VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0269-8
Math. Slovaca **64** (2014), No. 5, 1197–1216

SUBORDINATION AND SUPERORDINATION RESULTS ASSOCIATED WITH THE GENERALIZED HYPERGEOMETRIC FUNCTION

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(*Communicated by Stanisława Kanas*)

ABSTRACT. In the present investigation, we obtain some subordination and superordination results involving Hadamard products for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

1. Introduction

Let $\mathscr{H}$ be the class of analytic functions in the open unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C}, \ \ |z| < 1\}.$$

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ and $a \in \mathbb{C}$, let

$$\mathscr{H}[a, n] = \{f : \ f \in \mathscr{H} \ \text{ and } \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Let $\mathscr{A}$ be the subclass of $\mathscr{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For f and F be members of the function class $\mathscr{H}$, the function f is said to be subordinate to F in $\mathbb{U}$, or the function F is said to be superordinate to f in $\mathbb{U}$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = F(w(z))$ $(z \in \mathbb{U})$. In such a case, we write

2010 Mathematics Subject Classification: Primary 30C45; Secondary 30C80.
Keywords: univalent functions, differential subordination and superordination, Hadamard product (convolution).

$f \prec F$ $(z \in \mathbb{U})$ or $f \prec F$ $(z \in \mathbb{U})$. Furthermore if the function F is univalent in $\mathbb{U}$, then we have that the following equivalence holds (see [12, 19]);

$$f \prec F \ (z \in \mathbb{U}) \iff f(0) = F(0) \ \text{ and } \ f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $\phi(r, s, t; z)\colon \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination

$$\phi\big(p(z), zp'(z), z^2p''(z); z\big) \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential subordination (1.2). The univalent function q is called a dominant of the solutions of the differential subordination (1.2) or, more simply, a dominant if $p \prec q$ $(z \in \mathbb{U})$ for all p satisfying (1.2) is said to be the best dominant. A dominant $\widetilde{q}$ that satisfies the subordination relationship $\widetilde{q} \prec p$ $(z \in \mathbb{U})$ for all dominants q of (1.2) is said to be the best dominant. Let $\varphi(r, s, t; z)\colon \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in $\mathbb{U}$ and satisfying the following differential superordination

$$h(z) \prec \varphi\big(p(z), zp'(z), z^2p''(z); z\big) \qquad (z \in \mathbb{U}), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). An analytic function q is called a subordinant of the solutions of the differential superordination (1.3) or, more simply, a subordinant if $q \prec p$ $(z \in \mathbb{U})$ for all p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies the subordination relationship $q \prec \widetilde{q}$ $(z \in \mathbb{U})$ for all subordinants q of (1.3) is said to be the best subordinant.

For two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.4}$$

For positive real parameters $\alpha_1, A_1, \dots, \alpha_l, A_l$ and $\beta_1, B_1, \dots, \beta_m, B_m$ $(l, m \in \mathbb{N} = 1, 2, 3, \dots)$ such that

$$1 + \sum_{n=1}^{m} B_m - \sum_{n=1}^{l} A_n \geq 0 \tag{1.5}$$

the Wright generalization [33]

$$\begin{aligned} {}_l\Psi_m[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m); z] \\ = {}_l\Psi_m[(\alpha_n, A_n)_{1,l}(\beta_n, B_n)_{1,m}; z] \end{aligned}$$

of the hypergeometric function ${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$ is defined by

$$
\begin{aligned}
&{}_l\Psi_m\big[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z\big] \\
&= \sum_{n=0}^{\infty} \Big\{ \prod_{t=0}^{l} \Gamma(\alpha_t + nA_t \Big\} \Big\{ \prod_{t=0}^{m} \Gamma(\beta_t + nB_t \Big\}^{-1} \frac{z^n}{n!}, \qquad z \in \mathbb{U}.
\end{aligned}
$$

If $A_t = 1$ $(t = 1, 2, \dots, l)$ and $B_t = 1$ $(t = 1, 2, \dots, m)$ we have the relationship:

$$
\begin{aligned}
\Omega_l \Psi_m\big[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z\big] &\equiv {}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!}
\end{aligned}
\tag{1.6}
$$

($l \le m + 1$; $l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $z \in \mathbb{U}$) is the generalized hypergeometric function (see for details [33]) where $\mathbb{N}$ denotes the set of all positive integers and $(\lambda)_n$ is the Pochhammer symbol and

$$
\Omega = \Big(\prod_{t=0}^{l} \Gamma(\alpha_t) \Big)^{-1} \Big(\prod_{t=0}^{m} \Gamma(\beta_t) \Big). \tag{1.7}
$$

By using the generalized hypergeometric function Dziok and Srivastava [13] introduced the linear operator which was subsequently extended by Dziok and Raina [14] by using Wright generalized hypergeometric function.

Let $\Theta[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] \colon \mathscr{A} \to \mathscr{A}$ be a linear operator defined by

$$
\Theta[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] f(z) := z\ \Omega_l \Psi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z).
$$

We observe that, for $f(z)$ of the form (1.1), we have

$$
\Theta\big[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}\big] f(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)\ a_n z^n \tag{1.8}
$$

where $\sigma_n(\alpha_1)$ is defined by

$$
\sigma_n(\alpha_1) = \frac{\Omega \Gamma(\alpha_1 + A_1(n-1)) \dots \Gamma(\alpha_l + A_l(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \dots \Gamma(\beta_m + B_m(n-1))} \tag{1.9}
$$

and Ω is given by (1.7).

