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ON A STRENGTHENED MULTIDIMENSIONAL HILBERT-TYPE INEQUALITY

Mario Krnić

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ABSTRACT. The main objective of this paper is a study of the general refinement and converse of the multidimensional Hilbert-type inequality in the so-called quotient form. Such extensions are deduced with the help of the sophisticated use of the well-known Hölder's inequality. The obtained results are then applied to homogeneous kernels with the negative degree of homogeneity. Also, we establish the conditions under which the constant factors involved in the established inequalities are the best possible. Finally, we consider some particular settings with homogeneous kernels and weighted functions. In such a way we obtain both refinements and converses of some actual results, known from the literature.

1. Introduction

Hilbert's inequality is one of the most significant weighted inequalities in mathematical analysis and its applications. Through the years, Hilbert-type inequalities were discussed by numerous authors, who either reproved them using various techniques, or applied and generalized them in many different ways. For more details about Hilbert's inequality the reader is referred to [4] or [7].

Although classical, Hilbert's inequality is still of interest to numerous mathematicians. Some of the recent results concerning Hilbert's inequality include extension to multidimensional case, equipped with conjugate exponents p_i, that is, $\sum_{i=1}^{n} 1/p_i = 1$, $p_i > 1$, $n \geq 2$ (see papers [1], [2], [6], [8], [9]).

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Here we refer to the paper [1], which provides a unified treatment of the multidimensional Hilbert-type inequality in the setting with conjugate exponents. Suppose $(\Omega_i, \Sigma_i, \mu_i)$ are σ-finite measure spaces and $K\colon \prod_{i=1}^{n} \Omega_i \to \mathbb{R}$, $\phi_{ij}\colon \Omega_j \to \mathbb{R}$, $f_i\colon \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \dots, n$, are non-negative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x}) \le \prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}, \tag{1.1}$$

where

$$\omega_i(x_i) = \left[\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(x_j)\,\mathrm{d}\hat{\mu}^i(\mathbf{x})\right]^{1/p_i} \tag{1.2}$$

and

$$\begin{gathered}\mathbf{\Omega} = \prod_{i=1}^{n} \Omega_i, \quad \hat{\mathbf{\Omega}}^i = \prod_{j=1, j\neq i}^{n} \Omega_j, \quad \mathbf{x} = (x_1, x_2, \dots, x_n), \\ \mathrm{d}\mu(\mathbf{x}) = \prod_{i=1}^{n} \mathrm{d}\mu_i(x_i), \quad \mathrm{d}\hat{\mu}^i(\mathbf{x}) = \prod_{j=1, j\neq i}^{n} \mathrm{d}\mu_j(x_j).\end{gathered} \tag{1.3}$$

The abbreviations as in (1.3) will be valid throughout the whole paper. Also note that $\|\cdot\|_{p_i}$ denotes the usual norm in $L^{p_i}(\Omega_i)$, that is

$$\|\phi_{ii}\omega_i f_i\|_{p_i} = \left[\int_{\Omega_i} (\phi_{ii}\omega_i f_i)^{p_i}(x_i)\,\mathrm{d}\mu_i(x_i)\right]^{1/p_i}, \qquad i = 1, 2, \dots, n.$$

The main purpose of this paper is to establish the general refinement and converse of the multidimensional Hilbert-type inequality (1.1). More precisely, such extensions can be established with the help of the refined use of the well-known Hölder's inequality.

The paper is organized in the following way: After this Introduction, in Section 2 we establish our main results, that is, refinement and converse of the Hilbert-type inequality (1.1) in the so-called quotient form. In other words, we shall find the lower and the upper bound for the quotient between the left-hand side and the right-hand side of inequality (1.1). Further, in Section 3 we apply our main results to homogeneous kernels with the negative degree of homogeneity. Also, the considerable attention is given to the investigation of the best possible constant factors involved in established Hilbert-type inequalities. Finally, in Section 4 we consider our results equipped with some particular kernels and weight functions, and compare our results with the previously known from the literature.

The techniques that will be used in the proofs are mainly based on classical real analysis, especially on Fubini's theorem and Hölder's inequality.

2. Main results

In this section we establish our main results, that is, refinement and converse of the multidimensional Hilbert-type inequality. More precisely, we shall consider the quotient between the left-hand side and the right-hand side of inequality (1.1). In such a way we shall find the lower and the upper bound for the above mentioned quotient, expressed in terms of similar quotient. The lower bound will establish the converse, while the upper bound will give the refinement of the Hilbert-type inequality (1.1). Such improvements will be referred to as the refinement and converse of the Hilbert-type inequality in the quotient form.

