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SOME RESULTS ON THE ENTIRE FUNCTION SHARING PROBLEM

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ABSTRACT. Our main result is as follows: let f and a be two entire functions such that $\max\{\rho_2(f), \rho_2(a)\} < \frac{1}{2}$. If f and $f^{(k)}$ a CM, and if $\rho(a^{(k)} - a) < \rho(f - a)$, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c. This result is applied to improve a result of Gundersen and Yang.

1. Introduction and main results

In this paper, we assume the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [9,12,16]). For any given nonconstant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0, \qquad r \notin E,$$

where $E \subset (0, \infty)$ is of finite logarithmic measure. A meromorphic function $a(z)$ is said to be a small function of $f(z)$ if $T(r, a) = S(r, f)$. In addition, we say that two meromorphic functions $f(z)$ and $g(z)$ share a small function a CM when $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities.

Uniqueness of the entire function f sharing values with its derivative f' was first investigated by Rubel and Yang [15]. For the entire function f, they proved that $f \equiv f'$, if f and f' share two distinct finite constants CM. In 1996, Brück [2] proposed the following famous conjecture.

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Conjecture. *Let f be a nonconstant entire function. Suppose that*

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then $f' - a = c(f-a)$ for some nonzero constant c.

This conjecture has been well studied by many complex analysts and is still open. The case that $a = 0$ and that $N\big(r, \frac{1}{f'}\big) = S(r,f)$ have been proved by Brück himself [2] while the case that f is of finite order has been proved by Gundersen and Yang [11]. Chen and Shon [5] has pointed out that the conjecture is still true if $\rho_2(f) < \frac{1}{2}$. And then they improved their result by the following result.

Theorem A. ([6]) *Let f be a nonconstant entire function with $\rho_2(f) < \frac{1}{2}$. If f and $f^{(k)}$ share the finite value a CM, then $f^{(k)} - a = c(f-a)$ for some nonzero constant c.*

For the small function sharing problem, Chang and Zhu [3] have proved some interesting results. We only recall one of their results below.

Theorem B. ([3]) *Let f be an entire function of finite order and a an entire function of order less than f's. If f and f' share a CM, then $f' - a = c(f-a)$ for some nonzero constant c.*

As a continuation of Theorems A and B, we prove the following Theorem 1.1, in which the condition that the order of the shared function is less than f's is omitted.

Theorem 1.1. *Let f and a be two entire functions such that* $\max\{\rho_2(f), \rho_2(a)\} < \frac{1}{2}$. *If f and $f^{(k)}$ share a CM, and if $\rho(a^{(k)} - a) < \rho(f-a)$, then $f^{(k)} - a = c(f-a)$ for some nonzero constant c.*

Obviously, we always restrict $\rho(a^{(k)} - a) < \infty$ in Theorem 1.1. We wonder that whether this condition can be relaxed. The following example satisfies the assumption of Theorem 1.1 but it does not satisfy the assumption of Theorem B.

***Example* 1.** Let

$$f = e^{2z} + \cos z + 15, \qquad a = \cos z + 16,$$

then $\rho(f) = \rho(a) = 1$, $b = a^{(4)} - a = -16$, and $f^{(4)} - a = 16(f-a)$.

With Theorem 1.1, we have two corollaries.

Corollary 1.1.1. *Let f be an entire function such that $\rho_2(f) < \frac{1}{2}$ and a an entire function of order less than f's. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f-a)$ for some nonzero constant c.*

Corollary 1.1.2. *Let f be an entire function such that $\rho_2(f) < \frac{1}{2}$. If f and $f^{(k)}$ have the same fixed points with same multiplicities, then $f^{(k)} - z = c(f - z)$ for some nonzero constant c.*

***Example* 2.** Let
$$f = \mathrm{e}^{\frac{1+\sqrt{3}\mathrm{i}}{2}z} + \frac{1+\sqrt{3}\mathrm{i}}{2}z + 1,$$
then f and f' share $a = z$ CM, and f and f' have the same fixed points with same multiplicities. In this case, we have $b = a' - a = 1 - z$, and $f' - a = \frac{1+\sqrt{3}\mathrm{i}}{2}(f - a)$.

Finally, we apply Theorem 1.1 to improve the following result proved by Gundersen and Yang [11].

