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# ESTIMATES ON THE DIMENSION OF AN EXPONENTIAL ATTRACTOR FOR A DELAY DIFFERENTIAL EQUATION

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ABSTRACT. We study the long time behavior of delay differential equation, considered in a bounded domain in $\mathbb{R}^d$. Using the short trajectory method to prove the existence of the exponential attractor. Also we have estimates on the fractal dimension of an exponential attractor.



## 1. Introduction

In this note we restrict our attention to the DDE

$$\begin{aligned} x'(t) &= -\mu x(t) + f(x(t-\tau)), \\ x(\theta) &= \varphi(\theta), \qquad \theta \in [-\tau, 0]. \end{aligned} \tag{1.1}$$

Where $\mu > 0$, $\tau > 0$ is a fixed delay time and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function and $f(0) = 0$.

Also $\varphi$ is a given function in the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}$. This space is denoted by $C = C([-\tau, 0]; \mathbb{R})$ and endowed with the uniform norm topology.

$$\big(\|f\| = \max\big\{|f(t)| : \ t \in [-\tau, 0]\big\},$$

and

$$\operatorname{dist}_C(f(t), g(t)) = \max\big\{|f(t) - g(t)| : \ t \in [-\tau, 0]\big\}\big).$$

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Every $\varphi \in C$ uniquely determines a solution $x\colon [-\tau, 0] \to \mathbb{R}$ of ODD (1.1) such that $x(0) = \varphi$ (initial function) and $x$ satisfies DDE (1.1) for all $t > 0$. In fact $x(t)$ is found by the method of steps: $x$ is known for $t \in [-\tau, 0]$, if $k \in N_0$ and $n = k\tau$, then, for $n \le t \le n + \tau$, $x$ is defined by

$$x(t) := \mathrm{e}^{-\mu(t-n)} x(n) + \int_n^t \mathrm{e}^{-\mu(t-s)} f(x(s-\tau))\, \mathrm{d}s. \tag{1.2}$$

Let $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$, $t \ge 0$ represent the segment of a given solution $x(t)$ on the interval $[t-\tau, t]$.

Furthermore, it is assumed that $f\colon [0,\infty) \to [0,\infty)$ is a globally Lipschitz function

$$|f(x) - f(y)| \le L\,|x-y| \qquad \text{for all } \ x, y \ge 0 \tag{1.3}$$

and there exists a constant $\beta$ such that

$$f(x) \le \beta, \qquad \text{for all } \quad x \ge 0. \tag{H1}$$

Let $x(t)$ be the solution of (1.1). Multiply by $\mathrm{e}^{\mu t}$ and integrating this equation to get

$$x(t) = x(0)\mathrm{e}^{-\mu t} + \int_0^t \mathrm{e}^{-\mu(t-s)} f(x(s-\tau))\, \mathrm{d}s, \tag{1.4}$$

**Lemma 1.1.** *If $f$ is Lipschitzian then the solution is unique.*

Proof. Let $x(t)$, $\widetilde{x}(t)$ be two solutions on $[-\tau, T]$. We subtract equations for $x(t)$, $\widetilde{x}(t)$ to get

$$\begin{aligned}
&x(t) - \widetilde{x}(t) \\
&= (x(0) - \widetilde{x}(0))\mathrm{e}^{-\mu t} + \int_0^t \mathrm{e}^{-\mu(t-s)} \big[f(x(s-\tau)) - f(\widetilde{x}(s-\tau))\big]\, \mathrm{d}s, \\
&|x(t) - \widetilde{x}(t)| \\
&= \left|(x(0) - \widetilde{x}(0))\mathrm{e}^{-\mu t} + \int_0^t \mathrm{e}^{-\mu(t-s)} \big[f(x(s-\tau)) - f(\widetilde{x}(s-\tau))\big]\, \mathrm{d}s\right|, \\
&\hspace{18em} \text{for all} \quad t \in [0, T] \\
&|x(t) - \widetilde{x}(t)| \\
&\le |x(0) - \widetilde{x}(0))|\, \mathrm{e}^{-\mu t} + L \int_0^t \mathrm{e}^{-\mu(t-s)}\, |x(s-\tau)) - \widetilde{x}(s-\tau)|\, \mathrm{d}s, \\
&\hspace{18em} \text{for all} \quad t \in [0, T]
\end{aligned}$$