For convenience, we write

$$
\Theta[\alpha_1] f(z) = \Theta[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m)] f(z). \tag{1.10}
$$

In view of (1.8), we get,

$$
zA_1(\Theta[\alpha_1] f(z))' = \alpha_1 \Theta[\alpha_1 + 1] f(z) - (\alpha_1 - A_1) \Theta[\alpha_1] f(z), \quad A_1 > 0. \tag{1.11}
$$

It is popularly known in the literature as Wright's generalized operator. Several interesting operators, which are special cases of the linear operator (1.8) have been widely studied by Dziok and Srivastava [13], Carlson and Shaffer [6], Ruscheweyh [26], Cho et al., [7–9], Choi et al., [11] and others (also see [15] and [32]).

Making use of the principle of subordination various subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$ were investigated by Bulboacă [5] (see also [4]), Miller and Mocanu[19], Owa and Srivastava [24]. Further, Miller and Mocanu [20] and Bulboacă [4] extended the study to differential superordination as the dual problem of differential subordination, later the study has been taken by many researchers like Ali et al., [1], Aouf and Mostafa [2], Cho et al., [10], Magesh et al., [17, 18], Mostafa and Aouf [21], Murugusundaramoorthy and Magesh [22], and Shanmugam et al., [28] and others.

In the the present investigation, we study the subordination and superordination results of the linear operator defined by (1.8). Also we obtain the sufficient condition for certain normalized analytic functions $f(z)$ in $\mathbb{U}$ such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \left(\frac{\Theta[\alpha_1](f * \Phi)(z)}{z}\right)^{\eta} \left(\frac{z}{\Theta[\alpha_1 + 1](f * \Psi)(z)}\right)^{\mu} \prec q_2(z), \tag{1.12}$$

where q_1, q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$, $q_2(0) = 1$ and $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$, $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ are analytic functions in $\mathbb{U}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$ and $\lambda_n \geq \mu_n$. Further, we state the number of known results as their special cases. In order to establish our main results, we need the following definition and lemmas:

Lemma 1.1. ([25: p. 159, Theorem 6.2]) *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = +\infty$, is a subordination chain if*

$$\operatorname{Re}\left\{ z \frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0, \qquad z \in \mathbb{U}, \quad t \geq 0.$$

Definition 1.1. ([20: p. 817, Definition 2]) Denote by Q, the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - \mathbb{E}(f)$, where

$$\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - \mathbb{E}(f)$.

Lemma 1.2. ([19: p. 132, Theorem 3.4h]) *Let q be univalent in the unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) := zq'(z)\phi(q(z)) \qquad \text{and} \qquad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

(1) *$Q(z)$ is starlike univalent in $\mathbb{U}$ and*

(2) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in \mathbb{U}$.

If p *is analytic with* $p(0) = q(0)$, $p(\mathbb{U}) \subseteq \mathbb{D}$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.13}$$

then

$$p(z) \prec q(z)$$

and q *is the best dominant.*

Lemma 1.3. ([5: p. 289, Corollary 3.2]) *Let* q *be convex univalent in the unit disk* $\mathbb{U}$ *and* ϑ *and* φ *be analytic in a domain* $\mathbb{D}$ *containing* $q(\mathbb{U})$. *Suppose that*

(1) $\operatorname{Re}\{\vartheta'(q(z))/\varphi(q(z))\} > 0$ *for* $z \in \mathbb{U}$ *and*

(2) $\psi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathbb{U}$.

If $p(z) \in \mathscr{H}[q(0), 1] \cap Q$, *with* $p(\mathbb{U}) \subseteq \mathbb{D}$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathbb{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{1.14}$$

then $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2. Subordination results

Using Lemma 1.2, we first prove the following theorem.

Theorem 2.1. *Let* $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3)$, $\gamma_3 \neq 0$, $\mu, \eta \in \mathbb{C}$ *and* q *be convex univalent with* $q(0) = 1$, *and assume that*

$$\operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3}q(z) - \frac{zq'(z)}{q(z)} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \qquad (z \in \mathbb{U}). \tag{2.1}$$

If $f \in \mathscr{A}$ *satisfies*

$$\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi) = \Delta(f,\, \Phi,\, \Psi,\, \gamma_1, \gamma_2, \gamma_3) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)}, \tag{2.2}$$

where

$$\begin{aligned}&\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi)\\ &:= \gamma_1 + \gamma_2\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu}\\ &\quad + \frac{\gamma_3}{A_1}\left[\eta\alpha_1\frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} - \mu(\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + \alpha_1[(\mu+1)-\eta]\right],\end{aligned} \tag{2.3}$$

and $A_1 > 0$, *then*

$$\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \prec q(z)$$

and q *is the best dominant.*

P r o o f. Define the function p by

$$p(z) := \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \qquad (z\in\mathbb{U}). \tag{2.4}$$

