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EXISTENCE OF NON-TRIVIAL SOLUTIONS FOR SYSTEMS OF n FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, employing a very recent local minimum theorem for differentiable functionals due to Bonanno, the existence of at least one non-trivial solution for a class of systems of n fourth order partial differential equations coupled with Navier boundary conditions is established.

1. Introduction

This paper treats the following nonlinear elliptic system of n fourth order partial differential equations under Navier boundary conditions

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i \Delta_{p_i} u_i + \beta_i |u_i|^{p_i-2} u_i & \\ = \lambda F_{u_i}(x, u_1, \dots, u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

for $1 \le i \le n$, where $\Delta_{p_i} u_i = \operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator, $n \ge 1$ is an integer, $p_i > \max\{1, \frac{N}{2}\}$, α_i and β_i are non-negative constants for $1 \le i \le n$, $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$, $\lambda > 0$, $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, t_2 \dots, t_n) \to F(x, t_1, t_2, \dots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in \Omega$ and F_{t_i} is continuous in $\Omega \times \mathbb{R}^n$ for $i = 1, \dots, n$. Here, F_{t_i} denotes the partial derivative of F with respect to t_i. The system (1.1) is called $(p_1, \dots, p_n)$-biharmonic.

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Precisely, based on a very recent local minimum theorem for differentiable functionals due to Bonanno [1: Theorem 5.1] (Theorem 2.1), we establish the existence of at least one non-trivial solution for the system (1.1) for any fixed positive parameter λ belonging to an exact interval.

Fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. Many authors have studied the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems by using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree theory, and critical point theory and variational methods, and we refer the reader to the papers [2, 4–11] and references therein.

We also refer the reader to the recent papers [3, 12] where the local minimum theorem for differentiable functionals due to Bonanno [1: Theorem 5.1] was successfully applied to second order Neumann boundary value problems.

2. Results

First we want to point out that our main tool is a critical point theorem, very recently obtained by Bonanno [1: Theorem 5.1] that we here recall in its equivalent formulation [1: Proposition 2.1, Remark 2.1]) (see also [13] for the related result).

For a given non-empty set X, and two functionals $\Phi, \Psi\colon X \to \mathbb{R}$, we define the following functions

$$\vartheta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup\limits_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.1. ([1: Theorem 5.1]) *Let X be a real Banach space; $\Phi\colon X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on*

X^*, $\Psi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that

$$\vartheta(r_1, r_2) < \rho(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)}\right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Here and in the sequel, X will denote the Cartesian product of n Sobolev spaces $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega)$ for $i = 1, \dots, n$, i.e., $X = (W^{2,p_1}(\Omega) \cap W_0^{1,p_1}(\Omega)) \times \dots \times (W^{2,p_n}(\Omega) \cap W_0^{1,p_n}(\Omega))$ endowed with the norm

$$\|(u_1, \dots, u_n)\| = \sum_{i=1}^n \|u_i\|_{p_i}$$

where

$$\|u_i\|_{p_i} = \left(\int_\Omega |\Delta u_i(x)|^{p_i} \,\mathrm{d}x + \alpha_i \int_\Omega |\nabla u_i(x)|^{p_i} \,\mathrm{d}x + \beta_i \int_\Omega |u_i(x)|^{p_i} \,\mathrm{d}x \right)^{1/p_i}$$

for $1 \le i \le n$.

We need the following proposition in the proof of Theorem 2.2.

Proposition 2.1. *Let $T\colon X \to X^*$ be the operator defined by*

$$\begin{aligned} T(u_1, \dots, u_n)(h_1, \dots, h_n) = &\int_\Omega \sum_{i=1}^n |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta h_i(x) \,\mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^n \alpha_i |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla h_i(x) \,\mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^n \beta_i |u_i(x)|^{p_i-2} u_i(x) h_i(x) \,\mathrm{d}x \end{aligned}$$

for every $(u_1, \dots, u_n)$, $(h_1, \dots, h_n) \in X$. Then T admits a continuous inverse on X^*.

P r o o f. Since

$$\begin{aligned} &T(u_1, \dots, u_n)(u_1, \dots, u_n) \\ &= \sum_{i=1}^n \left(\int_\Omega |\Delta u_i(x)|^{p_i} \,\mathrm{d}x + \alpha_i \int_\Omega |\nabla u_i(x)|^{p_i} \,\mathrm{d}x + \beta_i \int_\Omega |u_i(x)|^{p_i} \,\mathrm{d}x \right) \\ &= \sum_{i=1}^n \|u_i\|_{p_i}^{p_i}, \end{aligned}$$

