

ON THE GENERALIZATION OF DENSITY TOPOLOGIES ON THE REAL LINE

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ABSTRACT. The paper concerns the density points with respect to the sequences of intervals tending to zero in the family of Lebesgue measurable sets. It shows that for some sequences analogue of the Lebesgue density theorem holds. Simultaneously, this paper presents proof of theorem that for any sequence of intervals tending to zero a relevant operator Φ_J generates a topology. It is almost but not exactly the same result as in the category aspect presented in [WIERTELAK, R.: *A generalization of density topology with respect to category*, Real Anal. Exchange **32** (2006/2007), 273–286]. Therefore this paper is a continuation of the previous research concerning similarities and differences between measure and category.

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1. Introduction

Throughout the paper we will use the standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathbb{N}$ the set of natural numbers and $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$. By $\lambda(A)$ we shall denote the Lebesgue measure of a measurable set A and by $|I|$ the length of an interval I .

It is well known definition that a point $x_0 \in \mathbb{R}$ is a density point of a set $A \in \mathcal{L}$ if

$$\lim_{h \rightarrow 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

This condition is equivalent to the following one

$$\lim_{\substack{h_1 \xrightarrow{n \rightarrow \infty} 0^+, h_2 \xrightarrow{n \rightarrow \infty} 0^+ \\ h_1 + h_2 > 0}} \frac{\lambda(A \cap [x_0 - h_1, x_0 + h_2])}{h_1 + h_2} = 1.$$

It is sometimes written in the form (see [6]):

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$$\forall_{\{J_n\}_{n \in \mathbb{N}}} \left[\left(0 \in \bigcap_{n \in \mathbb{N}} J_n \wedge |J_n| \xrightarrow{n \rightarrow \infty} 0 \right) \implies \lim_{n \rightarrow \infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1 \right].$$

We say that a sequence of closed intervals $J = \{J_n\}_{n \in \mathbb{N}}$ is *tending to zero*, if $\text{diam}\{\{0\} \cup J_n\} \xrightarrow{n \rightarrow \infty} 0$. We will consider only sequences of closed intervals.

Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of intervals tending to zero. We shall say that a point $x_0 \in \mathbb{R}$ is a *J-density point* of a set $A \in \mathcal{L}$, if

$$\lim_{n \rightarrow \infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 1.$$

If a point $x_0 \in \mathbb{R}$ is a *J-density point* of the set $\mathbb{R} \setminus A$, then x_0 is a *J-dispersion point* of A .

Equivalently we have that x_0 is a *J-dispersion point* of A if and only if

$$\lim_{n \rightarrow \infty} \frac{\lambda(A \cap (J_n + x_0))}{|J_n|} = 0.$$

If $A \in \mathcal{L}$, then we denote

$$\Phi_J(A) = \{x \in \mathbb{R} : x \text{ is a } J\text{-density point of } A\}.$$

This operator fulfills the following properties.

PROPERTY 1. If $A \in \mathcal{L}$, then $\Phi_J(A) \in F_{\delta\sigma}$ in particular $\Phi_J(A) \in \mathcal{L}$.

Proof. Let $A \in \mathcal{L}$. Then $x \in \Phi_J(A)$ if and only if

$$\forall_{k \in \mathbb{N}} \quad \exists_{m \in \mathbb{N}} \quad \forall_{n > m} \quad \frac{\lambda(A \cap (J_n + x))}{|J_n|} \geq 1 - \frac{1}{k}.$$

Hence

$$\Phi_J(A) = \bigcap_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n > m} \left\{ x \in \mathbb{R} : \frac{\lambda(A \cap (J_n + x))}{|J_n|} \geq 1 - \frac{1}{k} \right\}.$$