Then the function p is analytic in $\mathbb{U}$ and $p(0)=1$. Therefore, by making use of (2.4) and (1.11), we obtain

$$\begin{aligned}&\gamma_1+\gamma_2 p(z)+\gamma_3\frac{zp'(z)}{p(z)}\\ &=\gamma_1+\gamma_2\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu}\\ &\quad+\frac{\gamma_3}{A_1}\left[\eta\alpha_1\frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)}-\mu(\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}+\alpha_1[(\mu+1)-\eta]\right],\end{aligned} \tag{2.5}$$

By using (2.5) in (2.2), we have

$$\gamma_1+\gamma_2 p(z)+\gamma_3\frac{zp'(z)}{p(z)} \prec \gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)}. \tag{2.6}$$

By setting

$$\theta(w) := \gamma_1+\gamma_2\omega \qquad \text{and} \qquad \phi(\omega) := \frac{\gamma_3}{w},$$

it can be easily observed that $\theta(w)$, $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w)\neq 0$. Also we see that

$$Q(z) := zq'(z)\phi(q(z)) = \gamma_3\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z))+Q(z) = \gamma_1+\gamma_2 q(z)+\gamma_3\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$ and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3}q(z)-\frac{zq'(z)}{q(z)}+1+\frac{zq''(z)}{q'(z)}\right\} > 0.$$

By the hypothesis of Theorem 2.1, the result now follows by an application of Lemma 1.2. □

Corollary 2.1.1. *Let $\Phi,\Psi\in\mathscr{A}$, $\gamma_i\in\mathbb{C}$ $(i=1,\dots,3)$, $\gamma_3\neq 0$, $\mu\in\mathbb{C}$, $-1\le B<A\le 1$ and assume that* (2.1) *holds true. If $f\in\mathscr{A}$ and*

$$\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi) \prec \gamma_1+\gamma_2\frac{1+Az}{1+Bz}+\gamma_3\frac{(A-B)z}{(1+Az)(1+Bz)} \tag{2.7}$$

and $A_1>0$, then

$$\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $\eta = \gamma_1 = \gamma_3 = 1$, $\gamma_2 = A = \mu = 0$ and $B = \frac{(1-\upsilon)e^{i\theta}}{\upsilon}$ $(0 \leq \theta < 2\pi)$ for some υ $(\frac{1}{2} \leq \upsilon < 1)$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.1, we get the following result obtained by Kuroki and Owa [16: Corollary 3.3].

Corollary 2.1.2. *If $f \in \mathscr{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} \prec \frac{\upsilon}{\upsilon + (1-\upsilon)e^{i\theta}z}, \qquad z \in \mathbb{U}, \quad 0 \leq \theta < 2\pi$$

for some υ $(\frac{1}{2} \leq \upsilon < 1)$, then

$$\frac{f(z)}{z} \prec \frac{\upsilon}{\upsilon + (1-\upsilon)e^{i\theta}z}, \qquad z \in \mathbb{U}$$

and $\frac{\upsilon}{\upsilon+(1-\upsilon)e^{i\theta}z}$ is the best dominant.

Remark 1. From Corollary 2.1.2, we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \upsilon \implies \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \upsilon, \qquad z \in \mathbb{U}, \quad \upsilon \in \left[\tfrac{1}{2}, 1\right).$$

Taking $\eta = \gamma_1 = 1$, $\gamma_3 = \gamma$, $\gamma_2 = \mu = 0$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)^2}$ in Corollary 2.1.1, we get the following result.

Corollary 2.1.3. *If $f \in \mathscr{A}$ satisfies*

$$1 + \gamma\frac{zf''(z)}{f'(z)} \prec 1 + \frac{\alpha(A-B)z}{(1+Az)(1+Bz)}, \qquad z \in \mathbb{U}$$

then

$$f'(z) \prec \frac{1+Az}{1+Bz}, \qquad z \in \mathbb{U}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Taking $\gamma = 1$, $A = 0$ and $B = \frac{(1-\upsilon)e^{i\theta}}{\upsilon}$ $(0 \leq \theta < 2\pi)$ for some υ $(\frac{1}{2} \leq \upsilon < 1)$ in Corollary 2.1.3, we get the following result obtained by Kuroki and Owa [16: Corollary 3.6].

Corollary 2.1.4. *If $f \in \mathscr{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\upsilon}{\upsilon + (1-\upsilon)e^{i\theta}z}, \qquad z \in \mathbb{U}, \quad 0 \leq \theta < 2\pi$$

for some υ $(\frac{1}{2} \leq \upsilon < 1)$ then

$$f'(z) \prec \frac{\upsilon}{\upsilon + (1-\upsilon)e^{i\theta}z}, \qquad z \in \mathbb{U}$$

and $\frac{\upsilon}{\upsilon+(1-\upsilon)e^{i\theta}z}$ is the best dominant.

Remark 2. From Corollary 2.1.4, we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \upsilon \implies \operatorname{Re}\{f'(z)\} > \upsilon, \quad z \in \mathbb{U}, \quad \upsilon \in \left[\tfrac{1}{2}, 1\right).$$

Taking $\eta = 0$ in Theorem 2.1, we get the following result.