Theorem C. ([11]) *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If the value a is shared by $f, f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f \equiv f'$.*

Theorem 1.2. *Let f be a nonconstant entire function such that $\rho_2(f) < \frac{1}{2}$, let $a \neq 0$, be a finite constant, and let n be a positive integer. If the value a is shared by $f^{(n)}$ and $f^{(n+1)}$ CM, and there is some constant b such that $f(z) - b$ and $f^{(n+1)}(z) - b$ have more than n distinct common zeros, then $f \equiv f'$.*

From Theorem 1.2 and its proof, we can easily prove the following result.

Corollary 1.2.1. *Let f be a nonconstant entire function such that $\rho_2(f) < \frac{1}{2}$, let $a \neq 0$, be a finite constant, and let n be a positive integer. If the value a is shared by $f, f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f \equiv f'$.*

2. Lemmas

Lemma 2.1. ([10]) *Let f be a nonconstant meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$. that has linear measure zero, such that if $\varphi_0 \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = r(\varphi_0)$ such that for all z satisfying $\arg z = \varphi_0$ and $|z| > R_0$, we have*
$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |z|^{k(\rho-1+\varepsilon)}.$$

With a similar reasoning as in the proof of [3: Lemma 3], we can prove the following Lemma.

Lemma 2.2. ([3]) *Let f be an analytic function on some ray $\arg z = \theta$ starting from $z_0 = r_0\mathrm{e}^{\mathrm{i}\theta}$ and $K(x)$ a positive, decreasing, continuous function on the interval $[r_0, +\infty)$. Suppose that $|f^{(k)}(z)|K(|z|)$ is unbounded on the ray $\arg z = \theta$ starting from $z_0 = r_0\mathrm{e}^{\mathrm{i}\theta}$. Then there exists an infinite sequence of points $z_n = r_n\mathrm{e}^{\mathrm{i}\theta}$ where $r_n \to \infty$, such that $|f^{(k)}(z_n)|K(|z_n|) \to \infty$ and*
$$\left|\frac{f(z_n)}{f^{(k)}(z_n)}\right| \le (1 + o(1))||z_n|^k.$$

Lemma 2.3. *Let $a(z)$ be an entire function of finite order and $Q(z)$ a nonconstant polynomial. If f is an entire solution of the equation*

$$f^{(k)} - \mathrm{e}^{Q} f = a \tag{2.1}$$

such that $\rho(f) > \rho(a)$, then $\rho(f) = \infty$.

Proof. Assume that f is of order $\rho(f) = \rho < \infty$ and set $\rho(a) = \sigma$. From Lemma 2.1, for any given $\varepsilon > 0$ ($0 < \varepsilon < \frac{\rho-\sigma}{3}$), there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that if $\varphi_0 \in [0, 2\pi) \setminus E$, then there is a constant $R_0 = R(\varphi_0)$ such that for all z satisfying $\arg z = \varphi_0$ and $|z| > R_0$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \le |z|^{k(\rho-1+\varepsilon)}. \tag{2.2}$$

Set $Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, where $a_n = \alpha_n \mathrm{e}^{\mathrm{i}\varphi_n}$, $\alpha_n \ge 0$, $\varphi_n \in [0, 2\pi)$. Denote

$$\begin{aligned} \Omega_0 &= \big\{\theta \in [0, 2\pi) : \ \cos(\varphi_n + n\theta) = 0\big\} \cup E, \\ \Omega_+ &= \big\{\theta \in [0, 2\pi) : \ \cos(\varphi_n + n\theta) > 0\big\} \setminus E, \\ \Omega_- &= \big\{\theta \in [0, 2\pi) : \ \cos(\varphi_n + n\theta) < 0\big\} \setminus E. \end{aligned}$$

Let $\theta \in \Omega_+$, then from (2.1) and (2.2), we have

$$\left|\frac{a(r\mathrm{e}^{\mathrm{i}\theta})}{f(r\mathrm{e}^{\mathrm{i}\theta})}\right| \ge \left|\mathrm{e}^{Q(r\mathrm{e}^{\mathrm{i}\theta})}\right| - \left|\frac{f^{(k)}(r\mathrm{e}^{\mathrm{i}\theta})}{f(r\mathrm{e}^{\mathrm{i}\theta})}\right| \ge \exp\{\mathrm{Re}\{Q(r\mathrm{e}^{\mathrm{i}\theta})\} - |z|^{k(\rho-1+\varepsilon)} \to \infty,$$

which yields that

$$|f(r\mathrm{e}^{\mathrm{i}\theta})| \le |a(r\mathrm{e}^{\mathrm{i}\theta})| \le \exp\{r^{\sigma+\varepsilon}\}. \tag{2.3}$$

