathbb{R}^n$ is associated with $\bar{\gamma}$, i.e., $J^1_0 \bar{\gamma}_0 = \bar{\gamma}(0) = \chi(\gamma(x))$.

In addition, let us denote $\Upsilon * \mathcal{S} = J^r_x(\gamma \cdot (s \circ {\gamma_0}^{-1})) = J^r_x \delta$, where $\gamma_0 \colon \mathbb{R}^n \to X$ is associated with Υ. Then $\tau(\Upsilon * \mathcal{S}) = J^r_0 \bar{\delta}$, where $\bar{\delta}$ is defined by (6), and using (4), we get

$$\begin{aligned} \bar{\delta} &= \chi \circ (\gamma \cdot (s \circ {\gamma_0}^{-1})) \circ \varphi^{-1} \circ t_{-\varphi(x)} \\ &= (\chi \circ \gamma \circ \varphi^{-1} \circ t_{-\varphi(x)}) \cdot (s \circ {\gamma_0}^{-1} \circ \varphi^{-1} \circ t_{-\varphi(x)}). \end{aligned} \tag{15}$$

Using (6) and according to Corollary 1 and Lemma 3, we have

$$J^1_0 \bar{\gamma}_0 = \bar{\gamma}(0) = \chi(\gamma(x)) = \chi(J^1_0 \gamma_0) = J^1_0 \tilde{\gamma}_0,$$

where both $\bar{\gamma}_0$ and $\tilde{\gamma}_0$ are linear. Corollary 1 gives us that $\bar{\gamma}_0 = \tilde{\gamma}_0$, and (3) implies ${\gamma_0}^{-1} \circ \varphi^{-1} \circ t_{-\varphi(x)} = \bar{\gamma}_0^{-1}$. Using it, $J^r_0 \bar{\delta}$ for $\bar{\delta}$ (15) coincides with (14) which means that $\tau(\Upsilon * \mathcal{S}) = \tau(\Upsilon) * \mathcal{S}$. Since Υ and $\mathcal{S}$ are arbitrary, this completes the proof. □

4. Prolongation of associated bundles

Let Q be a left L^1_n-manifold and let $F_Q X$ be a bundle with fibre Q, associated with the principal L^1_n-bundle FX; a point of $F_Q X$ is, by definition, the equivalence class $[\Xi, q]$ of a pair $(\Xi, q) \in FX \times Q$ with respect to the right action

$$((\Xi, q), A) \mapsto (\Xi \cdot A, A^{-1} \cdot q)$$

of L^1_n on $FX \times Q$.

Let $(T^r_n L^1_n, *)$ be a Lie group as in Section 3. Consider the mapping

$$T^r_n L^1_n \times T^r_n Q \to T^r_n Q; \quad (J^r_0 s, J^r_0 f) \mapsto J^r_0 (s \cdot (f \circ s_0^{-1})). \tag{16}$$

Lemma 5. (16) *defines a left action of* $(T^r_n L^1_n, *)$ *on* $T^r_n Q$.

P r o o f. The proof is a modification of the proof of Lemma 2. □

The action (16) will be denoted by $J^r_0 s * J^r_0 f = J^r_0(s \cdot (f \circ s_0^{-1}))$.

Let $J^r FX$ be a principal $T^r_n L^1_n$-bundle with the structure group $(T^r_n L^1_n, *)$. Using (16) we can construct a bundle $(J^r FX)_Y$ with type fibre $Y = T^r_n Q$, associated with $J^r FX$. The group $(T^r_n L^1_n, *)$ acts on $J^r FX \times Y$ by the formula

$$((J^r_x \gamma, J^r_0 f), \mathcal{S}) \to (J^r_x \gamma * \mathcal{S}, \mathcal{S}^{-1} * J^r_0 f),$$

where $\mathcal{S}^{-1} = J^r_0 s^{-1}$ is the inverse of $\mathcal{S} = J^r_0 s \in (T^r_n L^1_n, *)$ defined in the proof of Lemma 2. The corresponding invertible linear mapping $\mathbb{R}^n \to \mathbb{R}^n$, associated with $\mathcal{S}^{-1}$, is s_0^{-1}. Thus we can write

$$\big(J^r_x \gamma * \mathcal{S}, \mathcal{S}^{-1} * J^r_0 f\big) = \big(J^r_x(\gamma \cdot (s \circ \gamma_0^{-1})),\, J^r_0(s^{-1} \cdot (f \circ s_0))\big), \tag{17}$$

where $\gamma_0 \colon \mathbb{R}^n \to X$ is the mapping associated with the r-jet $J^r_x \gamma$.