Corollary 2.1.5. *Let $\Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3)$, $\gamma_3 \neq 0$, $0 \neq \mu \in \mathbb{C}$ and q be convex univalent with $q(0) = 1$ and assume that* (2.1) *holds true. If $f \in \mathscr{A}$ satisfies*

$$\Delta_1^{(\gamma_i)_1^3}(f;\Psi) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}, \tag{2.8}$$

where

$$\begin{aligned}\Delta_1^{(\gamma_i)_1^3}(f;\Psi) := \gamma_1 &+ \gamma_2\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \\ &- \frac{\gamma_3}{A_1}\left[\mu(\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} - \alpha_1(\mu+1)\right],\end{aligned} \tag{2.9}$$

and $A_1 > 0$, then

$$\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

Putting $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = -1$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Psi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.5, we get the following corollary.

Corollary 2.1.6. *Let $\mu > 0$, be a real number. Let $q(z) \neq 0$ be a univalent function in $\mathbb{U}$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $\mathbb{U}$. If $f \in A$ $(f'(z) \neq 0)$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{-1}{\mu}\frac{zq'(z)}{q(z)}, \qquad z \in \mathbb{U}$$

then

$$\left(\frac{1}{f'(z)}\right)^{\mu} \prec q(z)$$

and q is the best dominant.

By setting $q(z) = \frac{\alpha(1-z)}{\alpha-z}$ in Corollary 2.1.6, we obtain the following corollary as stated in [30: Theorem 4.2].

Corollary 2.1.7. *If $f \in \mathscr{A}$, $f'(z) \neq 0$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{(\alpha-1)z}{\mu(\alpha-z)(1-z)}, \qquad z \in \mathbb{U}$$

then

$$\left(\frac{1}{f'(z)}\right)^{\mu} \prec \frac{\alpha(1-z)}{\alpha-z}, \qquad z \in \mathbb{U}$$

and $\frac{\alpha(1-z)}{\alpha-z}$ *is the best dominant, where* $\alpha > 1$ *and* $\mu > 0$ *are some real numbers.*

By setting $q(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$ in Corollary 2.1.6, we obtain the following corollary as stated in [30: Theorem 4.4].

Corollary 2.1.8. *If* $f \in \mathscr{A}$, $f'(z) \neq 0$ *satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(\beta-1)z}{\mu(1+(1-2\beta)z)(1-z)}, \qquad z \in \mathbb{U}$$

then

$$\left(\frac{1}{f'(z)}\right)^{\mu} \prec \frac{1+(1-2\beta)z}{1-z}, \qquad z \in \mathbb{U}$$

and $\frac{1+(1-2\beta)z}{1-z}$ *is the best dominant, where* $0 \leq \beta < 1$ *and* $\mu > 0$ *are some real numbers.*

Taking $\mu = 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1.9. *Let* $\Phi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 3)$, $\gamma_3 \neq 0$, $0 \neq \eta \in \mathbb{C}$ *and* q *be convex univalent with* $q(0) = 1$, *and assume that* (2.1) *holds true. If* $f \in \mathscr{A}$ *satisfies*

$$\Delta_2^{(\gamma_i)_1^3}(f;\Phi) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}, \tag{2.10}$$

where

$$\begin{aligned}\Delta_2^{(\gamma_i)_1^3}(f;\Phi) := \gamma_1 &+ \gamma_2\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta} \\ &+ \frac{\gamma_3}{A_1}\left[\eta\alpha_1\frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} + \alpha_1[1-\eta]\right],\end{aligned} \tag{2.11}$$

and $A_1 > 0$, *then*

$$\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta} \prec q(z)$$

and q *is the best dominant.*

Remark 3.

(1) Putting $\gamma_1 = 1$, $\gamma_2 = 0$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)^2}$, $q(z) = \frac{1}{(1-z)^{2ab}}$ $(b \in \mathbb{C} \setminus \{0\})$, $\eta = a$ and $\gamma_4 = \frac{1}{b}$ in Corollary 2.1.9, we get the result obtained by Obradovič et al., [23: Theorem 1].

(2) Putting $\gamma_1 = 1$, $\gamma_2 = 0$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{1-z}$, $q(z) = \frac{1}{(1-z)^{2b}}$ $(b \in \mathbb{C} \setminus \{0\})$, $\eta = 1$ and $\gamma_3 = \frac{1}{b}$ in Corollary 2.1.9 and then combining this together with Lemma1.2, we obtain the result of Srivastava and Lashin [31: Theorem 3].

(3) Taking $\gamma_1 = 1$, $\gamma_2 = 0$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{1-z}$, $\gamma_3 = \frac{\mathrm{e}^{\mathrm{i}\lambda}}{ab\cos\lambda}$ $(a, b \in \mathbb{C}$, $|\lambda| < \frac{\pi}{2})$, $\eta = a$ and $q(z) = (1-z)^{-2ab\cos\lambda \mathrm{e}^{-\mathrm{i}\lambda}}$ in Corollary 2.1.9, we obtain the result of Aouf et al. [3: Theorem 1].

(4) By taking $A_t = B_t = 1$, $\gamma_1 = 1$, $\gamma_2 = 0$ and $\Phi(z) = \frac{z}{1-z}$ in Corollary 2.1.9, we have the result obtained by the author [22: Theorem 3.5].

(5) Specializing the values of $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = \frac{1}{b}$ $(b \in \mathbb{C} \setminus \{0\})$, $q(z) = \frac{1}{(1-z)^{2ab}}$ $(a, b \in \mathbb{C} \setminus \{0\})$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.9, we obtain the results as stated in Srivastava and Lashin [31: Theorem].