It is well known that the Hilbert-type inequality is derived with the help of Hölder's inequality. Our main results will be derived with the help of the existing Hilbert-type inequality (1.1) and yet another sophisticated use of Hölder's inequality.

We start with the refinement of the Hilbert-type inequality in the quotient form.

Theorem 2.1. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces and let $K\colon \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij}\colon \Omega_j \to \mathbb{R}$, $f_i\colon \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \dots, n$, be non-negative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then*

$$\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, \mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}} \le \frac{\left[\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i^{p_i/n}(x_i) \prod_{i,j=1}^{n} \phi_{ij}^{p_i/n}(x_j)\, \mathrm{d}\mu(\mathbf{x})\right]^{n/\max\limits_{1\le i\le n}\{p_i\}}}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/\max\limits_{1\le i\le n}\{p_i\}}}, \tag{2.1}$$

where $p_i > 1$ are conjugate exponents and $\omega_i\colon \Omega_i \to \mathbb{R}$ is defined by (1.2), $i = 1, 2, \dots, n$.

Proof. The left-hand side of the Hilbert-type inequality (1.1) can be rewritten in the form

$$\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\, \mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \left[\prod_{i=1}^{n} F_i^{1/q_i}(\mathbf{x})\right]^{1-n/M} \cdot \left[\prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\right]^{n/M} \mathrm{d}\mu(\mathbf{x}),$$

where the functions $F_i\colon \mathbf{\Omega} \to \mathbb{R}$ are defined by

$$F_i(\mathbf{x}) = K(\mathbf{x}) f_i^{p_i}(x_i) \prod_{j=1}^{n} \phi_{ij}^{p_i}(x_j), \qquad i = 1, 2, \dots, n, \tag{2.2}$$

$M=\max_{1\le i\le n}\{p_i\}$, and

$$q_i=\frac{p_i(M-n)}{M-p_i},\qquad i=1,2,\dots,n.$$

Clearly, the above relation is meaningful because if $M=p_l$ for some $l\in\{1,2,\dots,n\}$, then $1/q_l=0$.

Further, the application of Hölder's inequality to the above form of the left-hand side of inequality (1.1) yields inequality

$$\begin{aligned}&\int_{\mathbf{\Omega}} K(\mathbf{x})\prod_{i=1}^n f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})\\ &\le\left[\int_{\mathbf{\Omega}}\prod_{i=1}^n F_i^{1/q_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})\right]^{1-n/M}\cdot\left[\int_{\mathbf{\Omega}}\prod_{i=1}^n F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})\right]^{n/M}.\end{aligned}\tag{2.3}$$

On the other hand, by using the well-known Fubini's theorem we have

$$\begin{aligned}\|F_i^{1/t}\|_t&=\left[\int_{\mathbf{\Omega}} K(\mathbf{x})(\phi_{ii}f_i)^{p_i}(x_i)\prod_{j=1,j\ne i}^n\phi_{ij}^{p_i}(x_j)\,\mathrm{d}\mu(\mathbf{x})\right]^{1/t}\\ &=\left[\int_{\Omega_i}(\phi_{ii}f_i)^{p_i}(x_i)\left(\int_{\hat{\mathbf{\Omega}}^i} K(\mathbf{x})\prod_{j=1,j\ne i}^n\phi_{ij}^{p_i}(x_j)\,\mathrm{d}\hat{\mu}^i(\mathbf{x})\right)\mathrm{d}\mu_i(x_i)\right]^{1/t}\\ &=\left[\int_{\Omega_i}(\phi_{ii}\omega_i f_i)^{p_i}(x_i)\,\mathrm{d}\mu_i(x_i)\right]^{1/t}\\ &=\|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/t},\qquad i=1,2,\dots,n,\quad t>0,\end{aligned}\tag{2.4}$$

and the right-hand side of Hilbert-type inequality (1.1) can be rewritten in the form

$$\prod_{i=1}^n\|\phi_{ii}\omega_i f_i\|_{p_i}=\left[\prod_{i=1}^n\|F_i^{1/q_i}\|_{q_i}\right]^{1-n/M}\cdot\left[\prod_{i=1}^n\|F_i^{1/n}\|_n\right]^{n/M}.$$