We assume $x = \widetilde{x}$ in $[-\tau, 0]$, we obtain

$$|x(t) - \widetilde{x}(t)| \leq L \int_0^t \mathrm{e}^{-\mu(t-s)} \, |x(s-\tau)) - \widetilde{x}(s-\tau)| \, \mathrm{d}s$$

since $\int_0^t \mathrm{e}^{-\mu(t-s)} \, |x(s-\tau)) - \widetilde{x}(s-\tau)| \, \mathrm{d}s \leq \int_0^t |x(s-\tau)) - \widetilde{x}(s-\tau)| \, \mathrm{d}s$,

$$|x(t) - \widetilde{x}(t)| \leq L \int_0^t \sup_{-\tau \leq u \leq s} |x(u) - \widetilde{x}(u)| \, \mathrm{d}s, \qquad t \in [0, T],$$

Then if $g(s) = \sup_{-\tau \leq u \leq s} |x(u) - \widetilde{x}(u)|$,

$$|x(t) - \widetilde{x}(t)| \leq L \int_0^t g(s) \, \mathrm{d}s, \qquad t \in [0, T]$$

$$\sup_{-\tau \leq t \leq \sigma} |x(t) - \widetilde{x}(t)| \leq \sup_{-\tau \leq t \leq \sigma} L \int_0^t g(u) \, \mathrm{d}u, \qquad \sigma \in [0, T],$$

$$g(\sigma) \leq L \int_0^\sigma g(u) \, \mathrm{d}u, \qquad \sigma \in [0, T],$$

It follows from integral form of Gronwall's inequality that $g(\sigma) = 0$ and hence $x(\sigma) = \widetilde{x}(\sigma)$, for all $\sigma \in [-\tau, T]$. $\square$

The mapping $S \colon \mathbb{R}^+ \times C \to C$, $S(t, \varphi) = x_t$, for all $t \geq 0$ (where $x$ is the unique solution with the initial condition $\varphi$) is well defined and have the semigroup property for DDE (1.1), i.e., $S(0) = I$ and $S(t+s) = S(t)S(s)$ also $S(t)\varphi$ is continuous in $t$ and is continuous respect to $\varphi$. Hence $(S(t), C([-\tau, 0], \mathbb{R}))$ for all $t \geq 0$ is a dynamical system.

Let us consider a sequence $\varphi_n \in C = C([-\tau, 0], \mathbb{R})$ such that $\varphi_n \to \varphi$ in $C$. We set $u_n(t) = S(t)\varphi_n$, $u(t) = S(t)\varphi$ for all $t \geq 0$. we must show $u_n(t) \to u(t)$ as $n \to \infty$ in $C$.

By uniqueness of solution and

$$u_n(t) = S(t)\varphi_n = u_n(0)\mathrm{e}^{-\mu t} + \int_0^t \mathrm{e}^{-\mu(t-s)} f(S(s-\tau))\varphi_n(\sigma) \, \mathrm{d}s$$

where $s, t \geq 0$, $-\tau \leq \sigma \leq 0$ and use of Lebesgue's convergence theorem

$$\lim_{n\to\infty} u_n(t) = \varphi(0)\mathrm{e}^{-\mu t} + \int_0^t \mathrm{e}^{-\mu(t-s)} f(S(s-\tau))\varphi(\sigma)\,\mathrm{d}s,$$

for $t \geq 0$, $-\tau \leq \sigma \leq 0$

$$\lim_{n\to\infty} S(t)\varphi_n = S(t)\varphi = u(t).$$

We recall several notions from the theory of dynamical systems and analysis. $(S(t), C)$ is called *dissipative*, if there is a bounded set $W \subset C$ such that, for any bounded set $B \subset C$, there is a time $t_0 = t_0(B)$ such that $S(t)B \subset W$ for all $t \geq t_0$.