(6) Specializing the values of $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = \frac{1}{b}$ $(b \in \mathbb{C} \setminus \{0\})$, $q(z) = \frac{1}{(1-z)^{2b}}$ $(b \in \mathbb{C} \setminus \{0\})$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)^2}$ in Corollary 2.1.9, we obtain the results as stated in Srivastava and Lashin [31: Theorem].

Putting $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.9, we get the following corollary.

Corollary 2.1.10. *Let $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3,\ \gamma_3 \neq 0)$, $\eta \in \mathbb{C}$ such that $\eta \neq 0$ and q be convex univalent with $q(0) = 1$, and (2.1) holds true. If $f \in \mathscr{A}$ satisfies*

$$\gamma_1 + \gamma_2 \left(f'(z)\right)^{\eta} + \gamma_3 \eta \frac{zf''(z)}{f'(z)} \prec \gamma_1 + \gamma_2 p(z) + \gamma_3 \frac{zp'(z)}{p(z)},$$

then

$$\left(f'(z)\right)^{\eta} \prec q(z)$$

and q is the best dominant.

Putting $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 1$ in Corollary 2.1.10, we obtain the following result.

Corollary 2.1.11. *Let $\eta > 0$, be a real number. Let $q(z) \neq 0$ be a univalent function in $\mathbb{U}$ such that $\frac{zq'(z)}{q(z)}$ is starlike in $\mathbb{U}$. If $f \in A$ $(f'(z) \neq 0)$ satisfies*

$$\frac{zf''(z)}{f'(z)} \prec \frac{1}{\eta} \frac{zq'(z)}{q(z)}, \qquad z \in \mathbb{U}$$

then

$$\left(f'(z)\right)^{\eta} \prec q(z)$$

and q is the best dominant.

By setting $q(z) = \frac{\alpha(1-z)}{\alpha-z}$ in Corollary 2.1.11, we obtain the following corollary as stated in [30: Theorem 4.1].

Corollary 2.1.12. *If $f \in \mathscr{A}$, $f'(z) \neq 0$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{(1-\alpha)z}{\eta(\alpha-z)(1-z)}, \qquad z \in \mathbb{U}$$

then

$$(f'(z))^{\eta} \prec \frac{\alpha(1-z)}{\alpha-z}, \qquad z \in \mathbb{U}$$

where $\alpha > 1$ and $\eta > 0$ are some real numbers and $\frac{\alpha(1-z)}{\alpha-z}$ is the best dominant.

By setting $q(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$ in Corollary 2.1.11, we obtain the following corollary as stated in [30: Theorem 4.3].

Corollary 2.1.13. *If $f \in \mathscr{A}$, $f'(z) \neq 0$ satisfies the differential subordination*

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(1-\beta)z}{\eta(1+(1-2\beta)z)(1-z)}, \qquad z \in \mathbb{U}$$

then

$$(f'(z))^{\eta} \prec \frac{1+(1-2\beta)z}{1-z}, \qquad z \in \mathbb{U}$$

for some real numbers $0 \leq \beta < 1$, $\eta > 0$ and $\frac{1+(1-2\beta)z}{1-z}$ is the best dominant.

Putting $\gamma_1 = \gamma - 1$, $\gamma_2 = 1 - \gamma$, $\gamma_3 = \gamma$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.9, we obtain the following corollary as stated in [29: Theorem 3.1].

Corollary 2.1.14. *Let $\gamma \neq 0$ be a complex number. Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{U}$ and satisfy the condition* (2.1)*. If $f \in \mathscr{A}$, $f'(z) \neq 0$, $z \in \mathbb{U}$, satisfies the differential subordination*

$$(1-\gamma)(f'(z)-1) + \gamma\frac{zf''(z)}{f'(z)} \prec (1-\gamma)(q(z)-1) + \gamma\frac{zq''(z)}{q'(z)}$$

then $f'(z) \prec q(z)$ and q is the best dominant.

Remark 4. Taking $q(z) = \frac{1+z}{1-z}$ in Corollary 2.1.14, we get the result as stated in [29: Corollary 3.4].

By fixing $\eta = \mu = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 2.1.15. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3,\ \gamma_3 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that* (2.1) *holds true. If $f \in \mathscr{A}$ satisfies*

$$\Delta_3^{(\gamma_i)_1^3}(f; \Phi, \Psi) \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)}, \tag{2.12}$$

where

$$\Delta_3^{(\gamma_i)_1^3}(f;\Phi,\Psi) := \gamma_1 + \gamma_2 \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + \frac{\gamma_3}{A_1}\left[\alpha_1 \frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} - (\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + \alpha_1\right], \tag{2.13}$$

and $A_1 > 0$*, then*

$$\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \prec q(z)$$

and q *is the best dominant.*

Taking $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \Psi(z) = \frac{z}{(1-z)}$ in Corollary 2.1.15, we get following result.

Corollary 2.1.16. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 3,\ \gamma_3 \neq 0)$*,* q *be convex univalent with* $q(0) = 1$*, and assume that* (2.1) *holds true. If* $f \in \mathscr{A}$ *satisfies*

$$\gamma_1 + \gamma_2 \frac{f(z)}{zf'(z)} - \gamma_3\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right] \prec \gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)},$$

then

$$\frac{f(z)}{zf'(z)} \prec q(z)$$

and q *is the best dominant.*

3. Superordination results

Now, by applying Lemma 1.3, we prove the following theorem.