Theorem 2. *The r-jet prolongation J^rF_QX of F_QX has a structure of fibre bundle with fibre T^r_nQ, associated with the principal $T^r_nL^1_n$-bundle J^rFX.*

Proof. Let $(J^rFX)_Y$ be a fibre bundle with fibre $Y = T^r_nQ$, associated with the principal $T^r_nL^1_n$-bundle J^rFX. We are going to show that there exists an isomorphism of manifolds $\Psi\colon (J^rFX)_Y \to J^rF_QX$, commuting with the projections onto the base X of FX.

Let $\gamma_0\colon \mathbb{R}^n \to X$ be the mapping associated with $J^r_x\gamma \in J^rFX$, where $\gamma\colon U \to FX$ is a local section over an open subset $U \subset X$, $x \in U$. Putting

$$\gamma_Q(z,q) = [\gamma\gamma_0(z), q] \tag{18}$$

we obtain a mapping $\gamma_Q\colon \gamma_0^{-1}(U) \times Q \to F_QX$. Consider

$$\Psi\colon (J^rFX)_Y \to J^rF_QX; \quad [J^r_x\gamma, J^r_0f] \mapsto J^r_x\beta,$$

where $\beta(y) = \gamma_Q(\gamma_0^{-1}(y), f(\gamma_0^{-1}(y)))$, i.e., $\beta = \gamma_Q \circ (\mathrm{id}_{\gamma_0^{-1}(U)} \times f) \circ \gamma_0^{-1}$. Clearly, β is a local section of F_QX defined on $U \subset X$. To show that Ψ is a well-defined mapping, take any pair $(J^r_x\gamma', J^r_0f') \in [J^r_x\gamma, J^r_0f]$. There exists $\mathcal{S} \in (T^r_nL^1_n, *)$, $\mathcal{S} = J^r_0s$, such that

$$(J^r_x\gamma', J^r_0f') = (J^r_x\gamma * \mathcal{S}, \mathcal{S}^{-1} * J^r_0f).$$

In (17), $\big(J^r_x\gamma * \mathcal{S}, \mathcal{S}^{-1} * J^r_0f\big) = \big(J^r_x(\gamma \cdot (s \circ \gamma_0^{-1})),\ J^r_0(s^{-1} \cdot (f \circ s_0))\big)$, denote $\delta = \gamma\cdot(s\circ\gamma_0^{-1})$ and $h = s^{-1}\cdot(f\circ s_0)$. Consider the r-jet $J^r_x(\delta_Q\circ(\mathrm{id}_{\delta_0^{-1}(U)} \times h)\circ\delta_0^{-1})$ and take its representative $y \mapsto \delta_Q(\delta_0^{-1}(y), h(\delta_0^{-1}(y)))$. In view of (18), we have

$$\begin{aligned}\delta_Q(\delta_0^{-1}(y), h(\delta_0^{-1}(y))) &= [\delta(y), h(\delta_0^{-1}(y))]\\ &= [(\gamma \cdot (s \circ \gamma_0^{-1}))(y), (s^{-1} \cdot (f \circ s_0))(\delta_0^{-1}(y))]\\ &= [\gamma(y) \cdot s(\gamma_0^{-1}(y)), s^{-1}(\delta_0^{-1}(y)) \cdot f(s_0(\delta_0^{-1}(y)))].\end{aligned}$$