T is coercive. Taking into account [14: (2.2)] for $p > 1$ there exists a positive constant C_p such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2,\\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2 \end{cases}$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^N$, for every $x, y \in \mathbb{R}^N$. Thus, it is easy to see that

$$\begin{aligned}
&\big\langle T(u_1,\dots,u_n) - T(v_1,\dots,v_n))(u_1 - v_1,\dots,u_n - v_n\big\rangle\\
&\geq \sum_{i\in I_1} C_{p_i}\int_\Omega \Big(|\Delta u_i(x) - \Delta v_i(x)|^{p_i}\,\mathrm{d}x\\
&\qquad\qquad + \alpha_i|\nabla u_i(x) - \nabla v_i(x)|^{p_i} + \beta_i|u_i(x) - v_i(x)|^{p_i}\Big)\,\mathrm{d}x\\
&\quad + \sum_{i\in I_2} C_{p_i}\int_\Omega\Bigg(\frac{|\Delta u_i(x) - \Delta v_i(x)|^2}{(|\Delta u_i(x)| + |\Delta v_i(x)|)^{2-p_i}} + \frac{\alpha_i|\nabla u_i(x) - \nabla v_i(x)|^2}{(|\nabla u_i(x)| + |\nabla v_i(x)|)^{2-p_i}}\\
&\qquad\qquad + \frac{\beta_i|u_i(x) - v_i(x)|^2}{(|u_i(x)| + |v_i(x)|)^{2-p_i}}\Bigg)\,\mathrm{d}x > 0
\end{aligned}$$

where $I_1 = \{i \in \{1,\dots,n\} :\ p_i \geq 2\}$ and $I_2 = \{i \in \{1,\dots,n\} :\ 1 < p_i < 2\}$, for every $(u_1,\dots,u_n), (v_1,\dots,v_n) \in X$, which means that T is strictly monotone. Moreover, arguing as in [9: Lemma 2], since $W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)$ for $i = 1,\dots,n$ is reflexive, for $(u_{1m},\dots,u_{nm}) \to (u_1,\dots,u_n)$ strongly in X as $m \to +\infty$, one has $T(u_{1m},\dots,u_{nm}) \to T(u_1,\dots,u_n)$ weakly in X^* as $m \to +\infty$. Hence T is demicontinuous, so by [15: Theorem 26.A(d)], the inverse operator T^{-1} of T exists. T^{-1} is continuous. Indeed, let $((f_{1m})_m,\dots,(f_{nm})_m)$ be a sequence of X^* such that $f_{im} \to f_i$ strongly in $(W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega))^*$ for $i = 1,\dots,n$ as $m \to +\infty$. Let $(u_{1m},\dots,u_{nm})$ and $(u_1,\dots,u_n)$ in X such that $T_{p_i}^{-1}(f_{im}) = u_{im}$ and $T_{p_i}^{-1}(f_i) = u_i$ where

$$\begin{aligned}
&\langle T_{p_i}(u_i), h_i\rangle\\
&= \int_\Omega |\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta h_i(x)\,\mathrm{d}x + \alpha_i\int_\Omega(|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla h_i(x))\,\mathrm{d}x\\
&\quad + \beta_i\int_\Omega(|u_i(x)|^{p_i-2}u_i(x)h_i(x))\,\mathrm{d}x
\end{aligned}$$

for every $u_i, h_i \in W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega)$ for $i = 1,\dots,n$. Taking in to account that T is coercive, one has that the sequence $(u_{1m},\dots,u_{nm})$ is bounded in the reflexive space X. For a suitable subsequence, we have $(u_{1m},\dots,u_{nm}) \to$

$(\widehat{u}_1, \dots, \widehat{u}_n)$ weakly in X as $m \to +\infty$, which concludes

$$\begin{aligned}\lim_{m\to+\infty} &\langle T(u_{1m}, \dots, u_{nm}) - T(u_1, \dots, u_n), (u_{1m}, \dots, u_{nm}) - (\widehat{u}_1, \dots, \widehat{u}_n)\rangle \\ &= \langle (f_{1m}, \dots, f_{nm}) - (f_1, \dots, f_n), (u_{1m}, \dots, u_{nm}) - (\widehat{u}_1, \dots, \widehat{u}_n)\rangle = 0.\end{aligned}$$

Note that if $(u_{1m}, \dots, u_{nm}) \to (\widehat{u}_1, \dots, \widehat{u}_n)$ weakly in X as $m \to +\infty$ and $T(u_{1m}, \dots, u_{nm}) \to T(\widehat{u}_1, \dots, \widehat{u}_n)$ strongly in X^* as $m \to +\infty$, one has $(u_{1m}, \dots, u_{nm}) \to (\widehat{u}_1, \dots, \widehat{u}_n)$ strongly in X as $m \to +\infty$, and since T is continuous we have $(u_{1m}, \dots, u_{nm}) \to (\widehat{u}_1, \dots, \widehat{u}_n)$ weakly in X as $m \to +\infty$ and $T(u_{1m}, \dots, u_{nm}) \to T(\widehat{u}_1, \dots, \widehat{u}_n) = T(u_1, \dots, u_n)$ strongly in X^* as $m \to +\infty$. Hence, taking into account that T is an injection, we have $(u_1, \dots, u_n) = (\widehat{u}_1, \dots, \widehat{u}_n)$. $\square$

We say that $u = (u_1, \dots, u_n)$ is a weak solution to the system (1.1) if $u = (u_1, \dots, u_n) \in X$ and

$$\begin{aligned}\int_\Omega \sum_{i=1}^n \Big(&|\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x) + \alpha_i|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x) \\ &+ \beta_i|u_i(x)|^{p_i-2}u_i(x)v_i(x)\Big)\,\mathrm{d}x - \lambda\int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x))v_i(x)\,\mathrm{d}x = 0\end{aligned}$$

for every $(v_1, \dots, v_n) \in X$.