A set $\mathcal{A} \subset C$ is called a *global attractor*, if

(1) $\mathcal{A}$ is compact;

(2) $S(t)\mathcal{A} = \mathcal{A}$, for all $t > 0$;

(3) $\mathrm{dist}_C(S(t)\boldsymbol{B}, \mathcal{A}) \to 0$ as $t \to \infty$, for any $\boldsymbol{B} \subset C$ bounded.

Of course, if a global attractor exists, then it is unique.

**Definition 1.1.** Let $\mathcal{X}$ be a Banach space. By $N(\mathcal{A}, \rho)$ we denote the smallest number of sets with diameter $\leq 2\rho$ that cover $\mathcal{A} \subset \mathcal{X}$. The *fractal dimension* is defined by

$$d^f_{\mathcal{X}}(\mathcal{A}) = \limsup_{\varepsilon\to 0+} \frac{\ln N(\mathcal{A}, \varepsilon)}{-\ln \varepsilon}.$$

**Definition 1.2.** The set $\mathcal{E} \subset \mathcal{X}$ is called *exponential attractor* for $(S(t), \mathcal{X})$ if

(1) $\mathcal{E}$ is compact;

(2) $S(t)\mathcal{E} \subset \mathcal{E}$, for all $t > 0$;

(3) $d^f_{\mathcal{X}}(\mathcal{E}) < \infty$;

(4) exists $\sigma > 0$ such that, for any $\boldsymbol{B} \subset \mathcal{X}$ bounded, $\mathrm{dist}_C(S(t)\boldsymbol{B}, \mathcal{E}) \leq C_1\mathrm{e}^{-\sigma t}$.

**Definition 1.3.** Set $\mathcal{E}^* \subset \mathcal{X}$ is called *exponential attractor* for the dynamical system $(S^n(t), \mathcal{X})$ if

(1) $\mathcal{E}^*$ is compact;

(2) $S(t)\mathcal{E}^* \subset \mathcal{E}$;

(3) $d^f_{\mathcal{X}}(\mathcal{E}^*) < \infty$;

(4) There exist $\sigma, C_1 > 0$ such that, $\mathrm{dist}_C(S^n\mathcal{X}, \mathcal{E}^*) \leq C_1\mathrm{e}^{-\sigma n}$, for all $n \in \mathbb{N}$.

The following lemma gives a useful help to get desirable result.

**Lemma 1.2.** *Let $S\colon \mathcal{X} \to \mathcal{X}$ be Lipschitz continuous, and let there exist $\theta \in (0,1)$ and a constant $K > 0$ such that for any $\rho > 0$, $F \subset \mathcal{X}$ with $\operatorname{diam}_{\mathcal{X}}(F) \le 2\rho$,*

$$N_{\mathcal{X}}(S(F), \theta\rho) \le K.$$

*Then, the dynamical system $(S^n, \mathcal{X})$ has an exponential attractor $\mathcal{E}^*$, and*

$$\dim^f_{\mathcal{X}}(\mathcal{E}^*) \le \frac{\ln K}{-\ln\theta}.$$

P r o o f. See e.g. [3]. □

**Lemma 1.3.** *Let $\mathcal{E}^*$ be an exponential attractor for $(S^n, \mathcal{X})$, where $S = S_{t^*}$ with some fixed $t^* > 0$. Assume that $S_t x$ is locally Lipschitz continuous w.r. to $t$ and $x$. Then there exist an exponential attractor $\mathcal{E}$ to $(S_t, X)$ such that*

$$\dim^f_{\mathcal{X}}(\mathcal{E}) \le \dim^f_{\mathcal{X}}(\mathcal{E}^*) + 1. \tag{1.5}$$

P r o o f. see e.g. [3]. □

**Theorem 1.1.** *The following are equivalent.*

(1) *Dynamical system $(S^n, B)$ has an exponential attractor $\mathcal{E}^*$;*

(2) *There exist $a, b > 0$, $\theta \in (0,1)$ and $K \ge 1$ such that*

$$N_C(S^n B, a\theta^n) \le bK^n, \qquad \text{for all} \quad n \in \mathbb{N}, \tag{1.6}$$