Let $\theta \in \Omega_-$. We first assume that $|f^{(k)}(z)| \exp\{-r^{\sigma+\varepsilon}\}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.2, there exists an infinite sequence of points $z_n = r_n \mathrm{e}^{\mathrm{i}\theta}$, where $r_n \to \infty$, such that $|f^{(k)}(z_n)| \exp\{-r_n^{\sigma+\varepsilon}\} \to \infty$ and

$$\left|\frac{f(z_n)}{f^{(k)}(z_n)}\right| \le (1 + o(1))||z_n|^k. \tag{2.4}$$

We obtain from (2.1) and (2.4) that

$$|f^{(k)}(r_n\mathrm{e}^{\mathrm{i}\theta})| = \left|\frac{a(r_n\mathrm{e}^{\mathrm{i}\theta})}{1 - \frac{f(r_n\mathrm{e}^{\mathrm{i}\theta})}{f^{(k)}(r_n\mathrm{e}^{\mathrm{i}\theta})}\mathrm{e}^{Q(r_n\mathrm{e}^{\mathrm{i}\theta})}}\right| \le \left|\frac{a(r_n\mathrm{e}^{\mathrm{i}\theta})}{1 - o(1)}\right| \le 2\exp\{r_n^{\sigma+\varepsilon}\},$$

which contradicts that $|f^{(k)}(z_n)| \exp\{-r_n^{\sigma+\varepsilon}\} \to \infty$.

Therefore, $|f^{(k)}(z)| \exp\{-r^{\sigma+\varepsilon}\}$ is bounded on the ray $\arg z = \theta$. Then there is some $M = M(\theta) > 0$ such that

$$|f^{(k)}(r\mathrm{e}^{\mathrm{i}\theta})| \exp\{-r^{\sigma+\varepsilon}\} \le M,$$

which gives

$$|f(r\mathrm{e}^{\mathrm{i}\theta})| \le 2Mr^k \exp\{r^{\sigma+\varepsilon}\} \le \exp\{r^{\sigma+2\varepsilon}\}. \tag{2.5}$$

Now we deduce from (2.3) and (2.4) that for each $\theta \in (\Omega_+ \cup \Omega_-)$ and sufficiently large r, we have

$$|f(re^{i\theta})| \le \exp\{r^{\sigma+2\varepsilon}\}. \tag{2.6}$$

Notice that $\Omega_0 = [0, 2\pi) \setminus (\Omega_+ \cup \Omega_-)$ has linear measure zero. Therefore, we can deduce from (2.6) and the Phragmén-Lindelöf theorem (see [8: pp. 138–139]) that (2.6) holds for each $\theta \in [0, 2\pi)$. Then we get a contradiction that $\rho = \rho(g) \le \sigma$. □

Using a similar proof as in the proof of [4: Remark 1], we can prove the following Lemma.

Lemma 2.4. ([4]) *Suppose that $f(z)$ is a transcendental entire function such that $\rho(f) = \infty$, $\rho_2(f) = \alpha < \infty$, and a set $E \subset [1, +\infty)$ has a finite logarithmic measure. Let $\nu(r, f)$ be the central index of $f(z)$. Then there exist a sequence of points: $\{z_k = r_k e^{i\theta_k}\}$ with $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, and a sequence of points $r_k \notin E, r_k \to \infty$ such that for any given $\varepsilon > 0$,*

(i) *If $\alpha > 0$, as r_k sufficiently large, we have*

$$\exp\{r_k^{\alpha-\varepsilon}\} < \nu(r_k, f) < \exp\{r_k^{\alpha+\varepsilon}\},$$

(ii) *If $\alpha = 0$, then for any given $M > 0$, as r_k sufficiently large, we have*

$$r_k^M < \nu(r_k, f) < \exp\{r_k^{\varepsilon}\}.$$

Lemma 2.5. ([14]) *Let*

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where n is a positive integer and $a_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given $0 < \varepsilon < \frac{\pi}{4n}$, consider $2n$ open angles S_j:

$$-\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon,$$

where $j = 0, \dots, 2n-1$. Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$,

$$\mathrm{Re}\{Q(z)\} > \alpha_n (1-\varepsilon) \sin(n\varepsilon) r^n$$

if $z \in S_j$ where j is even, while

$$\mathrm{Re}\{Q(z)\} < -\alpha_n (1-\varepsilon) \sin(n\varepsilon) r^n$$

if $z \in S_j$ where j is odd.