The function $f(x) = \lambda(A \cap (J_n + x))$ is continuous for a fixed $n \in \mathbb{N}$, even it satisfies Lipschitz condition. Indeed, for every x_1, x_2 we get

$$\begin{aligned} |f(x_1) - f(x_2)| &= |\lambda(A \cap (J_n + x_1)) - \lambda(A \cap (J_n + x_2))| \\ &\leq |\lambda((A \cap (J_n + x_1)) \triangle (A \cap (J_n + x_2)))| \\ &= |\lambda(A \cap ((J_n + x_1) \triangle (J_n + x_2)))| \\ &\leq |\lambda((J_n + x_1) \triangle (J_n + x_2))| \leq 2|x_1 - x_2|. \end{aligned}$$

Therefore $\Phi_J(A) \in F_{\delta\sigma}$. □

PROPERTY 2. For any sets $A, B \in \mathcal{L}$ and a sequence of intervals J tending to zero we have:

- (1) $\Phi_J(\emptyset) = \emptyset, \quad \Phi_J(\mathbb{R}) = \mathbb{R};$
- (2) $\lambda(A \triangle B) = 0 \implies \Phi_J(A) = \Phi_J(B);$
- (3) $\Phi_J(A \cap B) = \Phi_J(A) \cap \Phi_J(B).$

It turned out that analogue of the Lebesgue density theorem does not hold for every sequence of intervals J tending to zero. Studying paper [1] we can find that there exists a sequence of intervals J tending to zero and a set $A \in \mathcal{L}$ of positive measure such that $\lambda(\Phi_J(A) \cap A) = 0$.

2. The main results

For any sequence of intervals $J = \{J_n\}_{n \in \mathbb{N}}$ tending to zero we define

$$\alpha(J) = \limsup_{n \rightarrow \infty} \frac{\text{diam}\{\{0\} \cup J_n\}}{|J_n|}$$

The proof of the next theorem bases on the idea presented in the Oxtoby book [4].

THEOREM 3. *If $\alpha(J) < \infty$ and $A \in \mathcal{L}$, then*

$$\lambda(A \Delta \Phi_J(A)) = 0.$$

Proof. It may be assumed without loss of generality that the set A is bounded and the sequence $\{|J_n|\}_{n \in \mathbb{N}}$ is decreasing. Moreover, we assume that

$$\text{diam}\{\{0\} \cup J_n\} < 2\alpha(J)|J_n|. \quad (1)$$

By condition 3 of Property 2 it is sufficient to prove that $\lambda(A \setminus \Phi_J(A)) = 0$. We show that for any $0 < \varepsilon < 1$ the set

$$E_\varepsilon = \left\{ x \in A : \liminf_{n \in \mathbb{N}} \frac{\lambda(A \cap (J_n + x))}{|J_n|} < 1 - \varepsilon \right\}$$

has measure zero. Suppose, contrary to our claim, that $\lambda^*(E_\varepsilon) > 0$. Hence there exists an open set $G \supset E$ such that $(1 - \varepsilon)\lambda(G) < \lambda^*(E_\varepsilon)$. Let $\mathcal{E}$ be the family of all closed intervals $I \subset G$ such that $\lambda(A \cap I) < (1 - \varepsilon)|I|$ and $I = J_n + x$ for some $x \in E_\varepsilon$ and $n \in \mathbb{N}$. Observe that

- (i) every neighbourhood of each $x \in E_\varepsilon$ contains an interval $I \in \mathcal{E}$;
- (ii) for any sequence $\{I_n\}$ of disjoint intervals of $\mathcal{E}$ the inequality $\lambda^*(E_\varepsilon \setminus \bigcup I_n) > 0$ holds.