For all $\gamma > 0$ we define

$$K(\gamma) := \Big\{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n \tfrac{|t_i|^{p_i}}{p_i} \le \gamma\Big\}. \tag{2.1}$$

Put

$$k = \max\Bigg\{\sup_{u_i\in(W^{2,p_i}(\Omega)\cap W_0^{1,p_i}(\Omega))\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{||u_i||_{p_i}^{p_i}} : \quad \text{for } 1 \le i \le n\Bigg\}. \tag{2.2}$$

For $p_i > \max\{1, \frac{N}{2}\}$ for $1 \le i \le n$, since the embedding $W^{2,p_i}(\Omega) \cap W_0^{1,p_i}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ for $1 \le i \le n$ is compact, one has $k < +\infty$.

Fix $x^0 \in \Omega$ and pick $s > 0$ such that

$$S(x^0, s) \subset \Omega \tag{2.3}$$

where $S(x^0, s)$ denotes the ball with center at x^0 and radius of s.

Put

$$\sigma_n = \sigma_n(N, p_n, s) := \frac{2\pi^{\frac{N}{2}}}{\Gamma\big(\frac{N}{2}\big)} \int_{\frac{s}{2}}^{s} \left|\frac{12(N+1)}{s^3}\xi - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{\xi}\right|^{p_n} \xi^{N-1}\,\mathrm{d}\xi,$$

$$\theta_n = \theta_n(N, p_n, x^0, s)$$
$$:= \int\limits_{S(x^0,s)\setminus S(x^0,\frac{s}{2})} \left[\sum_{i=1}^{N}\left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right)^2\right]^{\frac{p_n}{2}} \mathrm{d}x$$

where Γ denotes the Gamma function and $l = \sqrt{\sum_{i=1}^{N}(x_i - x_i^0)^2}$, and

$$\varrho_n = \varrho_n(N, p_n, s)$$
$$:= \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left(\frac{(\frac{s}{2})^N}{N} + \int\limits_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}\xi^3 - \frac{12}{s^2}\xi^2 + \frac{9}{s}\xi - 1\right|^{p_n}\xi^{N-1}\,\mathrm{d}\xi\right).$$

Set

$$L_n := \sigma_n + \alpha_n\theta_n + \beta_n\varrho_n,$$

and for a given non-negative constant ν and a given positive constant τ with $\frac{\nu^{p_n}}{k\prod_{i=1}^{n-1} p_i} \neq L_n\tau^{p_n}$ where k is given by (2.2), put

$$a_\tau(\nu) := \frac{\int_\Omega \sup\limits_{(t_1,\ldots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x, t_1, \ldots, t_n)\,\mathrm{d}x - \int_{S(x^0,\frac{s}{2})} F(x, 0, \ldots, 0, \tau)\,\mathrm{d}x}{\nu^{p_n} - k\left(\prod_{i=1}^{n-1} p_i\right)L_n\tau^{p_n}}.$$

We formulate our main result as follows:

Theorem 2.2. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition (2.3), and there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\frac{\nu_1}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}} < \sqrt[p_n]{L_n}\tau < \frac{\nu_2}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}}$ such that*

(A1) *$F(x, 0, \ldots, 0, t_n) \geq 0$ for each $x \in \overline{\Omega} \setminus S(x^0, \frac{s}{2})$, $t_n \in [0, \tau]$;*

(A2) *$a_\tau(\nu_2) < a_\tau(\nu_1)$.*

Then, for each $\lambda \in \left]\left(\frac{1}{k\prod_{i=1}^{n} p_i}\right)\frac{1}{a_\tau(\nu_1)}, \left(\frac{1}{k\prod_{i=1}^{n} p_i}\right)\frac{1}{a_\tau(\nu_2)}\right[$ the system (1.1) admits at least one non-trivial weak solution $u_0 = (u_{01}, \ldots, u_{0n}) \in X$ such that $\frac{\nu_1}{k} < \sum_{i=1}^{n}\prod_{j=1, j\neq i}^{n} p_j\|u_{0i}\|_{p_i}^{p_i} < \frac{\nu_2}{k}$.

Proof. In order to apply Theorem 2.1 to our problem, arguing as in [7], we introduce the functionals $\Phi, \Psi\colon X \to \mathbb{R}$ for each $u = (u_1, \ldots, u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$$

and

$$\Psi(u) = \int_\Omega F(x, u_1(x), \ldots, u_n(x))\,\mathrm{d}x.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \ldots, u_n) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\begin{aligned}\Phi'(u)(v) = &\int_\Omega \sum_{i=1}^{n} |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x)\,\mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^{n} \alpha_i |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x)\,\mathrm{d}x \\ &+ \int_\Omega \sum_{i=1}^{n} \beta_i |u_i(x)|^{p_i-2} u_i(x) v_i(x)\,\mathrm{d}x\end{aligned}$$

and

$$\Psi'(u)(v) = \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x)\,\mathrm{d}x$$

for every $v = (v_1, \ldots, v_n) \in X$, respectively, while Proposition 2.1 gives that Φ' admits a continuous inverse on X^*. Moreover, Φ is sequentially weakly lower semicontinuous and coercive. Furthermore, $\Psi'\colon X \to X^*$ is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on X. For this, for fixed $(u_1, \ldots, u_n) \in X$ let $(u_{1m}, \ldots, u_{nm}) \to (u_1, \ldots, u_n)$ weakly in X as $m \to +\infty$, then we have $(u_{1m}, \ldots, u_{nm})$ converges uniformly to $(u_1, \ldots, u_n)$ on $\overline{\Omega}$ as $m \to +\infty$(see [15]). Since $F(x, ., \ldots, .)$ is C^1 in $\mathbb{R}^n$ for every $x \in \overline{\Omega}$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \overline{\Omega}$, so for $1 \le i \le n$, $F_{u_i}(x, u_{1m}, \ldots, u_{nm}) \to F_{u_i}(x, u_1, \ldots, u_n)$ strongly as $m \to +\infty$ which follows $\Psi'(u_{1m}, \ldots, u_{nm}) \to \Psi'(u_1, \ldots, u_n)$ strongly as $m \to +\infty$. Thus we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator by [15: Proposition 26.2].