Theorem 3.1. *Let* $\Phi, \Psi \in \mathscr{A}$*,* $\gamma_i \in \mathbb{C}$ $(i = 1, \ldots, 3)$*,* $\gamma_3 \neq 0$*,* $\mu, \eta \in \mathbb{C}$ *and* q *be convex univalent with* $q(0) = 1$*, and assume that*

$$\operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3} q(z)\right\} \geq 0. \tag{3.1}$$

If $f \in \mathscr{A}$*,* $\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \in \mathscr{H}[q(0),1] \cap Q$*. Let* $\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi)$ *be univalent in* $\mathbb{U}$ *and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi), \tag{3.2}$$

where $\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi)$ *is given by* (2.3)*, then*

$$q(z) \prec \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu}$$

and q *is the best subordinant.*

P r o o f. Define the function p by

$$p(z) := \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu}. \tag{3.3}$$

Simple computation from (3.3), we get,

$$\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi) = \gamma_1 + \gamma_2 p(z) + \gamma_3\frac{zp'(z)}{p(z)},$$

then

$$\gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 p(z) + \gamma_3\frac{zp'(z)}{p(z)}.$$

By setting $\vartheta(w) = \gamma_1 + \gamma_2 w$ and $\phi(w) = \frac{\gamma_3}{w}$, it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$. Also, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and $\phi(w) \neq 0$.

If we let

$$\begin{aligned} L(z,t) &= \vartheta(q(z)) + \phi(q(z))tzq'(z) = \gamma_1 + \gamma_2 q(z) + \gamma_3 t\frac{zq'(z)}{q(z)} \\ &= a_1(t)z + \dots \end{aligned} \tag{3.4}$$

Differentiating (3.4) with respect to z and t, we have

$$\begin{aligned} \frac{\partial L(z,t)}{\partial z} &= 2\gamma_2 q'(z) + t\gamma_3\left[\frac{zq''(z)}{q(z)} + \frac{q'(z)}{q(z)} - z\left(\frac{q'(z)}{q(z)}\right)^2\right] \\ &= a_1(t) + \dots \end{aligned}$$

and

$$\frac{\partial L(z,t)}{\partial t} = \gamma_3\frac{zq'(z)}{q(z)}.$$

Also,

$$\frac{\partial L(0,t)}{\partial z} = \gamma_3 q'(0)\left[\frac{\gamma_2}{\gamma_3} + t\frac{1}{q(0)}\right].$$

From the univalence of q we have $q'(0) \neq 0$ and $q(0) = 1$, it follows that $a_1(t) \neq 0$ for $t \geq 0$ and $\lim\limits_{t\to\infty}|a_1(t)| = +\infty$.

A simple computation yields,

$$\operatorname{Re}\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3}q(z) + t\left(1 + \frac{zq''(z)}{q'(z)} - zq'(z)\right)\right\}.$$

Using the fact that q is convex univalent function in $\mathbb{U}$ and $\gamma_3 \neq 0$, we have,

$$\operatorname{Re}\left\{z\frac{\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad \text{if} \quad \operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3}q(z)\right\} > 0, \qquad z \in \mathbb{U}, \quad t \geq 0.$$

Now Theorem 3.1 follows by applying Lemma 1.3. $\square$

By taking $q(z) = (1 + Az)/(1 + Bz)$ $(-1 \le B < A \le 1)$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1.1. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3,\ \gamma_3 \ne 0)$, $\mu, \eta \in \mathbb{C}$ and q be convex univalent with $q(0) = 1$, and $\operatorname{Re}\left\{\frac{\gamma_2}{\gamma_3}\left(\frac{1+Az}{1+Bz}\right)\right\} > 0$. If $f \in \mathscr{A}$, $\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \in \mathscr{H}[q(0), 1] \cap Q$. Let $\Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi)$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 \frac{1 + Az}{1 + Bz} + \gamma_3 \frac{(A - B)z}{(1 + Az)(1 + Bz)} \prec \Delta^{(\gamma_i)_1^3}(f; \Phi, \Psi),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \left(\frac{\Theta[\alpha_1](f * \Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1 + 1](f * \Psi)(z)}\right)^{\mu}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Taking $\eta = 0$ in Theorem 3.1, we get the following result.

Corollary 3.1.2. *Let $\Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3,\ \gamma_3 \ne 0)$, $0 \ne \mu \in \mathbb{C}$ and q be convex univalent with $q(0) = 1$, and assume that (3.1) holds true. If $f \in \mathscr{A}$, $\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \in \mathscr{H}[q(0), 1] \cap Q$. Let $\Delta_1^{(\gamma_i)_1^3}(f; \Psi)$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \Delta_1^{(\gamma_i)_1^3}(f; \Psi), \tag{3.5}$$

where

$$\begin{aligned}\Delta_1^{(\gamma_i)_1^3}(f; \Psi) := \gamma_1 &+ \gamma_2 \left(\frac{z}{\Theta[\alpha_1 + 1](f * \Psi)(z)}\right)^{\mu} \\ &- \frac{\gamma_3}{A_1}\left[\mu(\alpha_1 + 1)\frac{\Theta[\alpha_1 + 2](f * \Psi)(z)}{\Theta[\alpha_1 + 1](f * \Psi)(z)} - \alpha_1(\mu + 1)\right],\end{aligned} \tag{3.6}$$

and $A_1 > 0$, then

$$q(z) \prec \left(\frac{z}{\Theta[\alpha_1 + 1](f * \Psi)(z)}\right)^{\mu}$$

and q is the best subordinant.