Lemma 2.6. ([7]) *Let $g(z)$ be an entire function of infinite order and $\rho_2(g) = \alpha < \infty$, and $\nu(r, g)$ be the central index of $g(z)$, then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log \nu(r, g)}{\log r} = \alpha.$$

Definition 2.1. The lower logarithmic density $\underline{\log\mathrm{dens}}\, H$ of subset $H \subset (1,\infty)$ is defined by

$$\underline{\log\mathrm{dens}}\, H = \varliminf_{r\to\infty}\bigg(\int_1^r (\chi_H(t)/t)\,\mathrm{d}t\bigg)\Big/\log r,$$

and the upper logarithmic density $\overline{\log\mathrm{dens}}\, H$ of subset $H \subset (1,\infty)$ is defined by

$$\overline{\log\mathrm{dens}}\, H = \varlimsup_{r\to\infty}\bigg(\int_1^r (\chi_H(t)/t)\,\mathrm{d}t\bigg)\Big/\log r,$$

where $\chi_H(t)$ is the characteristic function of the set H.

Lemma 2.7. ([1]) *Let $h(z)$ be an entire function with order $\rho = \rho(h) < \frac{1}{2}$ and set*

$$A(r) = \inf_{|z|=r} \log|h(z)|, \qquad B(r) = \sup_{|z|=r} \log|h(z)|.$$

If $\rho < \alpha < 1$, then

$$\underline{\log\mathrm{dens}}\{A(r) > (\cos\alpha\pi)B(r)\} \geq 1 - \frac{\rho}{\alpha}.$$

Remark 1. By Definition 2.1, we see that if $\overline{\log\mathrm{dens}}\, H > 0$, then $\mathrm{lm}\, H = \infty$.

3. Proof of Theorem 1.1

As $f-a$ and $f^{(k)}-a$ share the value 0 CM, by the normal Hadamard theorem (for the case $\rho(f) < \infty$) and the Hadamard theorem of entire functions of infinite order (see [13]), we have

$$\frac{g^{(k)} - b}{g} = \frac{f^{(k)} - a}{f - a} = \mathrm{e}^{Q}, \tag{3.1}$$

where $g = f - a$, $b = a^{(k)} - a$ and Q is an entire function such that $\rho(\mathrm{e}^Q) \leq \max\{\rho_2(f), \rho_2(a)\} < \frac{1}{2}$. Set $\rho_2(g) = \alpha$. Then $0 \leq \alpha \leq \max\{\rho_2(f), \rho_2(a)\} < \frac{1}{2}$.

We divide our proof into three cases for $Q(z)$.

***Case* 1**. $Q(z)$ is a constant function. Then our assertion holds.

***Case* 2**. $Q(z)$ is a polynomial with degree $\deg Q = n \geq 1$. Now by (3.1) and Lemma 2.3, we see that $\rho(g) = \infty$. From Wiman-Valiron theory (see [12, 13]), there exists a set $E \subset [1,\infty)$ of finite logarithmic measure such that

$$\frac{g^{(k)}(z)}{g(z)} = (1 + o(1))\left(\frac{\nu(r,g)}{z}\right)^k \tag{3.2}$$

holds for z satisfying $|z| = r \notin E$ and $|g(z)| = M(r,g)$.

Since $\rho(b) < \rho(g)$, we get from (3.1) and (3.2) that

$$(1+o(1))\left(\frac{\nu(r,g)}{z}\right)^k = \mathrm{e}^{Q(z)} \tag{3.3}$$

holds for z satisfying $|z| = r \notin E$ and $|g(z)| = M(r,g)$.