Set $w(x) = (0, \ldots, 0, w_n(x))$ such that

$$w_n(x) = \begin{cases} 0 & \text{if } \ x \in \overline{\Omega} \setminus S(x^0, s) \\ \tau & \text{if } \ x \in S(x^0, \frac{s}{2}) \\ \tau\left(\frac{4}{s^3}l^3 - \frac{12}{s^2}l^2 + \frac{9}{s}l - 1\right) & \text{if } \ x \in S(x^0, s) \setminus S(x^0, \frac{s}{2}), \end{cases}$$

$r_1 = \frac{\nu_1^{p_n}}{k\prod_{i=1}^{n} p_i}$ and $r_2 = \frac{\nu_2^{p_n}}{k\prod_{i=1}^{n} p_i}$. We have, for $1 \le i \le n$,

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus S(x^0, s) \cup S(x^0, \frac{s}{2}) \\ \tau\left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right) & \text{if } x \in S(x^0, s) \setminus S(x^0, \frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus S(x^0, s) \cup S(x^0, \frac{s}{2}) \\ \tau\left(\frac{12}{s^3}\frac{(x_i - x_i^0)^2 + l^2}{l} - \frac{24}{s^2} + \frac{9}{s}\frac{l^2 - (x_i - x_i^0)^2}{l^3}\right) & \text{if } x \in S(x^0, s) \setminus S(x^0, \frac{s}{2}), \end{cases}$$

and so that

$$\sum_{i=1}^{N} \frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus S(x^0, s) \cup S(x^0, \frac{s}{2}) \\ \tau\left(\frac{12l(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9}{s}\frac{N-1}{l}\right) & \text{if } x \in S(x^0, s) \setminus S(x^0, \frac{s}{2}). \end{cases}$$

It is easy to see that $w = (0, \ldots, 0, w_n) \in X$ and, in particular, since

$$\int_\Omega |\Delta w_n(x)|^{p_n}\,\mathrm{d}x = \tau^{p_n}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}\xi - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{\xi}\right|^{p_n}\xi^{N-1}\,\mathrm{d}\xi,$$

$$\begin{aligned} &\int_\Omega |\nabla w_n(x)|^{p_n}\,\mathrm{d}x \\ &= \int_{S(x^0,s)\setminus S(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\tau^2\left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right)^2\right]^{\frac{p_n}{2}}\mathrm{d}x \\ &= \tau^{p_n}\int_{S(x^0,s)\setminus S(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right)^2\right]^{\frac{p_n}{2}}\mathrm{d}x \end{aligned}$$

and

$$\int_\Omega |w_n(x)|^{p_n}\,\mathrm{d}x = \tau^{p_n}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}\xi^3 - \frac{12}{s^2}\xi^2 + \frac{9}{s}\xi - 1\right|^{p_n}\xi^{N-1}\,\mathrm{d}\xi\right),$$

we observe

$$\|w_n\|_{p_n}^{p_n} = L_n\tau^{p_n}.$$

From the condition $\frac{\nu_1}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}} < \sqrt[p_n]{L_n}\tau < \frac{\nu_2}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}}$, we obtain

$$r_1 < \Phi(w) < r_2.$$

Moreover, from (2.2) for each $(u_1, \ldots, u_n) \in X$, we have

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \leq k\|u_i\|_{p_i}^{p_i}$$

for $i = 1, \ldots, n$. Then

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \leq k \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{2.4}$$

for each $u = (u_1, \ldots, u_n) \in X$. This, together with the definition of Φ, yields

$$\begin{aligned}
\Phi^{-1}(]-\infty, r_2[) &= \big\{u = (u_1, \ldots, u_n) \in X : \ \Phi(u) < r_2\big\} \\
&= \Big\{u \in X : \ \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} < r_2\Big\} \\
&\subseteq \left\{u \in X : \ \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} < \frac{\nu_2^{p_n}}{\prod_{i=1}^{n} p_i} \text{ for all } x \in \Omega\right\}.
\end{aligned}$$

So,

$$\begin{aligned}
\sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{(u_1,\ldots,u_n)\in\Phi^{-1}(]-\infty,r_2[)} \int_\Omega F(x, u_1(x), \ldots, u_n(x))\,\mathrm{d}x \\
&\leq \int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\nu_2^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x, t_1, \ldots, t_n)\,\mathrm{d}x.
\end{aligned}$$

Since $0 \leq w_n(x) \leq \tau$ for each $x \in \Omega$, the assumption (A1) ensures that

$$\int_{\overline{\Omega}\smallsetminus S(x^0,s)} F(x, 0, \ldots, 0, w_n(x))\,\mathrm{d}x + \int_{S(x^0,s)\smallsetminus S(x^0,\frac{s}{2})} F(x, 0, \ldots, 0, w_n(x))\,\mathrm{d}x \geq 0.$$