The property (i) is evident from the definition. The property (ii) follows from the fact that

$$\begin{aligned} \lambda^*\left(E_\varepsilon \cap \bigcup I_n\right) &\leq \sum \lambda(A \cap I_n) \leq (1 - \varepsilon) \sum |I_n| \\ &= (1 - \varepsilon)\lambda\left(\bigcup I_n\right) \leq (1 - \varepsilon)\lambda(G) < (1 - \varepsilon)\lambda^*(E_\varepsilon). \end{aligned}$$

We construct inductively a sequence $\{I_n\}_{n \in \mathbb{N}}$ of disjoint intervals of $\mathcal{E}$ as follows. Define

$$k_0 = \min \left\{ n \in \mathbb{N} : \exists_{x \in E_\varepsilon} J_n + x \in \mathcal{E} \right\}.$$

We choose interval I_1 from $\mathcal{E}$ with length $|J_{k_0}|$. Having chosen intervals I_i for $i \in \{1, 2, \dots, n\}$ let $\mathcal{E}_n$ be the subset of $\mathcal{E}$ which consists of intervals that are disjoint from $I_1, \dots, I_n$. Properties (ii) and (i) imply that $\mathcal{E}_n \neq \emptyset$. Let us define

$$k_n = \min \left\{ i \in \mathbb{N} : \bigcup_{x \in E_\varepsilon} J_i + x \in \mathcal{E}_n \right\}.$$

We choose interval I_{n+1} from $\mathcal{E}_n$ with length $|J_{k_n}|$.

Putting $B = E_\varepsilon \setminus \bigcup_{n \in \mathbb{N}} I_n$ we obtain by property (ii) that $\lambda^*(B) > 0$. Hence there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} |I_n| < \frac{\lambda^*(B)}{4\alpha(J) + 1}. \quad (2)$$

Let K_n denote the interval concentric with I_n such that $|K_n| = (4\alpha(J) + 1)|I_n|$ for each $n > N$. The inequality (2) implies that $\bigcup_{n \in \mathbb{N}} K_n$ does not cover the

set B . So, there exists a point $x \in B \setminus \bigcup_{n > N} K_n$. Therefore $x \in E_\varepsilon \setminus \bigcup_{n=1}^N I_n$. By

properties (ii) and (i) there exists an interval $I_x \in \mathcal{E}_N$, $I_x = J_{n_x} + x$. It is clear that $I_x \cap I_n \neq \emptyset$ for some $n > N$. Let $n_0 = \min\{n \in \mathbb{N} : I_n \cap I_x \neq \emptyset\}$. Then $|J_{n_x}| = |I_x| \leq |I_{n_0}|$. Using (1) we obtain

$$\text{dist}(x, I_{n_0}) \leq \text{diam}\{x \cup I_x\} = \text{diam}\{\{0\} \cup J_{n_x}\} < 2\alpha(J)|J_{n_x}| \leq 2\alpha(J)|I_{n_0}|.$$

It follows that $x \in K_{n_0}$. It is contrary to the fact $x \in B \setminus \bigcup_{n > N} K_n$. This and the inclusion

$$A \setminus \Phi_J(A) \subset \bigcup_{\varepsilon \in (0,1) \cap \mathbb{Q}} E_\varepsilon$$

mean that $\lambda(A \Delta \Phi_J(A)) = 0$. □

Combining Property 2 with Theorem 3 implies that operator Φ_J is the lower density operator. By the general lifting theorem [3] we have the following theorem.

THEOREM 4. *Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of intervals tending to zero such that $\alpha(J) < \infty$. Then the family*

$$\mathcal{T}_J = \{A \in \mathcal{S} : A \subset \Phi_J(A)\}$$

is a topology on $\mathbb{R}$. Moreover, we have that $\mathcal{T}_{\text{nat}} \subsetneq \mathcal{T}_J$ and $\mathcal{T}_d \subset \mathcal{T}_J$.

This theorem is the essential generalization of the case of $\langle s \rangle$ -topology (see [2]). There are considered sequences of symmetrical intervals of the form $J_n = [-\frac{1}{s_n}, \frac{1}{s_n}]$, where $s = \{s_n\}_{n \in \mathbb{N}}$ is an unbounded and nondecreasing sequence of positive real numbers.