*where $\kappa$ is Lipschitz constant of $L$.*

P r o o f. The existence of $\mathcal{E}^*$ implies (1.6) with $\theta = \mathrm{e}^{-\sigma}$ and $K = \mathrm{e}^{d\sigma}$, where $\sigma$ is the constant appearing in above definition and $d > 0$ is arbitrary number such that $\dim^C_f(\mathcal{E}^*) < d$. conversely, if (1.6) holds, we can construct an exponential attractor such that

$$\dim^f_{\mathcal{X}}(\mathcal{E}^*) \le \frac{\ln K}{-\ln\theta}.$$

and definition holds with $\sigma = -\ln\theta$. For more details see [4]. □

$F \subset C$ is *equicontinuous* means

$$\forall\varepsilon \in (0,\infty)\ \ \exists\delta \in (0,\infty)\ \ \forall f \in F\ \ \forall t_1, t_2 \in [-\tau, 0]:$$
$$|t_1 - t_2| < \delta \implies |f(t_1) - f(t_2)| < \varepsilon.$$

By Arzela-Ascoli theorem a subset $F$ of $C([-\tau,0],\mathbb{R})$ is compact iff it is closed, bounded and equicontinuous.

A continuous map $T\colon X \to X$ is *conditionally compact continuous* if $A \subset X$ bounded and $TA$ bounded imply $\overline{TA}$ is compact. $T$ is *completely continuous* if it is conditionally compact continuous and also a bounded map.

## 2. Global attractor

**Lemma 2.1.** *Under hypotheses $(H_1)$, the semigroup $S(t)$ generated by* (1.1) *is a bounded map and is dissipative. Thus, there is compact global attractor $\mathcal{A}$.*

P r o o f. Let us first show that $S(t)$ is dissipative. If we multiply Eq. (1.1) by $e^{\mu t}$

$$x'(t) = -\mu x(t) + f(x(t-\mu)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}x(t)) = f(x(t-\mu))\mathrm{e}^{\mu t},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}x(t)) \leq \beta\mathrm{e}^{\mu t},$$

integrating this equation we obtain

$$\mathrm{e}^{\mu t}x(t) - x(0) \leq \int_0^t \beta\mathrm{e}^{\mu s}\,\mathrm{d}s,$$
$$x(t) \leq x(0)\mathrm{e}^{-\mu t} + \frac{\beta}{\mu}(1-\mathrm{e}^{-\mu t}), \qquad \text{for all} \quad t \geq 0,$$

and then $\limsup\limits_{t\to\infty} x(t) \leq 2\frac{\beta}{\mu}$. Since $x(t)$ is bounded above then $-f(x(t-\tau))$ is bounded below by constant $\widetilde{K}$. By the same arguing as above

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mu t}x(t)) \geq -\frac{\widetilde{K}}{\mu}\mathrm{e}^{\mu t},$$
$$x(t) \geq x(0)\mathrm{e}^{-\mu t} - \frac{\widetilde{K}}{\mu}(1-\mathrm{e}^{-\mu t}) \qquad \text{for all} \quad t \geq 0,$$

one obtains that $\liminf\limits_{t\to\infty} x(t) \geq -2\frac{\widetilde{K}}{\mu}$.

The set $B = \{u(t) \in C\colon |u(t)| \leq \rho,\ \ \rho = \frac{2\beta}{\mu}\}$ is bounded invariant and absorb in set for $S(t,\varphi)$ in $C$: i.e. $u(t) \in \beta$, for all $t \geq T_0$ with $T_0$ defined by the condition $|\varphi(0)| = \frac{2\beta}{\mu}e^{-\mu T_0}$ if $\varphi(0) > 0$ and $T_0 = 0$ for $\varphi(0) \leq 0$.

The inequality immediately proves dissipativeness of equation (1.1).

Let $B \in C([-\tau,0],\mathbb{R})$ is bounded and $F = S(t)B$, We want to show $\overline{S(t)B}$ is compact.