Therefore, one has

$$\begin{aligned}
\vartheta(r_1, r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\
&\leq \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\nu_2^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x, t_1, \ldots, t_n)\,\mathrm{d}x - \Psi(w)}{r_2 - \Phi(w)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(k\prod_{i=1}^{n}p_i\right) \\
&\times \frac{\int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\nu_2^{p_n}}{\prod_{i=1}^{n}p_i}\right)} F(x,t_1,\ldots,t_n)\,\mathrm{d}x - \int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,\tau)\,\mathrm{d}x}{\nu_2^{p_n} - k\Big(\prod_{i=1}^{n-1}p_i\Big)L_n\tau^{p_n}} \\
&= \left(k\prod_{i=1}^{n}p_i\right)a_\tau(\nu_2).
\end{aligned}
$$

On the other hand, arguing as before, one has

$$
\begin{aligned}
\rho(r_1,r_2) &\geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{\Phi(w)-r_1} \\
&\geq \frac{\Psi(w) - \int_\Omega \sup_{(t_1,\ldots,t_n)\in K\left(\frac{\nu_1^{p_n}}{\prod_{i=1}^{n}p_i}\right)} F(x,t_1,\ldots,t_n)\,\mathrm{d}x}{\Phi(w)-r_1} \\
&\geq \left(k\prod_{i=1}^{n}p_i\right) \\
&\times \frac{\int_{S(x^0,\frac{s}{2})} F(x,0,\ldots,0,\tau)\,\mathrm{d}x - \int_\Omega \sup_{(t_1,\ldots,t_n)\in K(\frac{\nu_1^{p_n}}{\prod_{i=1}^{n}p_i})} F(x,t_1,\ldots,t_n)\,\mathrm{d}x}{k(\prod_{i=1}^{n-1}p_i)L_n\tau^{p_n} - \nu_1^{p_n}} \\
&= \left(k\prod_{i=1}^{n}p_i\right)a_\tau(\nu_1).
\end{aligned}
$$

Hence, from Assumption (A2), one has $\vartheta(r_1,r_2) < \rho(r_1,r_2)$. Therefore, applying Theorem 2.1, for each $\lambda \in \left]\left(\frac{1}{k\prod_{i=1}^{n}p_i}\right)\frac{1}{a_\tau(\nu_1)},\ \left(\frac{1}{k\prod_{i=1}^{n}p_i}\right)\frac{1}{a_\tau(\nu_2)}\right[$, the functional $\Phi-\lambda\Psi$ admits at least one critical point $u_0=(u_{01},\ldots,u_{0n})\in X$ such that $r_1<\Phi(u_0)<r_2$, that is

$$\frac{\nu_1^{p_n}}{k} < \sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n} p_j\|u_{0i}\|_{p_i}^{p_i} < \frac{\nu_2^{p_n}}{k}.$$

Hence, taking into account that the weak solutions of the system (1.1) are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u)=0$, we achieve the stated assertion. $\square$

An immediate consequence of Theorem 2.2 is the following.

Theorem 2.3. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition* (2.3), *and there exist two positive constants ν and τ with $\sqrt[p_n]{L_n}\tau < \frac{\nu}{\sqrt[p_n]{k \prod_{i=1}^{n-1} p_i}}$ such that Assumption* (A1) *in* Theorem 2.2 *holds. Furthermore, suppose that*

$$\text{(A3)}\quad \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\nu^{p_n}} < \frac{\int_{S(x^0,\frac{s}{2})} F(x,0,\dots,0,\tau)\,\mathrm{d}x}{k(\prod_{i=1}^{n-1} p_i)L_n\tau^{p_n}};$$

(A4) $F(x,0,\dots,0) = 0$ *for very $x \in \Omega$.*

Then, for each

$$\lambda \in \left]\left(\frac{1}{p_n}\right)\frac{L_n\tau^{p_n}}{\int_{S(x^0,\frac{s}{2})} F(x,0,\dots,0,\tau)\,\mathrm{d}x}, \left(\frac{1}{k\prod_{i=1}^n p_i}\right)\frac{\nu^{p_n}}{\int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}\right[$$

the system (1.1) *admits at least one non-trivial weak solution $u_0 = (u_{01},\dots,u_{0n}) \in X$ such that $\sup_{x\in\Omega}\sum_{i=1}^n \frac{|u_{0i}(x)|^{p_i}}{p_i} < \frac{\nu^{p_n}}{\prod_{i=1}^n p_i}$.*

P r o o f. The conclusion follows from Theorem 2.2 by taking $\nu_1 = 0$ and $\nu_2 = \nu$. Indeed, owing to our assumptions, one has

$$\begin{aligned}
a_\tau(\nu) &< \frac{\left(1 - \frac{k(\prod_{i=1}^{n-1} p_i)L_n\tau^{p_n}}{\nu^{p_n}}\right)\int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\nu^{p_n} - k\left(\prod_{i=1}^{n-1} p_i\right)L_n\tau^{p_n}} \\
&= \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^n p_i}\right)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\nu^{p_n}} \\
&< \frac{\int_{S(x^0,\frac{s}{2})} F(x,0,\dots,0,\tau)\,\mathrm{d}x}{k\left(\prod_{i=1}^{n-1} p_i\right)L_n\tau^{p_n}} = a_\tau(0).
\end{aligned}$$

In particular, one has

$$a_\tau(\nu) < \frac{\int_\Omega \sup_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(x,t_1,\dots,t_n)\,\mathrm{d}x}{\nu^{p_n}}.$$

Hence, Theorem 2.2, bearing (2.4) in mind, ensures the desired conclusion. □

We now point out the following relevant consequences of the main results, when F does not depend on $x \in \Omega$.