THEOREM 5. *If $J = \{J_n\}_{n \in \mathbb{N}}$ is a sequence of intervals tending to zero such that $\alpha(J) < \infty$, then there exists a sequence $K = \{K_n\}_{n \in \mathbb{N}}$ of symmetrical intervals tending to zero such that for every set $A \in \mathcal{L}$ we have that $\Phi_K(A) \subset \Phi_J(A)$.*

P r o o f. Let $J_n = [a_n, b_n]$ for $n \in \mathbb{N}$ and $\alpha(J) < \infty$. Define $s_n = \max\{|a_n|, |b_n|\}$ and $K_n = [-s_n, s_n]$ for $n \in \mathbb{N}$. Assume that $A \in \mathcal{L}$, $0 \in \Phi_K(A)$ and $\varepsilon > 0$. There exists $n_1 \in \mathbb{N}$ such that

$$\frac{\text{diam}\{\{0\} \cup J_n\}}{|J_n|} < 2\alpha(J)$$

for every $n > n_1$ and there exists $n_2 \in \mathbb{N}$ such that

$$\frac{\lambda(A \cap K_n)}{|K_n|} > 1 - \frac{\varepsilon}{4\alpha(J)}$$

for every $n > n_2$. Hence for every $n > \max\{n_1, n_2\}$ we have

$$\lambda(A' \cap J_n) \leq \lambda(A' \cap K_n) < \frac{\varepsilon}{4\alpha(J)}|K_n| < \varepsilon|J_n|$$

as

$$|K_n| \leq 2 \text{diam}\{\{0\} \cup J_n\} < 4\alpha(J)|J_n|.$$

Therefore

$$\frac{\lambda(A \cap J_n)}{|J_n|} > 1 - \varepsilon.$$

It implies that $0 \in \Phi_J(A)$. □

THEOREM 6. *If $J = \{J_n\}_{n \in \mathbb{N}}$ is a sequence of intervals tending to zero and $\alpha(J) = \infty$, then there exists an open set A such that $0 \in \Phi_d(A)$ and $0 \notin \Phi_J(A)$.*

P r o o f. By assumption that $\alpha(J) = \infty$ there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ such that

$$\text{diam}\{\{0\} \cup J_{n_k}\} > (k+1)|J_{n_k}| \quad (3)$$

for every $k \in \mathbb{N}$. Moreover it can be assumed that

$$\text{diam}\{\{0\} \cup J_{n_{k+1}}\} \leq |J_{n_k}| \quad (4)$$

and that all intervals J_{n_k} for $k \in \mathbb{N}$ are situated on one side of zero.

Putting

$$A = (-1, 1) \setminus \left(\bigcup_{k \in \mathbb{N}} J_{n_k} \cup \{0\} \right)$$

we obtain that A is an open set and $0 \notin \Phi_J(A)$.

For every $m \in \mathbb{N}$ we define

$$j(m) = \min\left\{k \in \mathbb{N} : \lambda\left(\left[-\frac{1}{m}, \frac{1}{m}\right] \cap J_{n_k}\right) > 0\right\}.$$

The sequence $\{j(m)\}_{m \in \mathbb{N}}$ is unbounded and nondecreasing and by (3)

$$\text{diam}\{\{0\} \cup J_{n_{j(m)}}\} \leq \frac{1}{m} + |J_{n_{j(m)}}| \leq \frac{1}{m} + \frac{1}{2} \text{diam}\{\{0\} \cup J_{n_{j(m)}}\}.$$