Taking $\mu = 0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1.3. *Let $\Phi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1, \dots, 3)$, $\gamma_3 \ne 0$, $\mu \in \mathbb{C}$ and q be convex univalent with $q(0) = 1$, and assume that (3.1) holds true. If $f \in \mathscr{A}$, $\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta} \in \mathscr{H}[q(0), 1] \cap Q$. Let $\Delta_2^{(\gamma_i)_1^3}(f; \Phi)$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \Delta_2^{(\gamma_i)_1^3}(f; \Phi) \tag{3.7}$$

where

$$\Delta_2^{(\gamma_i)_1^3}(f;\Phi) := \gamma_1 + \gamma_2\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^\eta + \frac{\gamma_3}{A_1}\left[\eta\alpha_1\frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} + \alpha_1[1-\eta]\right], \tag{3.8}$$

and $A_1 > 0$, then

$$q(z) \prec \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^\eta$$

and q is the best subordinant.

By fixing $\Phi(z) = \frac{z}{(1-z)^2}$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\beta_1 = 1$ in Corollary 3.1.3, we obtain the following corollary.

Corollary 3.1.4. *Let $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,3)$, $\gamma_3 \neq 0$, $0 \neq \mu \in \mathbb{C}$ and q be convex univalent with $q(0) = 1$, and (3.1) holds true. If $f \in \mathscr{A}$, $(f'(z))^\mu \in \mathscr{H}[q(0),1] \cap Q$. Let $\gamma_1 + \gamma_2\,(f'(z))^\mu + \gamma_3\mu\frac{zf''(z)}{f'(z)}$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2\,(f'(z))^\mu + \gamma_3\mu\frac{zf''(z)}{f'(z)},$$

then

$$q(z) \prec (f'(z))^\mu$$

and q is the best subordinant.

Putting $\gamma_1 = \gamma - 1$, $\gamma_2 = 1 - \gamma$, $\gamma_3 = \gamma$, $\eta = 1$, $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \frac{z}{(1-z)}$ in Corollary 3.1.3, we obtain the following corollary.

Corollary 3.1.5. *Let $\gamma \neq 0$ be a complex number. Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{U}$ and satisfy the condition (3.1). If $f \in \mathscr{A}$, $0 \neq f'(z) \in \mathscr{H}[q(0),1] \cap Q$. Let $(1-\gamma)(f'(z)-1) + \gamma\frac{zf''(z)}{f'(z)}$ be univalent in $\mathbb{U}$ and*

$$(1-\gamma)(q(z)-1) + \gamma\frac{zq''(z)}{q'(z)} \prec (1-\gamma)(f'(z)-1) + \gamma\frac{zf''(z)}{f'(z)}$$

then $q(z) \prec f'(z)$ and q is the best subordinant.

By fixing $\eta = \mu = 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1.6. *Let $\Phi, \Psi \in \mathscr{A}$, $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,3,\ \gamma_3 \neq 0)$, q be convex univalent with $q(0) = 1$, and assume that (3.1) holds true. If $f \in \mathscr{A}$, $\frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} \in \mathscr{H}[q(0),1] \cap Q$. Let $\Delta_3^{(\gamma_i)_1^3}(f;\Phi,\Psi)$ be univalent in $\mathbb{U}$ and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3\frac{zq'(z)}{q(z)} \prec \Delta_3^{(\gamma_i)_1^3}(f;\Phi,\Psi), \tag{3.9}$$

where

$$\Delta_3^{(\gamma_i)_1^3}(f;\Phi,\Psi) := \gamma x_1 + \gamma_2 \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + \frac{\gamma_3}{A_1}\Bigg[\alpha_1 \frac{\Theta[\alpha_1+1](f*\Phi)(z)}{\Theta[\alpha_1](f*\Phi)(z)} - (\alpha_1+1)\frac{\Theta[\alpha_1+2](f*\Psi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)} + \alpha_1\Bigg], \tag{3.10}$$

and $A_1 > 0$, *then*

$$q(z) \prec \frac{\Theta[\alpha_1](f*\Phi)(z)}{\Theta[\alpha_1+1](f*\Psi)(z)}$$

and q *is the best subordinant.*

Taking $A_t = B_t = 1$, $l = 2$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\Phi(z) = \Psi(z) = \frac{z}{(1-z)}$ in Corollary 3.1.6, we get following result.