For a given non-negative constant ν and a given positive constant τ with $\frac{\nu^{p_n}}{k\prod_{i=1}^{n-1} p_i} \neq L_n\tau^{p_n}$ where k is given by (2.2), put

$$b_\tau(\nu) := \frac{m(\Omega) \max_{(t_1,\dots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(t_1,\dots,t_n) - (\frac{s}{2})^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})} F(0,\dots,0,\tau)}{\nu^{p_n} - k\Big(\prod_{i=1}^{n-1} p_i\Big) L_n\tau^{p_n}}.$$

Corollary 2.1. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition (2.3), and there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\frac{\nu_1}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}} < \sqrt[p_n]{L_n}\tau < \frac{\nu_2}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}}$ such that*

(B1) *$F(0,\dots,0,t_n) \geq 0$ for each $t_n \in [0,\tau]$;*

(B2) *$b_\tau(\nu_2) < b_\tau(\nu_1)$.*

Then, for each $\lambda \in \Big]\Big(\frac{1}{k\prod_{i=1}^{n} p_i}\Big)\frac{1}{b_\tau(\nu_1)}, \Big(\frac{1}{k\prod_{i=1}^{n} p_i}\Big)\frac{1}{b_\tau(\nu_2)}\Big[$ the system

$$\begin{cases} \Delta(|\Delta u_i|^{p_i-2}\Delta u_i) - \alpha_i\Delta_{p_i}u_i + \beta_i|u_i|^{p_i-2}u_i & \\ = \lambda F_{u_i}(u_1,\dots,u_n) & \text{in } \Omega, \\ u_i = \Delta u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

for $1 \leq i \leq n$, admits at least one non-trivial weak solution $u_0 = (u_{01},\dots,u_{0n}) \in X$ such that $\frac{\nu_1^{p_n}}{k} < \sum_{i=1}^{n}\prod_{j=1,j\neq i}^{n} p_j\|u_{0i}\|_{p_i}^{p_i} < \frac{\nu_2^{p_n}}{k}$.

P r o o f. Set $F(x,t_1,\dots,t_n) = F(t_1,\dots,t_n)$ for all $x\in\Omega$ and $t_i\in\mathbb{R}$ for $1 \leq i \leq n$. Since $m(S(x^0,\frac{s}{2})) = (\frac{s}{2})^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}$, Theorem 2.2 ensures the conclusion. □

Corollary 2.2. *Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition* (2.3)*, and there exist two positive constants ν and τ with $\sqrt[p_n]{L_n}\tau < \frac{\nu}{\sqrt[p_n]{k\prod_{i=1}^{n-1} p_i}}$ such that* Assumption (B1) *in* Corollary 2.1 *holds. Furthermore, suppose that*

(B3) $$\frac{m(\Omega) \max_{(t_1,\ldots,t_n)\in K\left(\frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i}\right)} F(t_1,\ldots,t_n)}{\nu^{p_n}} < \frac{\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)} F(0,\ldots,0,\tau)}{k\left(\prod_{i=1}^{n-1} p_i\right) L_n \tau^{p_n}};$$

(B4) $F(0,\ldots,0) = 0$.

Then, for each

$$\lambda \in \left]\left(\frac{1}{p_n}\right)\frac{L_n\tau^{p_n}}{\left(\frac{s}{2}\right)^N \frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})} F(0,\ldots,0,\tau)}, \left(\frac{1}{k\prod_{i=1}^{n} p_i}\right)\frac{\nu^{p_n}}{m(\Omega) \max_{(t_1,\ldots,t_n)\in K(\frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i})} F(t_1,\ldots,t_n)}\right[$$

the system (2.5) *admits at least one non-trivial weak solution* $u_0 = (u_{01},\ldots,u_{0n}) \in X$ *such that* $\sup_{x\in\Omega}\sum_{i=1}^{n}\frac{|u_{0i}(x)|^{p_i}}{p_i} < \frac{\nu^{p_n}}{\prod_{i=1}^{n} p_i}$.

Proof. It is enough to apply Theorem 2.3 to the function $F(x,t_1,\ldots,t_n) = F(t_1,\ldots,t_n)$ for all $x \in \Omega$ and $t_i \in \mathbb{R}$ for $1 \le i \le n$. $\square$

We here want to point out the following consequences, which follow from Theorem 2.2 and Theorem 2.3, respectively.

Let $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function, namely,

(κ_1) $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;

(κ_2) $t \to f(x,t)$ is continuous for almost every $x \in \Omega$;

(κ_3) for every $\varsigma > 0$ there exists a function $l_\varsigma \in L^2(\Omega)$ such that

$$\sup_{|t|\le\varsigma} |f(x,t)| \le l_\varsigma(x)$$

for almost every $x \in \Omega$.