Hence

$$\text{diam}\{\{0\} \cup J_{n_{j(m)}}\} \leq \frac{2}{m}. \quad (5)$$

Now we show that $0 \in \Phi_d(A)$. Indeed, from (4), (3), (5) we get that

$$\begin{aligned} \lambda\left(\left[-\frac{1}{m}, \frac{1}{m}\right] \setminus A\right) &= \lambda\left(\left[-\frac{1}{m}, \frac{1}{m}\right] \cap \bigcup_{k=j(m)}^{\infty} J_{n_k}\right) \leq \lambda\left(\bigcup_{k=j(m)}^{\infty} J_{n_k}\right) \\ &\leq \text{diam}\{\{0\} \cup J_{n_{j(m)+1}}\} + |J_{n_{j(m)}}| \leq 2|J_{n_{j(m)}}| \\ &\leq \frac{2}{j(m)+1} \text{diam}\{\{0\} \cup J_{n_{j(m)}}\} \leq \frac{4}{(j(m)+1)m}. \end{aligned}$$

Therefore

$$\frac{\lambda\left(\left[-\frac{1}{m}, \frac{1}{m}\right] \setminus A\right)}{2/m} \leq 1 - \frac{2}{j(m)+1} \xrightarrow{m \rightarrow \infty} 0.$$

It means that $0 \in \Phi_d(A)$. □

The next theorem we obtain as the conclusion of the last two theorems.

THEOREM 7. *For every sequence of intervals $J = \{J_n\}_{n \in \mathbb{N}}$ tending to zero the following conditions are equivalent:*

- i) $\alpha(J) < \infty$;
- ii) $\mathcal{T}_d \subset \mathcal{T}_J$.

It has been already mentioned that analogue of Lebesgue Density Theorem does not hold for any sequence of intervals tending to zero. Nevertheless, the following theorem is true.

THEOREM 8. *If a sequence of intervals $J = \{J_n\}_{n \in \mathbb{N}}$ is tending to zero and $A \in \mathcal{L}$, then*

$$\lambda(A \cap \Phi_J(\mathbb{R} \setminus A)) = 0.$$

To prove it we need a supporting lemma.

LEMMA 9. *Let $A \in \mathcal{L}$, $\{(A_i, B_i)\}_{i=1}^n$ be sequence of pairs of measurable sets such that $\lambda(A \cap A_i) < \varepsilon|A_i|$, where $\varepsilon < 1/4$, and $\lambda(B_i) = \lambda(A_i)$. If the sets A_i are pairwise disjoint, then*

$$\lambda\left(A \cap \bigcup_{i=1}^n (A_i \cup B_i)\right) < (1 - \varepsilon)\lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right).$$

Proof.

$$\begin{aligned} \frac{\lambda\left(A \cap \bigcup_{i=1}^n (A_i \cup B_i)\right)}{\lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right)} &\leq \frac{\varepsilon \sum_{i=1}^n \lambda(A_i) + \lambda\left(\bigcup_{i=1}^n B_i \setminus \bigcup_{i=1}^n A_i\right)}{\lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right)} \\ &= \frac{\varepsilon \sum_{i=1}^n \lambda(A_i) + \lambda\left(\bigcup_{i=1}^n A_i\right) - \lambda\left(\bigcup_{i=1}^n A_i\right) + \lambda\left(\bigcup_{i=1}^n B_i \setminus \bigcup_{i=1}^n A_i\right)}{\lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right)} \end{aligned}$$

$$= 1 - \frac{(1 - \varepsilon) \sum_{i=1}^n \lambda(A_i)}{\lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right)} \leq 1 - \frac{(1 - \varepsilon) \sum_{i=1}^n \lambda(A_i)}{2 \sum_{i=1}^n \lambda(A_i)} = \frac{1 + \varepsilon}{2} < 1 - \varepsilon.$$

Therefore

$$\lambda\left(A \cap \bigcup_{i=1}^n (A_i \cup B_i)\right) < (1 - \varepsilon) \lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right). \quad \square$$

Proof of Theorem 8. Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of intervals tending to zero. If $\alpha(J) < \infty$, then theorem holds by virtue of Theorem 3. So, suppose that $\alpha(J) = \infty$. Hence there exists subsequence $\{J_{n_k}\}_{k \in \mathbb{N}}$ such that $\text{diam}\{\{0\} \cup J_{n_k}\} > k|J_{n_k}|$ for $k \in \mathbb{N}$. Moreover we can assume that $|J_{n_k}| > \text{diam}\{\{0\} \cup J_{n_{k+1}}\}$ for $k \in \mathbb{N}$ and all intervals J_{n_k} are located on one side of zero. For the sake of the simplicity we denote $J'_k = J_{n_k}$ for $k \in \mathbb{N}$.