From Lemma 2.4, there exist a sequence of points: $\{z_m = r_m \mathrm{e}^{\mathrm{i}\theta_m}\}$ with $|g(z_m)| = M(r_m, g)$, $\theta_m \in [0, 2\pi)$, $\lim\limits_{m\to\infty} \theta_m = \theta_0 \in [0, 2\pi)$ and a sequence of points $r_m \notin E$, $r_m \to \infty$ such that for any given ε $\left(0 < 3\varepsilon < \min\{\frac{1}{2} - \alpha, \frac{\pi}{4n}\}\right)$, and $M > k+1$, as r_m sufficiently large, we have

$$r_m^M < \nu(r_m, g) < \exp\{r_m^{\alpha+\varepsilon}\}. \tag{3.4}$$

Let

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n = \alpha_n \mathrm{e}^{\mathrm{i}\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. By Lemma 2.5, for the ε above, there are $2n$ open angles S_j:

$$-\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon,$$

where $j = 0, \ldots, 2n-1$.

There are three cases for θ_0:

(i) $\theta_0 \in S_j$, where j is odd;

(ii) $\theta_0 \in S_j$, where j is even;

(iii) $\theta_0 = \frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j \in \{1, 2, \ldots, n-1\}$.

Case (i). $\theta_0 \in S_j$, where j is odd. Sine S_j is open, from Lemma 2.5, as m sufficiently large, we have

$$\mathrm{Re}\{Q(z_m)\} < -d r_m^n, \tag{3.5}$$

where $d = \alpha_n(1-\varepsilon)\sin(n\varepsilon) > 0$.

Now combining (3.3)–(3.5), we obtain

$$r_m < r_m^{M(k-1)} \le \left|(1+o(1))\frac{\nu(r_m,g)}{z_m}\right|^k \le 2\left|\mathrm{e}^{Q(z_m)}\right| < 2\exp\{-d r_m^n\},$$

a contradiction.

Case (ii). $\theta_0 \in S_j$, where j is even. Sine S_j is open, from Lemma 2.5, as m sufficiently large, we have

$$\mathrm{Re}\{Q(z_m)\} > d r_m^n, \tag{3.6}$$

where $d = \alpha_n(1-\varepsilon)\sin(n\varepsilon) > 0$.

We obtain from (3.3), (3.4) and (3.6) that

$$\exp\{d r_m^n\} \le |\mathrm{e}^{Q(z_m)}| = \left|(1+o(1))\left(\frac{\nu(r_m,g)}{z_m}\right)^k\right| \le 2\exp\{r_m^{\alpha+\varepsilon}\},$$

which yields that $n \le \alpha + \varepsilon < 1$, a contradiction.

Case (iii). $\theta_0 = \frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n}$ for some $j \in \{1, 2, \ldots, n-1\}$. We claim that there is some $N > 0$ such that for $m > N$, $\theta_m = \theta_0$. Otherwise, there exists an infinite subsequence $\{\theta_{m_1}\} \subset \{\theta_m\}$ such that $\theta_{m_1} \neq \theta_0$. However, by Lemma 2.5, there exists some $R > 0$ such that, for $r_{m_1} > R$, either $\mathrm{Re}\{Q(z_{m_1})\} < -d_2 r_{m_1}^n$ or $\mathrm{Re}\{Q(z_{m_1})\} > d_2 r_{m_1}^n$, where $d_2 > 0$ is some constant. We can suppose that $\mathrm{Re}\{Q(z_{m_1})\} < -d_2 r_{m_1}^n$ holds for all $r_{m_1} > R$. Then with a similar reasoning as in the case (ii), we can deduce a similar contradiction.

Therefore, we see that there is some $N > 0$ such that for $m > N$, $\theta_m = \theta_0$. In the following, we discuss two subcases:

(iii) a. there exists some $s \in \{1, \ldots, n-1\}$ such that $\mathrm{Re}\{b_{n-1}(re^{i\theta_0})^{n-1}\} = \cdots$ $\cdots = \mathrm{Re}\{b_{s+1}(re^{i\theta_0})^{s+1}\} = 0$ and $\mathrm{Re}\{b_s(re^{i\theta_0})^s\} \neq 0$;

(iii) b. $\mathrm{Re}\{b_{n-1}(re^{i\theta_0})^{n-1}\} = \cdots = \mathrm{Re}\{b_0(re^{i\theta_0})^{s+1}\} = 0$.