Using $s^{-1}(y) = (s \circ s_0(y))^{-1}$ and (11) we obtain

$$\begin{aligned}&\delta_Q(\delta_0^{-1}(y), h(\delta_0^{-1}(y)))\\ &= \big[\gamma(y) \cdot s(\gamma_0^{-1}(y)),\ \big((s \circ s_0)(\delta_0^{-1}(y))\big)^{-1} \cdot f\big(s_0(\delta_0^{-1}(y))\big)\big]\\ &= \big[\gamma(y) \cdot s(\gamma_0^{-1}(y)),\ \big((s \circ s_0 \circ s_0^{-1} \circ \gamma_0^{-1})(y)\big)^{-1} \cdot (f \circ s_0 \circ s_0^{-1} \circ \gamma_0^{-1})(y)\big]\\ &= \big[\gamma(y) \cdot s(\gamma_0^{-1}(y)),\ \big(s(\gamma_0^{-1}(y))\big)^{-1} \cdot (f \circ \gamma_0^{-1})(y)\big]\\ &= \big[\gamma(y),\ f \circ \gamma_0^{-1}(y)\big] = \big(\gamma_Q \circ (\mathrm{id}_{\gamma_0^{-1}(U)} \times f) \circ \gamma_0^{-1}\big)(y).\end{aligned}$$

This proves the independence of the r-jet $J^r_x(\gamma_Q \circ (\mathrm{id}_{\gamma_0^{-1}(U)} \times f) \circ \gamma_0^{-1})$ of the choice of $(J^r_x\gamma', J^r_0f') \in [J^r_x\gamma, J^r_0f]$. Thus

$$\Psi\colon (J^rFX)_Y \to J^rF_QX; \quad [J^r_x\gamma, J^r_0f] \mapsto J^r_x\big(\gamma_Q \circ (\mathrm{id}_{\gamma_0^{-1}(U)} \times f) \circ \gamma_0^{-1}\big)$$

is a well-defined mapping. Moreover, it can be verified that this mapping has the inverse Ψ^{-1} defined by the formula

$$\Psi^{-1} \colon J^r F_Q X \to (J^r FX)_Y; \quad J^r_x \beta \mapsto \big[J^r_x \gamma, J^r_0(p_2 \circ \gamma_Q^{-1} \beta \gamma_0)\big],$$

where γ is any local section of FX over $U \subset X$, $x \in U$, and $p_2 \colon \mathbb{R}^n \times Q \to Q$ is the second projection. Thus Ψ is a bijection. The differentiability of both Ψ and Ψ^{-1} follows from the differentiability of γ_Q and the composition of jets. The commutativity of Ψ with the projections onto X is obvious. □

5. Reduction of $W^r FX$ to $J^r FX$

Let P (resp. P_1) be a principal G-bundle (resp. G_1-bundle) over a manifold X. We say that P is a reduction of P_1 if there exists a pair (ν_X, ν), where $\nu \colon G \to G_1$ is an injective homomorphism of Lie groups and $\nu_X \colon P \to P_1$ is a homomorphism of principal fibre bundles over id_X, i.e., ν_X is smooth with $\operatorname{proj} \nu_X = \mathrm{id}_X$ and $\nu_X(p \cdot g) = \nu_X(p) \cdot \nu(g)$ for all $p \in P$ and $g \in G$.

The aim of this section is to show that the principal $T^r_n L^1_n$-bundle $J^r FX$ with the structure group $(T^r_n L^1_n, *)$ is a reduction of the principal $(L^1_n)^r_n$-bundle $W^r FX$.

Consider the mapping ν assigning to $\mathcal{S} \in T^r_n L^1_n$, $\mathcal{S} = J^r_0 s$, the element $\nu(\mathcal{S}) \in (L^1_n)^r_n$ defined by the formula

$$\nu(\mathcal{S}) = (J^r_0 s_0, J^r_0 s),$$

where s_0 is the mapping associated with $\mathcal{S}$. Clearly, ν is a well-defined mapping.