(1) $S(t)B$ is bounded; let $\varphi \in B$ is bounded then by (1.5)

$$|x(t)| \leq \mathrm{e}^{-\mu(t)}\,|x(0)| + \int_0^t \mathrm{e}^{-\mu(t-s)}\,|f(x(s-\tau))|\,\mathrm{d}s,$$

by $(H_1)$ and boundedness of $x(0)$ we get as wish. Or if $t = 1$ in right hand

$$|x(t)| \leq \mathrm{e}^{-\mu} |x(t-1)| + \int_0^1 \mathrm{e}^{-\mu(1-s)} |f(x(s-\tau))| \, \mathrm{d}s,$$

$$|x(t)| \leq \mathrm{e}^{-\mu} |\varphi(0)| + \frac{\beta}{\mu}(\mathrm{e}^{-\mu} - 1),$$

$S(1)\varphi$ is bounded then $S(t)\varphi$ is bounded for all $t \geq 0$.

(2) $S(t)B$ is equally continuous, i.e., fo rall $\varepsilon > 0$ there exist $\delta > 0$ s.t. for $t_1, t_2 \in [-\tau, 0]\colon |t_1 - t_2| < \delta$

$$|x(t_1) - x(t_2)| = \left| \int_{t_1}^{t_2} x'(s) \, \mathrm{d}s \right| = \left| \int_{t_1}^{t_2} -\mu x(s) + f(x(s-\mu)) \, \mathrm{d}s \right|,$$

$$\leq \int_{t_2}^{t_1} |-\mu x(s)| + |f(x(s-\mu))| \, \mathrm{d}s,$$

by $(H_1)$ and boundedness of $x(s)$ in $s \in [t_1, t_2]$,

$$|x(t_1) - x(t_2)| \leq |t_1 - t_2| \, M, \qquad M = \mu N + \beta$$

where $N = \sup |x(s)|$ for all $s \in [t_1, t_2]$, it is enough we choose $\delta = \frac{\varepsilon}{M}$.

By (1.1) and (3.3) and Arzela-Ascoli theorem $\overline{S(t)B}$ is compact, then $S(t)$ is completely continuous for any $t \geq -\tau$. Thus, the existence of the global attractor of the equation (1.1) is an immediate consequence of [1: Theorem (3.4.8)]. □

## 3. Exponential attractor

The aim of this section is to prove that $(\{S(t)\}_{t\geq 0},\ C([-\tau, 0], \mathbb{R}))$ has an exponential attractor, also we will show the estimate of its fractal dimension, the main idea of the proof is to use the so called method of trajectories, cf. [2]

One usually first constructs the exponential attractor for a certain discrete dynamical system $(S^n, C)$ subgroup of $S_t$ generated by $L = S(\tau)$ with fixed delay time $\tau$. Then use of Lemma (1.3) to extend it for the entire dynamics.

We estimate the difference of the time derivatives. we establish a kind of “smoothing property”. $C$ is replaced with the smaller Banach space $C^1 = C^1([0, \tau], \mathbb{R})$ of continuously differentiable functions $f\colon [0, \tau] \to \mathbb{R}$ with the norm given by $\|f\|_{C^1} = \|f\|_C + \|f'\|_C$, Then a result of the compact embedding of $C^1$ into $C$ will complete the proof.

**Lemma 3.1.** *The mapping $L$ is well defined and (global) Lipschitz continuous.*

Proof. We assume as Lemma (1.1) that $x(t)$, $\widetilde{x}(t)$ be two solutions on $[0,\tau]$ with initial function $\psi$ and $\widetilde{\psi}$. We subtract equations for $x(t)$, $\widetilde{x}(t)$ to get

$$|x(t)-\widetilde{x}(t)| \leq |(x(0)-\widetilde{x}(0))| + L\int_0^t |x(s-\tau)-\widetilde{x}(s-\tau)|\,\mathrm{d}s, \qquad t\in[0,\tau],$$

$$|x(t)-\widetilde{x}(t)| \leq |x(0)-\widetilde{x}(0)| + L\int_{-\tau}^{t-\tau} |x(u)-\widetilde{x}(u)|\,\mathrm{d}u, \qquad t\in[0,\tau],$$

$$|x(t)-\widetilde{x}(t)| \leq \left\|\psi-\widetilde{\psi}\right\|_C + Lt\left\|\psi-\widetilde{\psi}\right\|_C \leq (1+Lt)\left\|\psi-\widetilde{\psi}\right\|_C, \qquad t\in[0,\tau],$$