Let F be the function defined by $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$ for each $(x,t) \in \Omega\times\mathbb{R}$. Let α_1 and β_1 are two non-negative constants and $p > \max\{1, \frac{N}{2}\}$.

Given a non-negative constant ν and a positive constant τ with $(\frac{\nu}{k})^p \neq L_1\tau^p$ where

$$k = \sup_{u\in(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}} |u(x)|}{\|u\|_p}$$

where

$$\|u\|_p = \left(\int_\Omega |\Delta u(x)|^p \,\mathrm{d}x + \alpha_1 \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \beta_1 \int_\Omega |u(x)|^p \,\mathrm{d}x \right)^{1/p}$$

put

$$c_\tau(\nu) := \frac{\int_\Omega \sup_{|t|\le\nu} F(x,t)\,\mathrm{d}x - \int_{S(x^0,\frac{s}{2})} F(x,\tau)\,\mathrm{d}x}{\nu^p - L_1(k\tau)^p}.$$

Theorem 2.4. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition* (2.3)*, and there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\frac{\nu_1}{k} < \sqrt[p]{L_1}\tau < \frac{\nu_2}{k}$ such that*

(C1) *$F(x,t) \ge 0$ for each $(x,t) \in \big(\overline{\Omega} \setminus S(x^0, \frac{s}{2})\big) \times [0,\tau]$;*

(C2) *$c_\tau(\nu_2) < c_\tau(\nu_1)$.*

Then, for each $\lambda \in \left]\left(\frac{1}{pk^p}\right)\frac{1}{c_\tau(\nu_1)}, \left(\frac{1}{pk^p}\right)\frac{1}{c_\tau(\nu_2)}\right[$ the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \alpha_1 \Delta_p u + \beta_1 |u|^{p-2} u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

admits at least one non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\frac{\nu_1}{k} < \left(\int_\Omega |\Delta u_0(x)|^p \,\mathrm{d}x + \alpha_1 \int_\Omega |\nabla u_0(x)|^p \,\mathrm{d}x + \beta_1 \int_\Omega |u_0(x)|^p \,\mathrm{d}x \right)^{1/p} < \frac{\nu_2}{k}.$$

Theorem 2.5. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition* (2.3)*, and there exist two positive constants ν and τ with $\sqrt[p]{L_1}\tau < \frac{\nu}{k}$ such that* Assumption (C1) *in* Theorem 2.4 *holds. Furthermore, suppose that*

(C3) $\dfrac{\int_\Omega \sup_{|t|\le\nu} F(x,t)\,\mathrm{d}x}{\nu^p} < \dfrac{1}{k^p L_1} \dfrac{\int_{S(x^0,\frac{s}{2})} F(x,\tau)\,\mathrm{d}x}{\tau^p}$.

Then, for each $\lambda \in \left]\frac{L_1\tau^p}{p\int_{S(x^0,\frac{s}{2})} F(x,\tau)\,\mathrm{d}x}, \frac{\nu^p}{pk^p \int_\Omega \sup_{|t|\le\nu} F(x,t)\,\mathrm{d}x}\right[$ the problem (2.6) *admits at least one non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\|u_0\|_\infty < \nu$.*

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and put $F(t) = \int_0^t f(\xi)\,\mathrm{d}\xi$ for all $t \in \mathbb{R}$. We have the following results as direct consequences of Theorem 2.4 and Theorem 2.5, respectively.

Theorem 2.6. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition (2.3), and there exist a non-negative constant ν_1 and two positive constants ν_2 and τ with $\frac{\nu_1}{k} < \sqrt[p]{L_1}\tau < \frac{\nu_2}{k}$ such that*

(D1) *$f(t) \geq 0$ for each $t \in \big[-\nu_2, \max\{\nu_2, \tau\}\big]$;*

(D2) $\frac{m(\Omega)F(\nu_2)-(\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)}F(\tau)}{\nu_2^p-L_1(k\tau)^p} < \frac{m(\Omega)F(\nu_1)-(\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma\left(1+\frac{N}{2}\right)}F(\tau)}{\nu_1^p-L_1(k\tau)^p}$

Then, for each

$$\lambda \in \Bigg]\Big(\frac{1}{pk^p}\Big)\frac{\nu_1^p - L_1(k\tau)^p}{m(\Omega)F(\nu_1) - (\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}F(\tau)}, \Big(\frac{1}{pk^p}\Big)\frac{\nu_2^p - L_1(k\tau)^p}{m(\Omega)F(\nu_2) - (\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}F(\tau)}\Bigg[$$

the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \alpha_1\Delta_p u + \beta_1|u|^{p-2}u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

admits at least one non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that

$$\frac{\nu_1}{k} < \Bigg(\int\limits_\Omega |\Delta u_0(x)|^p\,\mathrm{d}x + \alpha_1\int\limits_\Omega |\nabla u_0(x)|^p\,\mathrm{d}x + \beta_1\int\limits_\Omega |u_0(x)|^p\,\mathrm{d}x\Bigg)^{1/p} < \frac{\nu_2}{k}.$$