Let $\mathcal{S} = J^r_0 s$, $\mathcal{T} = J^r_0 t$ be elements of the Lie group $(T^r_n L^1_n, *)$. Since $\mathcal{S} * \mathcal{T} = J^r_0 u$, where $u = s \cdot (t \circ s_0^{-1})$ and for the mapping $u_0 \colon \mathbb{R}^n \to \mathbb{R}^n$, associated with $J^r_0 u$, we have $u_0 = s_0 \circ t_0$ (see proof of Lemma 2), we can write

$$\nu(\mathcal{S} * \mathcal{T}) = (J^r_0(s_0 \circ t_0), J^r_0(s \cdot (t \circ s_0^{-1}))).$$

Additionally, with respect to the operation defined on $(L^1_n)^r_n$, we have

$$\nu(\mathcal{S}) \cdot \nu(\mathcal{T}) = (J^r_0 s_0, J^r_0 s) \cdot (J^r_0 t_0, J^r_0 t) = (J^r_0(s_0 \circ t_0), J^r_0(s \cdot (t \circ s_0^{-1}))).$$

Thus ν is a homomorphism of groups. Clearly, ν is an injective smooth mapping, and therefore we can conclude that ν is an injective immersion of the Lie group $(T^r_n L^1_n, *)$ to $(L^1_n)^r_n$.

Now, consider

$$\nu_X \colon J^r FX \to W^r FX; \quad J^r_x \gamma \mapsto (J^r_0 \gamma_0, J^r_x \gamma),$$

where $\gamma_0 \colon \mathbb{R}^n \to X$ is the mapping associated with $J^r_x \gamma$. It is easy to see that ν_X is a well-defined injective smooth mapping and $\operatorname{proj} \nu_X = \mathrm{id}_X$. We are going to show that

$$\nu_X(\Upsilon * \mathcal{S}) = \nu_X(\Upsilon) \cdot \nu(\mathcal{S}) \tag{19}$$

for all $\Upsilon \in J^r FX$ and $\mathcal{S} \in (T^r_n L^1_n, *)$.

First, we notice that the mapping associated with $\Upsilon * \mathcal{S}$, where $\Upsilon = J^r_x \gamma$, $\mathcal{S} = J^r_0 s$, is equal to $\gamma_0 \circ s_0$ (see Proof of Lemma 4). Now, we can write

$$\begin{aligned}\nu_X(\Upsilon * \mathcal{S}) &= \nu_X(J^r_x \gamma * J^r_0 s) = \nu_X\big(J^r_x(\gamma \cdot (s \circ \gamma_0^{-1}))\big)\\ &= \big(J^r_0(\gamma_0 \circ s_0),\, J^r_x(\gamma \cdot (s \circ \gamma_0^{-1}))\big)\end{aligned}$$

and (using the action of $(L^1_n)^r_n$ on $W^r FX$)

$$\nu_X(\Upsilon) \cdot \nu(\mathcal{S}) = (J^r_0 \gamma_0, J^r_x \gamma) \cdot (J^r_0 s_0, J^r_0 s) = \big(J^r_0(\gamma_0 \circ s_0),\, J^r_x(\gamma \cdot (s \circ \gamma_0^{-1}))\big).$$

Thus (19) is true.

Summarizing, we obtain the following main result of this paper.

Theorem 3. *The principal bundle $J^r FX$ with the structure group $(T^r_n L^1_n, *)$ is a reduction of the principal $(L^1_n)^r_n$-bundle $W^r FX$.*

This is analogous to the result on reduction of $W^r FX$ to the principal bundle $F^{r+1}X$ with the structure group L^{r+1}_n (see [12]).

Remark 1. We have an injective homomorphism of Lie groups

$$\iota\colon L^{r+1}_n \to T^r_n L^1_n, \quad J^{r+1}_0 \alpha \mapsto J^r_0 \tilde{\alpha}, \tag{20}$$

where $\tilde{\alpha}\colon \mathbb{R}^n \to L^1_n$ is for any $z \in \mathbb{R}^n$ given by

$$\tilde{\alpha}(z) = J^1_0(t_z \circ \alpha \circ t_{-\alpha_0^{-1}(z)}),$$

and $\alpha_0\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping satisfying $J^1_0 \alpha_0 = J^1_0 \alpha$.