$$|x(t)-\widetilde{x}(t)| \leq k_1\left\|\psi-\widetilde{\psi}\right\|_C, \qquad k_1=(1+L\tau),\ \ t\in[0,\tau]$$

$$\sup_{0\leq t\leq\tau} |x(t)-\widetilde{x}(t)| \leq k_1\left\|\psi-\widetilde{\psi}\right\|_C,$$

$$\left\|x(t)-\widetilde{x}(t)\right\|_{C[0,\tau]} \leq k_1\left\|\psi-\widetilde{\psi}\right\|_C,$$

$$\left\|S(\tau)\psi-S(\tau)\widetilde{\psi}\right\|_C \leq k_1\left\|\psi-\widetilde{\psi}\right\|_C.$$

Which asserts that $L=S(\tau)$ is Lipschitz continuous.

Let $x=s(\tau)\psi$ and $\widetilde{x}=S(\tau)\widetilde{\psi}$ be two solutions

$$\left|\dot{x}(t)-\dot{\widetilde{x}}(t)\right| \leq \mu\left|x(t)-\widetilde{x}(t)\right| + |f(x(t-\tau))-f(\widetilde{x}(t-\tau))|, \qquad t\in[0,\tau],$$

$$\left|\dot{x}(t)-\dot{\widetilde{x}}(t)\right| \leq \mu\,|x(t)-\widetilde{x}(t)| + L\,|x(t-\tau)-\widetilde{x}(t-\tau)|, \qquad s=t-\tau\in[-\tau,0]$$

$$\leq \mu\,|x(t)-\widetilde{x}(t)| + L\left|\psi(s)-\widetilde{\psi}(s)\right|,$$

take supremum respect to $t$ on $[0,\tau]$

$$\left\|\dot{x}(t)-\dot{\widetilde{x}}(t)\right\|_C \leq \mu k_1\left\|\psi-\widetilde{\psi}\right\|_C + L\left\|\psi-\widetilde{\psi}\right\|_C,$$

$$\left\|\dot{x}(t)-\dot{\widetilde{x}}(t)\right\|_C \leq k_2\left\|\psi-\widetilde{\psi}\right\|_C, \qquad k_2=\mu k_1+L,$$

then

$$\left\|L\psi-L\widetilde{\psi}\right\|_{C^1} \leq \kappa\left\|\psi-\widetilde{\psi}\right\|_C.$$

Where $\kappa=k_1+k_2$. □

**Lemma 3.2.** *Let $M \geq \frac{1}{5}$, $B > 0$. The set*

$$\mathcal{A} = \big\{\mathcal{X} \in C([0,t^*],\mathbb{R}) : \ |\mathcal{X}| \leq M, \ \operatorname{Lip}\mathcal{X} \leq B\big\}$$

*can covered by $K$ balls of radius $1$ in the space $C([0,t^*],\mathbb{R})$, where*

$$\ln K \leq C(t^* B + 1)(\ln M + 1). \qquad C > 0 \tag{3.1}$$

P r o o f. We consider points $t_0 = 0 < t_1 < \cdots < t_n = t^*$ such that $t_{i+1} - t_i \leq \delta := \frac{1}{5B}$. Then $n \leq \frac{t^*}{\delta} + 1 \leq 5Bt^* + 1$.

Furthermore, there exist point $x_j$, $j = 1,\dots,m$, in $\mathbb{R}$ such that $B_{\mathbb{R}}\big(x_j, \frac{1}{5}\big)$ cover $B_{\mathbb{R}}(0,M)$, then $m \leq (\frac{3M}{\frac{1}{5}}) = 15M$.