Suppose, contrary to our claim, that the set

$$E = A \cap \Phi_J(\mathbb{R} \setminus A)$$

is not a nullset, hence $\lambda(E) > 0$. Let $0 < \varepsilon < 1/4$ and $A \in \mathcal{L}$. Without loss of generality we can assume that the set A is bounded. There exists an open set $G \supset E$ such that $(1 - \varepsilon)\lambda(G) < \lambda(E)$.

For every $x \in E$ and $k \in \mathbb{N}$ we define intervals $A_{x,k} = J'_k + x$ and $B_{x,k} = \left[x - \frac{|J'_k|}{2}, x + \frac{|J'_k|}{2}\right]$. Then $|A_{x,k}| = |B_{x,k}| = |J'_k|$. Let $\mathcal{E}$ denote the family of pairs of intervals $A_{x,k}, B_{x,k} \subset G$ such that $\lambda(A \cap A_{x,k}) < \varepsilon|A_{x,k}|$. It is easy to see that an arbitrary neighbourhood of $x \in E$ contains a pair of intervals $(A_{x,k}, B_{x,k})$ of the family $\mathcal{E}$.

Now we construct a sequence of pairs of intervals $(A_n, B_n) \in \mathcal{E}$ in the following way. Let

$$k_0 = \min\{k \in \mathbb{N} : (A_{x,k}, B_{x,k}) \in \mathcal{E}\}.$$

We choose the pair of intervals (A_1, B_1) from $\mathcal{E}$ with length $|J'_{k_0}|$. Suppose that pairs of intervals (A_i, B_i) , for $i \in \{1, 2, \dots, n\}$ have been defined and intervals A_i are pairwise disjoint. By lemma 9 we get that

$$\lambda\left(E \cap \bigcup_{i=1}^n (A_i \cup B_i)\right) \leq (1 - \varepsilon) \lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right).$$

It means that

$$\lambda\left(E \cap \bigcup_{i=1}^n (A_i \cup B_i)\right) \leq (1 - \varepsilon) \lambda\left(\bigcup_{i=1}^n (A_i \cup B_i)\right) \leq (1 - \varepsilon) \lambda(G) < \lambda(E). \quad (6)$$

Therefore the set $\bigcup_{i=1}^n (A_i \cup B_i)$ does not cover the set E .

Define the family $\mathcal{E}_n$ composed of pairs of intervals $(A_{x,k}, B_{x,k}) \in \mathcal{E}$, such that every interval $A_{x,k}$ is disjoint with each interval A_i for $i \in \{1, 2, \dots, n\}$. By condition (6) we get that $\mathcal{E}_n \neq \emptyset$. Let

$$k_n = \min\{k \in \mathbb{N} : A_{x,k}, B_{x,k} \in \mathcal{E}_n\}.$$

We choose the pair of intervals (A_{n+1}, B_{n+1}) from $\mathcal{E}_n$ with length $|J'_{k_n}|$.