Proof. Set $f(x,t) = f(t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$. From the condition $\frac{\nu_1}{k} < \sqrt[p]{L_1}\tau < \frac{\nu_2}{k}$, we get $\nu_1 < \nu_2$. Therefore, Assumptions (D1) means $f(t) \geq 0$ for each $t \in [-\nu_1, \nu_1]$ and $f(t) \geq 0$ for each $t \in [-\nu_2, \nu_2]$, which follow $\max\limits_{t\in[-\nu_1,\nu_1]} F(t) = F(\nu_1)$ and $\max\limits_{t\in[-\nu_2,\nu_2]} F(t) = F(\nu_2)$. So, from Assumptions (D1) and (D2) we arrive at Assumptions (C1) and (C2), respectively. Hence, we achieve the conclusion by applying Theorem 2.4 observing that the problem (2.6) reduces to the problem (2.7). □

Theorem 2.7. *Assume that $x^0 \in \Omega$ and $s > 0$ satisfy the condition (2.3), and there exist two positive constants ν and τ with $\sqrt[p]{L_1}\tau < \frac{\nu}{k}$ such that*

(D3) *for each $t \in [-\nu, \max\{\nu, \tau\}]$;*

(D4) $m(\Omega)\frac{F(\nu)}{\nu^p} < \frac{(\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}}{k^pL_1}\frac{F(\tau)}{\tau^p}$.

Then, for each $\lambda \in \Big]\big(\frac{1}{p}\big)\frac{L_1\tau^p}{(\frac{s}{2})^N\frac{\pi^{N/2}}{\Gamma(1+\frac{N}{2})}F(\tau)},\ \big(\frac{1}{pk^p}\big)\frac{\nu^p}{m(\Omega)F(\nu)}\Big[$ the problem (2.7) admits at least one non-trivial weak solution $u_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\|u_0\|_\infty < \nu$.

We give a special case of our main result as follows.

Theorem 2.8. *Let α_1, β_1 be two non-negative constants, $h\colon \Omega \to \mathbb{R}$ be a positive and essentially bounded function and $g\colon \mathbb{R} \to \mathbb{R}$ be a non-negative function such that*

$$\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty.$$

Then, for each $\lambda \in \left]0, \left(\frac{1}{pk^p \int_\Omega h(x)\,\mathrm{d}x}\right) \sup_{\nu>0} \frac{\nu^p}{\int_0^\nu g(\xi)\,\mathrm{d}\xi}\right[$, the problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \alpha_1 \Delta_p u + \beta_1 |u|^{p-2}u = \lambda h(x) g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least one non-trivial weak solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

P r o o f. For fixed $\lambda \in \left]0, \left(\frac{1}{pk^p \int_\Omega h(x)\,\mathrm{d}x}\right) \sup_{\nu>0} \frac{\nu^p}{\int_0^\nu g(\xi)\,\mathrm{d}\xi}\right[$, there exists positive constant ν such that

$$\lambda < \left(\frac{1}{pk^p \int_\Omega h(x)\,\mathrm{d}x}\right) \frac{\nu^p}{\int_0^\nu g(\xi)\,\mathrm{d}\xi}.$$

Moreover, the condition $\lim_{t\to 0^+} \frac{g(t)}{t^{p-1}} = +\infty$ yields $\lim_{t\to 0^+} \frac{\int_0^t g(\xi)\,\mathrm{d}\xi}{t^p} = +\infty$. Therefore, we can choose positive constant τ satisfying $\sqrt[p]{L_1}\tau < \frac{\nu}{k}$ such that

$$\left(\frac{L_1}{\lambda p}\right) \frac{1}{\int_{S(x^0,\frac{s}{2})} h(x)\,\mathrm{d}x} < \frac{\int_0^\tau g(\xi)\,\mathrm{d}\xi}{\tau^p}.$$

Hence, Theorem 2.5 leads to the conclusion. □

Remark 2.1. For fixed γ put $\lambda_\gamma := \left(\frac{1}{pk^p \int_\Omega h(x)\,\mathrm{d}x}\right) \sup_{\nu\in]0,\gamma[} \frac{\nu^p}{\int_0^\nu g(\xi)\,\mathrm{d}\xi}$. The result of Theorem 2.8 for every $\lambda \in]0, \lambda_\gamma[$ holds with $|u_0(x)| < \gamma$ for all $x \in \Omega$ where u_0 is the ensured non-trivial weak solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [2: Remark 4.3]).

Finally, we present the following example to illustrate the result.

Example 2.1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 < 9\}$ and $p = 4$. Consider the problem

$$\begin{cases} \Delta(|\Delta u|^2 \Delta u) = \lambda(1 + \mathrm{e}^{-u^+}(u^+)^2(3 - u^+)) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

where $u^+ = \max\{u, 0\}$. Let

$$g(t) = 1 + \mathrm{e}^{-t^+}(t^+)^2(3 - t^+)$$

for all $t \in \mathbb{R}$ where $t^+ = \max\{t, 0\}$ and $h(x) = 1$ for every $x \in \Omega$. It is clear that $\lim\limits_{t\to 0^+} \frac{g(t)}{t^3} = +\infty$. Hence, taking Remark 2.1 into account, by applying Theorem 2.8, choosing $\alpha_1 = \beta_1 = 0$, since $k^4 = \frac{9\times 6^4}{289\pi^3}$, for every $\lambda \in \left]0, \frac{289\pi^2}{9\times 6^6}\frac{\mathrm{e}}{1+\mathrm{e}}\right[$ the problem (2.8) has at least one non-trivial weak solution $u_0 \in W^{2,4}(\Omega) \cap W_0^{1,4}(\Omega)$ such that $\|u_0\|_\infty < 1$.

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