We consider the set

$$\mathcal{N} = \big\{\mathcal{X} \colon [0,t^*] \to \mathbb{R} : \ \mathcal{X}(t_i) = x_j \ \text{ and linear in } \ [t_i,t_{i+1}]\big\},$$

if $K$ is cardinality of $\mathcal{N}$ then,

$$\ln k = \ln m^{n+1} \leq \ln(15M)(5Bt^* + 2) \leq C(t^* B + 1)(\ln M + 1).$$

We claim that the balls $B(\mathcal{X},1)$, $\mathcal{X} \in \mathcal{N}$ cover the set $\mathcal{A}$. Let $\psi \in \mathcal{A}$ be arbitrary function, there exists $\mathcal{X}(t) \in \mathcal{N}$ such that $|\mathcal{X}(t_i) - \psi(t_i)| \leq \frac{1}{5}$, $i = 0,\dots,n$ we want to show for all $t \in [0,t^*]$, $|\mathcal{X}(t) - \psi(t)| \leq 1$.

It suffices to show this for fixed $t \in [t_i,t_{i+1}]$.

$$\begin{aligned} |\mathcal{X}(t) - \psi(t)| &\leq |\mathcal{X}(t) - \mathcal{X}(t_i)| + |\mathcal{X}(t_i) - \psi(t_i)| + |\psi(t_i) - \psi(t)| \\ &\leq |\mathcal{X}(t) - \mathcal{X}(t_i)| + \frac{1}{5} + \delta B \leq |\mathcal{X}(t) - \mathcal{X}(t_i)| + \frac{2}{5}. \end{aligned}$$

By piecewise linearity of $\mathcal{X}$ in $[t_i,t_{i+1}]$ we get

$$|\mathcal{X}(t) - \mathcal{X}(t_i)| \leq |\mathcal{X}(t_{i+1}) - \mathcal{X}(t_i)|\,.$$

Also

$$\begin{aligned} |\mathcal{X}(t_{i+1}) - \mathcal{X}(t_i)| &\leq |\mathcal{X}(t_{i+1}) - \psi(t_{i+1})| + |\psi(t_{i+1}) - \psi(t_i)| + |\psi(t_i) - \mathcal{X}(t_i)| \\ &\leq \frac{1}{5} + \delta B + \frac{1}{5} \leq \frac{3}{5}. \end{aligned}$$

and we done as desired. □

**Theorem 3.1.** *Let $L = S(\tau)$, then the semigroup $(L^n, B)$ has exponential attractor $\mathcal{E}^*$, and*

$$\dim^f_{\mathcal{X}}(\mathcal{E}^*) \leq \frac{\ln K}{-\ln\theta}.$$

*Where $K = N_C\big(B_{C^1}(0;1), \frac{1}{4\kappa}\big)$.*

P r o o f. The key step of the proof is to show that, for $F \subset B$ with $\operatorname{diam}_C(F) \le 2\rho$, there exist $\theta \in (0,1)$ and $K > 0$ independent of $F$ and $\rho$ s.t.

$$N_C(L(F), \rho\theta) \le K. \tag{3.2}$$

Once this is done, we establish by induction that

$$N_C(L^n B, \theta^n R) \le K^n$$

where $R = \operatorname{diam}_C(B)$. And the conclusion follows from Theorem 1.1.

To prove (3.2), by $\operatorname{diam}_C(F) \le 2\rho$, we have $F \subset B_C(f_0, 2\rho)$ for arbitrary function $f_0 \in F$, then $L(B_C(f_0; 2\rho) \cap B) \subset B_{C^1}(Lf_0; 2\kappa\rho)$, where $\kappa$ is Lipschitz constant of $L$ on $C^1$ (Lemma 3.1), which can be covered by $K$ balls with radii $\frac{e^{-\gamma}\rho}{2}$ centered in $C$ which can in turn be replaced by the balls with a radii $e^{-\gamma}\rho$ centered in $B$ with $\gamma = -\ln\theta$ and $\theta = \frac{1}{2}$, i.e $LF \subset \bigcup_j B_C(f_j; \theta\rho)$, $f_j \in B$. Thus, we have shown that

$$N_C(LF, \theta\rho) \le K,$$

and $K$ independent of $F$ and $\rho$. Also we know that

$$N_C(B_{C^1}(f_j; 2\kappa\rho), \theta\rho) = N_{C^1}\big(B_C(0;1), \tfrac{1}{4\kappa}\big).$$