Arguing as in the Case (ii), we can easily see that the subcase (iii) a is impossible. In the subcase (iii) b, there is some constant $K > 0$ such that $-K < \mathrm{Re}\{Q(re^{i\theta_0})\} < K$, that is

$$e^{\frac{1}{K}} < \left|e^{Q(re^{i\theta_0})}\right| < e^K. \tag{3.7}$$

From (3.3), (3.4) and (3.7), we have

$$r_m < r_m^{M(k-1)} \leq \left|(1+o(1))\frac{\nu(r_m, g)}{z_m}\right|^k \leq 2\left|e^{Q(z_m)}\right| < e^K,$$

a contradiction.

Case 3. $Q(z)$ is a transcendental entire function such that $\rho(Q) \leq \alpha < \frac{1}{2}$. Since $\rho(e^Q) = \infty$, from (3.1), we have $\rho(g) = \infty$. Now we can use (3.2) with a restriction that $\nu(r, g) \geq |z|^M$. And hence (3.3) holds for z satisfying $|z| = r \notin E$, $|g(z)| = M(r, g)$ and $\nu(r, g) \geq |z|^M$. Now take a principal branch of $\mathrm{Log}\left((1+o(1))\left(\frac{\nu(r,g)}{z}\right)^k\right)$ in (3.3), and we get

$$Q(z) = k\log\left((1+o(1))\left(\frac{\nu(r, g)}{z}\right)\right),$$

which gives

$$|Q(z)| \leq \left|k\log\left((1+o(1))\left(\frac{\nu(r, g)}{z}\right)\right)\right| + 2\pi \leq 2k\log\nu(r, g). \tag{3.8}$$

On one hand, from Lemma 2.6, we have

$$\frac{\log\log\nu(r, g)}{\log r} \leq \alpha + 1. \tag{3.9}$$

On the other hand, by Lemma 2.7 and its Remark, there exists a set $H \subset [1, +\infty)$ with $\text{lm}(H) = +\infty$ such that for any z satisfying $|z| = r \in H$,

$$|Q(z)| \geq M(r, Q)^{d_3}, \tag{3.10}$$

where $d_3 \in (0, 1)$. Now for all z satisfying $|z| = r \in H \setminus E$ and $|g(z)| = M(r, g)$, by (3.8)–(3.10), we have

$$\frac{M(r, Q)^{d_3}}{2kr^{\alpha+1}} \leq 1,$$

which implies that $Q(z)$ is a polynomial, a contradiction. This completes our proof.

4. Proof of Theorem 1.2

Set $g = f^{(n)}$. We can see that Theorem 1.1 is valid for g and g'. Thus, there is some nonzero constant c such that

$$g' - a = c(g - a), \tag{4.1}$$

which gives that

$$g = b + d\mathrm{e}^{cz}, \tag{4.2}$$

for some constants b and $d \neq 0$. From integration of (4.2), we obtain

$$f(z) = P(z) + \frac{d}{c^n}\mathrm{e}^{cz}, \tag{4.3}$$

where $P(z)$ is a polynomial with degree $\deg P \leq n$.

Let z_0 be a point such that $f(z_0) = f^{(n+1)}(z_0) = b$. Then we have

$$P(z_0) + \frac{d}{c^n}\mathrm{e}^{cz_0} = cd\mathrm{e}^{cz_0} = b.$$

This gives

$$P(z_0) + \frac{b}{c^{n+1}} = b.$$

Thus z_0 is a zero of $P(z) + \frac{b}{c^{n+1}} - b$. By assumption, $P(z) + \frac{b}{c^{n+1}} - b$ has more than n zeros. Therefore, $P(z) \equiv b - \frac{b}{c^{n+1}}$. Then from (4.3), we have

$$f(z) = b - \frac{b}{c^{n+1}} + \frac{d}{c^n}\mathrm{e}^{cz}, \tag{4.4}$$

and hence

$$f^{(n)}(z) = d\mathrm{e}^{cz} \quad \text{and} \quad f^{(n+1)}(z) = cd\mathrm{e}^{cz}. \tag{4.5}$$

As $a \neq 0$, we obtain from (4.1) and (4.5) that $c = 1$. Thus from (4.4), we can see that $f(z) = d\mathrm{e}^{z}$ and hence $f \equiv f'$. This completes the proof of Theorem 1.2.

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