Corollary 3.1.7. *Let* $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,3)$, $\gamma_3 \neq 0$, q *be convex univalent with* $q(0) = 1$, *and assume that* (3.1) *holds true. If* $f \in \mathscr{A}$, $\frac{f(z)}{zf'(z)} \in \mathscr{H}[q(0),1] \cap Q$. *Let* $\gamma_1 + \gamma_2\frac{f(z)}{zf'(z)} - \gamma_3\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right]$ *be univalent in* $\mathbb{U}$ *and*

$$\gamma_1 + \gamma_2 q(z) + \gamma_3 \frac{zq'(z)}{q(z)} \prec \gamma_1 + \gamma_2 \frac{f(z)}{zf'(z)} - \gamma_3\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right],$$

then

$$q(z) \prec \frac{f(z)}{zf'(z)}$$

and q *is the best subordinant.*

4. Sandwich results

There is a complete analog of Theorem 2.1 for differential subordination and Theorem 3.1 for differential superordination. We can combine the results of Theorem 2.1 with Theorem 3.1 and obtain the following sandwich theorem.

Theorem 4.1. *Let* q_1 *and* q_2 *be convex univalent in* $\mathbb{U}$, $\gamma_i \in \mathbb{C}$ $(i = 1,\ldots,3)$, $\gamma_3 \neq 0$, $\mu,\eta \in \mathbb{C}$ *and let* q_2 *satisfy* (2.1) *and* q_1 *satisfy* (3.1). *For* $f,\Phi,\Psi \in \mathscr{A}$, *let* $\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \in \mathscr{H}[1,1] \cap Q$ *and* $\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi)$ *defined by* (2.3) *be univalent in* $\mathbb{U}$ *satisfying*

$$\gamma_1 + \gamma_2 q_1(z) + \gamma_3 \frac{zq_1'(z)}{q_1(z)} \prec \Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi) \prec \gamma_1 + \gamma_2 q_2(z) + \gamma_3 \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and best dominant.*

By taking $q_1(z) = \frac{1+A_1z}{1+B_1z}$ $(-1 \leq B_1 < A_1 \leq 1)$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \leq B_2 < A_2 \leq 1)$ in Theorem 4.1 we obtain the following result.

Corollary 4.1.1. *For* $f, \Phi, \Psi \in \mathscr{A}$, *let* $\left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \in \mathscr{H}[1,1]\cap Q$ *and* $\Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi)$ *defined by* (2.3) *be univalent in* $\mathbb{U}$ *satisfying*

$$\begin{aligned}&\gamma_1+\gamma_2\frac{1+A_1z}{1+B_1z}+\gamma_4\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\\ &\prec \Delta^{(\gamma_i)_1^3}(f;\Phi,\Psi) \prec \gamma_1+\gamma_2\frac{1+A_2z}{1+B_2z}+\gamma_4\frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}\end{aligned}$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{\Theta[\alpha_1](f*\Phi)(z)}{z}\right)^{\eta}\left(\frac{z}{\Theta[\alpha_1+1](f*\Psi)(z)}\right)^{\mu} \prec \frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

We remark that, one can easily restate Theorem 4.1 for the different choices of $\Phi(z)$, $\Psi(z)$, A_t, B_t, l, m, $\alpha_1, \alpha_2, \ldots \alpha_l$, $\beta_1, \beta_2, \ldots \beta_m$ and for $\gamma_1, \gamma_2, \gamma_3$.

5. Conclusion

We conclude this paper by remarking that in view of the function class defined by the subordination relation (1.12) and expressed in terms of the convolution (1.4) involving arbitrary coefficients, the main results would lead to additional new results. In fact, by appropriately selecting the arbitrary sequences ($\Phi(z)$ and $\Psi(z)$) and specializing the parameters $A_t = 1$ $(t = 1, \ldots, l)$ and $B_t = 1$ $(t = 1, \ldots, m)$, l, m, η, μ, γ_1, γ_2, γ_3 and the function $q(z)$ the results presented in this paper would find further applications for the classes which incorporate generalized forms of linear operators illustrated below

(1)

$$\Phi(z) = z + \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}}\frac{z^n}{(n-1)!}$$

and

$$\Psi(z) = z + \sum_{n=2}^{\infty}n\frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}}\frac{z^n}{(n-1)!}$$

where $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$, $(l, m \in \mathbb{N} = 1, 2, 3, \ldots)$ are complex parameters $\beta_j \notin \{0, -1, -2, \ldots\}$ for $j = 1, 2, \ldots m$, $l \leq m+1$ (results for the Dziok-Srivastava operator [13])

(2) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$ and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n \frac{(a)_{n-1}}{(c)_{n-1}} \frac{z^n}{(n-1)!}$, where $c \neq 0, -1, -2, \ldots$ (results for the Carlson and Shaffer operator [6])

(3) $\Phi(z) = \frac{z}{(1-z)^{\lambda+1}}, \lambda \geq -1$, and $\Psi(z) = \frac{z}{(1-z)^{\lambda+2}}$, $\lambda \geq -1$ (results for the Ruscheweyh derivative operator [26])

(4) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} n^k z^n$, and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n^{k+1} z^n$, $k \geq 0$ (results for the Salagean operator [27])

(5) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n$, where $\lambda \geq 0$, $k \in \mathcal{Z}$ (results for the Multiplier transformation [7,8])

(6) $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \ldots z^n$, and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} n \ldots z^n$, where $\lambda \geq 0$, $k \in \mathcal{Z}$ (results for the Choi-Saigo-Srivastava operator [11])

in Theorem 2.1, Theorem 3.1 and Theorem 4.1 would eventually lead further new results. These considerations can fruitfully be worked out and we skip the details in this regard.

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Received 5. 12. 2011
Accepted 29. 6. 2012

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