By Lemma (3.1) the set $L(F)$ is contained in $L(f_0) + \widetilde{\mathcal{A}}$, where

$$\widetilde{\mathcal{A}} = \big\{\mathcal{X} \in C : \ |\mathcal{X}| \le 2\kappa\rho, \ \operatorname{Lip}\mathcal{X} \le 2\kappa\rho\big\},$$

This covering is equivalent to the problem of this satisfy in condition of Lemma 3.2 with $M = 4\kappa$, $B = 4\kappa$. Then

$$\ln K \le C(\tau B + 1)(\ln M + 1) \le C(4\tau\kappa + 1)(\ln 4\kappa + 1) \le C_1(\tau\kappa + 1)(\ln\kappa + 1).$$

Then

$$\dim^f_{\mathcal{X}}(\mathcal{E}^*) \le C'(\tau\kappa + 1)(\ln\kappa + 1). \tag{3.3}$$

□

**Theorem 3.2.** *The dynamical system* $(S(t), C)$ *associated with DDE* (1.1) *has exponential attractor* $\mathcal{E}$. *its fractal dimension is estimated as*

$$\dim^f_{\mathcal{X}}(\mathcal{E}) \le C'(\tau\kappa + 1)(\ln\kappa + 1) + 1. \tag{3.4}$$

P r o o f. we must the first prove that $S(t)$ is local Lipschitz continuous w.r. to $x$ and $t$. By which is shown in Lemma 2.1

$$\|S_{t_1}x - S_{t_2}x\|_C \le M\,|t_1 - t_2|\,, \quad t_1, t_2 \in [0, T]$$

Let $T > \tau$ and $x(t_1)$ and $\widetilde{x}(t_1)$ are two solutions

$$|x(t_1) - \widetilde{x}(t_1)| \leq |x(0) - \widetilde{x}(0)| \, \mathrm{e}^{-\mu t_1} + L \int_0^{\tau} \mathrm{e}^{-\mu(t_1 - s)} \, |x(s-\tau) - \widetilde{x}(s-\tau)| \, \mathrm{d}s$$

$$+ L \int_{\tau}^{t_1} \mathrm{e}^{-\mu(t_1 - s)} \, |x(s-\tau) - \widetilde{x}(s-\tau)| \, \mathrm{d}s, \qquad t_1 \in [0, T],$$

By the same arguing as Lemma 3.1

$$|x(t_1) - \widetilde{x}(t_1)| \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C + L \int_{\tau}^{t_1} \sup_{0 \leq u \leq s} |x(u) - \widetilde{x}(u)| \, \mathrm{d}s, \qquad t_1 \in [0, T],$$

Then if $g(s) = \sup\limits_{0 \leq u \leq s} |x(u) - \widetilde{x}(u)|$,

$$|x(t_1) - \widetilde{x}(t_1)| \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C + L \int_{\tau}^{t_1} g(s) \, \mathrm{d}s,$$

$$\sup_{0 \leq t_1 \leq t} |x(t_1) - \widetilde{x}(t_1)| \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C + \sup_{0 \leq t_1 \leq t} L \int_{\tau}^{t_1} g(s) \, \mathrm{d}s \qquad t \in [0, T],$$

$$g(t) \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C + L \int_{\tau}^{t} g(s) \, \mathrm{d}s,$$

use of integral form of Gronwall's inequality

$$g(t) \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C \mathrm{e}^{L(t-\tau)},$$

$$\|x(t) - \widetilde{x}(t)\|_{C[0,\tau]} \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C \mathrm{e}^{L(t-\tau)},$$

$$\|x(t) - \widetilde{x}(t)\|_{C[0,\tau]} \leq k_1 \big\| \psi - \widetilde{\psi} \big\|_C \mathrm{e}^{L(T-\tau)},$$

$$\|x(t) - \widetilde{x}(t)\|_{C[0,\tau]} \leq k_2 \big\| \psi - \widetilde{\psi} \big\|_C,$$

Then

$$\Big\| S(t)\psi - S(t)\widetilde{\psi} \Big\|_{C[0,\tau]} \leq k_2 \big\| \psi - \widetilde{\psi} \big\|_C, \qquad t \in [0, T], \ T > \tau.$$

for evaluation the dimension we substitute (3.3) into (1.5), yielding (3.4) and this prove the lemma. □

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