As a result we obtain the sequence $\{(A_n, B_n)\}_{n \in \mathbb{N}}$ of pairs of intervals of the family $\mathcal{E}$ such that the intervals of the sequence $\{A_n\}_{n \in \mathbb{N}}$ are pairwise disjoint and the sequence $\{|A_n|\}_{n \in \mathbb{N}}$ is not increasing. Moreover

$$\lambda\left(E \cap \bigcup_{i=1}^{\infty} (A_i \cup B_i)\right) \leq \lambda\left(A \cap \bigcup_{i=1}^{\infty} (A_i \cup B_i)\right) \leq (1 - \varepsilon) \lambda\left(\bigcup_{i=1}^{\infty} (A_i \cup B_i)\right).$$

Therefore

$$\lambda\left(E \cap \bigcup_{n=1}^{\infty} (A_n \cup B_n)\right) < (1 - \varepsilon) \lambda\left(\bigcup_{i=1}^{\infty} (A_i \cup B_i)\right) < (1 - \varepsilon) \lambda(G) < \lambda(E).$$

Putting $X = E \setminus \bigcup_{n=1}^{\infty} (A_n \cup B_n)$ we obtain that $\lambda(X) > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\lambda\left(\bigcup_{i=N+1}^{\infty} (A_i \cup B_i)\right) < \lambda(X)/3.$$

For each $n > N$ we define the intervals C_n, D_n concentric with A_n, B_n respectively and with length $3|A_n|$. Then

$$\lambda\left(X \setminus \bigcup_{n=N+1}^{\infty} (C_n \cup D_n)\right) > 0.$$

Hence, there exists $x_0 \in \left(X \setminus \bigcup_{n=N+1}^{\infty} (C_n \cup D_n)\right)$. In particular $x_0 \in G \setminus \bigcup_{i=1}^N (A_i \cup B_i)$.

Moreover, there exists a neighbourhood $V \ni x_0$ such that $V \subset G \setminus \bigcup_{i=1}^N (A_i \cup B_i)$.

It implies that there is $k \in \mathbb{N}$ such that $(A_{x_0,k}, B_{x_0,k}) \in \mathcal{E}_N$ and $A_{x_0,k} \subset V, B_{x_0,k} \subset V$. It is clear that $(A_{x_0,k}, B_{x_0,k})$ does not belong to the sequence $\{(A_n, B_n)\}_{n \in \mathbb{N}}$. Simultaneously, there exists $n_0 > N$ such that $A_{x_0,k} \cap A_{n_0} \neq \emptyset$ and $|A_{x_0,k}| \leq |A_{n_0}|$. Otherwise the pair $(A_{x_0,k}, B_{x_0,k})$ should belong to the sequence $\{(A_n, B_n)\}_{n \in \mathbb{N}}$. Now we consider two cases.

If $|A_{x_0,k}| < |A_{n_0}|$, then

$$\text{dist}(x_0, A_{n_0}) \leq \text{diam}\{\{x_0\} \cup A_{x_0,k}\} = \text{diam}\{\{0\} \cup J'_k\} < |J'_{k-1}| \leq |A_{n_0}|.$$

Hence $x_0 \in C_{n_0}$.

If $|A_{x_0,k}| = |A_{n_0}|$, then $B_{x_0,k} \cap B_{n_0} \neq \emptyset$. Moreover $x_0 \in B_{x_0,k}$, so

$$\text{dist}(x_0, B_{n_0}) \leq |B_{x_0,k}| = |B_{n_0}|.$$

Hence $x_0 \in D_{n_0}$.

In both cases we obtain that $x_0 \in C_{n_0} \cup D_{n_0}$. On the other hand $x_0 \in \left(X \setminus \bigcup_{n=N+1}^{\infty} (C_n \cup D_n)\right)$. This contradiction ends the proof. $\square$

THEOREM 10. *Let $J = \{J_n\}_{n \in \mathbb{N}}$ be a sequence of intervals tending to zero. Then*

$$\mathcal{T}_J = \{A \in \mathcal{S} : A \subset \Phi_J(A)\}$$

is a topology on $\mathbb{R}$, which will be called J -density topology.

Moreover, we have that $\mathcal{T}_{nat} \subsetneq \mathcal{T}_J$.