Using (20) and the corresponding local trivializations of principal bundles $F^{r+1}X$ and $J^r FX$ we obtain that $F^{r+1}X$ is a reduction of $J^r FX$. Thus, we have the sequence of reductions

$$F^{r+1}X \longrightarrow J^r FX \longrightarrow W^r FX.$$

Remark 2. Let $\bar{F}^2 X$ be the *semi-holonomic* frame bundle of order 2. In [14], it is stated that there exists a natural diffeomorphism from $J^1 FX$ onto the principal bundle $\bar{F}^2 X$ (without any reference to the principal bundle structure on $J^1 FX$). Considering the *holonomic* frame bundle $F^2 X$, this statement transforms into the following one: The mapping

$$\iota_X\colon F^2 X \to J^1 FX; \quad \iota_X(J^2_0 \zeta) = J^1_x(J^1 \zeta \circ \zeta^{-1}),$$

is a homomorphism of principal fibre bundles over id_X.

Remark 3. Let Q be a left L^1_n-manifold. By the *general prolongation theory*, $T^r_n Q$ has a (canonical) structure of a left L^{r+1}_n-manifold. For any $J^{r+1}_0 \alpha \in L^{r+1}_n$, $J^r_0 f \in T^r_n Q$, a left action of L^{r+1}_n on $T^r_n Q$ is given by

$$J^{r+1}_0 \alpha \cdot J^r_0 f = J^r_0(\bar{\alpha} \cdot (f \circ \alpha^{-1})), \tag{21}$$

where $\bar{\alpha}$ is defined by

$$\bar{\alpha}(z) = J^1_0(t_z \circ \alpha \circ t_{-\alpha^{-1}(z)}).$$

Denoting $\iota(J_0^{r+1}\alpha) = J_0^r\tilde{\alpha}$, and $\alpha_0 \circ \alpha^{-1} = \beta$, we have

$$\begin{aligned} J_0^{r+1}\alpha \cdot J_0^r f &= J_0^r(\bar{\alpha} \cdot (f \circ \alpha^{-1})) \\ &= J_0^r((\bar{\alpha} \circ \alpha \circ \alpha_0^{-1} \circ \alpha_0 \circ \alpha^{-1}) \cdot (f \circ \alpha_0^{-1} \circ \alpha_0 \circ \alpha^{-1})) \\ &= J_0^r((\tilde{\alpha} \cdot (f \circ \alpha_0^{-1})) \circ (\alpha_0 \circ \alpha^{-1})) \\ &= (J_0^r\tilde{\alpha} * J_0^r f) \cdot J_0^r\beta. \end{aligned} \tag{22}$$

Let us denote by $\pi_n^{r+1,1} \colon L_n^{r+1} \to L_n^1$ the canonical jet projection, by $\iota_n^{1,r+1} \colon L_n^1 \to L_n^{r+1}$ the canonical injective Lie group morphism, and put $K_n^{r+1,1} = \operatorname{Ker} \pi_n^{r+1,1}$. Then L_n^{r+1} is the interior semi-direct product of $\iota_n^{1,r+1}(L_n^1)$ and $K_n^{r+1,1}$.

Consider the subgroup $\iota(L_n^{r+1})$ of $T_n^r L_n^1$, defined by ι (20). Then (22) means that the left action (21) of L_n^{r+1} on $T_n^r Q$ corresponds with the action (16) of $\iota(L_n^{r+1})$ on $T_n^r Q$ through the element $J_0^r\beta \in K_n^{r+1,1}$.

Remark 4. The action (16) of $T_n^r L_n^1$ on $T_n^r Q$ is in some sense more general than the left action (21) of L_n^{r+1} on $T_n^r Q$ given by the general prolongation theory. Consider a vector bundle with type fibre Q with $\dim Q = m$. Let Q be a left L_m^1-manifold. Let (z^I) denote coordinates on Q, $1 \le I \le m$. Then (16) allows us to consider actions of L_m^1 on Q of the form

$$\bar{z}^I = P_J^I(x^k) z^J,$$

where $P_J^I \colon U \to L_m^1$ are arbitrary smooth mappings.

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