Proof. By the Property 2 we have that $\emptyset \in \mathcal{T}_J$, $\mathbb{R} \in \mathcal{T}_J$ and the family $\mathcal{T}_J$ is closed under finite intersections.

Let $\{A_t\}_{t \in T} \subset \mathcal{T}_J$. We will prove that $\bigcup_{t \in T} A_t \in \mathcal{T}_J$. Let B denote a measurable kernel of the set $\bigcup_{t \in T} A_t$. Then $\lambda((B \cap A_t) \triangle A_t) = 0$ for $t \in T$ and

$$B \subset \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi_J(A_t) = \bigcup_{t \in T} \Phi_J(B \cap A_t) \subset \Phi_J(B).$$

By the Theorem 8 we have $\lambda(\Phi_J(B) \setminus B) = 0$. Hence $\bigcup_{t \in T} A_t \in \mathcal{L}$ and moreover

$$\bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi_J(A_t) \subset \Phi_J\left(\bigcup_{t \in T} A_t\right).$$

Therefore $\bigcup_{t \in T} A_t \in \mathcal{T}_J$ and $\mathcal{T}_J$ is a topology.

The inclusion $\mathcal{T}_{nat} \subset \mathcal{T}_J$ is evident from the definition of J -density topology. It is proper because $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_J \setminus \mathcal{T}_{nat}$. $\square$

The next theorem shows that we have obtained a more general definition of density point.

THEOREM 11. *There exists a sequence of intervals $J = \{J_n\}_{n \in \mathbb{N}}$ tending to zero and measurable sets A and B such that $0 \in \Phi_J(A)$, $0 \notin \Phi_d(A)$, $0 \notin \Phi_J(B)$, $0 \in \Phi_d(B)$.*

Proof. It is enough to consider the sequence $J = \left\{\left[\frac{n}{(n+1)!}, \frac{1}{n!}\right]\right\}_{n \in \mathbb{N}}$ and sets $A = \bigcup_{n \in \mathbb{N}} \left[\frac{n}{(n+1)!}, \frac{1}{n!}\right]$, $B = \mathbb{R} \setminus A$. Then $0 \in \Phi_J(A)$, $0 \notin \Phi_d(A)$, $0 \notin \Phi_J(B)$, $0 \in \Phi_d(B)$. $\square$

3. Conclusions

In paper [5] are presented the following definitions.

Let $\mathcal{I}$ be the σ -ideal of first category sets in $\mathbb{R}$. The point 0 is called an $\mathcal{I}(J)$ -density point of a Baire set A if

$$\chi_{\frac{2}{|J_n|}(A-s(J_n))\cap[-1,1]}(x) \xrightarrow[n \rightarrow \infty]{\mathcal{I}} \chi_{[-1,1]}(x),$$

which means that

$$\forall \{n_k\}_{k \in \mathbb{N}} \quad \exists \{n_{k_m}\}_{m \in \mathbb{N}} \quad \exists \Theta \in \mathcal{I} \quad \forall x \notin \Theta \quad \chi_{\frac{2}{|J_{n_{k_m}}|}(A-s(J_{n_{k_m}}))\cap[-1,1]}(x) \xrightarrow[m \rightarrow \infty]{} \chi_{[-1,1]}(x).$$

If A is a Baire set then we denote

$$\Phi_{\mathcal{I}(J)}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}(J)\text{-density point of } A\}.$$

If $J = \{J_n\}_{n \in \mathbb{N}}$ is a sequence of intervals tending to zero each of operators Φ_J and $\Phi_{\mathcal{I}(J)}$ generates the topology, the J -density topology and $\mathcal{I}(J)$ -density topology, respectively. Therefore it is the next similarity between measure and category. Simultaneously the operator $\Phi_{\mathcal{I}(J)}$ is the lower density operator, while the operator Φ_J is not because the analogue of the Lebesgue theorem does not hold. Hence it is the next difference between measure and category.

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