Therefore, inequality (2.3) can be expressed in the following form:

$$\frac{\int\limits_{\mathbf{\Omega}} K(\mathbf{x}) \prod\limits_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod\limits_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}} \leq \left[\frac{\int\limits_{\mathbf{\Omega}} \prod\limits_{i=1}^{n} F_i^{1/q_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod\limits_{i=1}^{n} \|F_i^{1/q_i}\|_{q_i}}\right]^{1-n/M} \left[\frac{\int\limits_{\mathbf{\Omega}} \prod\limits_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod\limits_{i=1}^{n} \|F_i^{1/n}\|_{n}}\right]^{n/M}. \tag{2.5}$$

Obviously $M \geq n$. If $M > n$, then $q_i > 0$ and

$$\sum_{i=1}^{n} \frac{1}{q_i} = \sum_{i=1}^{n} \frac{M-p_i}{p_i(M-n)} = \frac{1}{M-n}\left[M\sum_{i=1}^{n}\frac{1}{p_i} - n\right] = 1,$$

that is, q_i are also conjugate exponents and Hölder's inequality yields inequality $\int\limits_{\mathbf{\Omega}} \prod\limits_{i=1}^{n} F_i^{1/q_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) \leq \prod\limits_{i=1}^{n} \|F_i^{1/q_i}\|_{q_i}$. Hence, relation (2.5) implies inequality

$$\frac{\int\limits_{\mathbf{\Omega}} K(\mathbf{x}) \prod\limits_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod\limits_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}} \leq \left[\frac{\int\limits_{\mathbf{\Omega}} \prod\limits_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod\limits_{i=1}^{n} \|F_i^{1/n}\|_{n}}\right]^{n/M},$$

which is also valid if $M = n$. Finally, by substituting the functions F_i in the last inequality, we get (2.1) as required. □

Remark 1. Bearing in mind the notations as in the proof of Theorem 2.1, we have $\int\limits_{\mathbf{\Omega}} \prod\limits_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) \leq \prod\limits_{i=1}^{n} \|F_i^{1/n}\|_{n}$, by Hölder's inequality. Therefore, the quotient on the right-hand side of inequality (2.1) is not greater than 1, which means that (2.1) represents the refinement of inequality (1.1).

In a similar way, we also obtain the lower bound for the quotient between the left-hand side and the right-hand side of inequality (1.1).

Theorem 2.2. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ-finite measure spaces and let $K\colon \mathbf{\Omega} \to \mathbb{R}$, $\phi_{ij}\colon \Omega_j \to \mathbb{R}$, $f_i\colon \Omega_i \to \mathbb{R}$, $i, j = 1, 2, \ldots, n$, be non-negative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then*

$$\begin{aligned} &\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}} \\ &\geq \frac{\Big[\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i^{p_i/n}(x_i) \prod_{i,j=1}^{n} \phi_{ij}^{p_i/n}(x_j)\,\mathrm{d}\mu(\mathbf{x})\Big]^{n/\min\limits_{1\le i\le n}\{p_i\}}}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}^{p_i/\min\limits_{1\le i\le n}\{p_i\}}}, \end{aligned} \tag{2.6}$$

where $p_i > 1$ are conjugate exponents and $\omega_i\colon \Omega_i \to \mathbb{R}$ is defined by (1.2), $i = 1, 2, \ldots, n$.

Proof. The starting point in obtaining (2.6) is the relation

$$\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) = \int_{\mathbf{\Omega}} \left[K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\right]^{m/n} \left[\prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})\right]^{1-m/n} \mathrm{d}\mu(\mathbf{x}),$$

where the functions $F_i\colon \mathbf{\Omega} \to \mathbb{R}$ are defined by (2.2), $m = \min\limits_{1\le i\le n}\{p_i\}$, and

$$r_i = \frac{p_i(n-m)}{p_i - m}, \qquad i = 1, 2, \ldots, n.$$

If $m = p_l$ for some $l \in \{1, 2, \ldots, n\}$, then $1/r_l = 0$, which means that the above decomposition is meaningful.

Now, the application of Hölder's inequality yields relation

$$\begin{aligned} &\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}) \\ &\leq \left[\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})\right]^{m/n} \left[\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})\right]^{1-m/n}. \end{aligned} \tag{2.7}$$

On the other hand, regarding relation (2.4), we have

$$\prod_{i=1}^{n} \|F_i^{1/n}\|_n = \left[\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}\right]^{m/n} \cdot \left[\prod_{i=1}^{n} \|F_i^{1/r_i}\|_{r_i}\right]^{1-m/n}.$$

Now, if we divide inequality (2.7) with the previous relation, we get inequality

$$\begin{aligned}
&\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|F_i^{1/n}\|_n} \\
&\leq \left[\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}}\right]^{m/n} \left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|F_i^{1/r_i}\|_{r_i}}\right]^{1-m/n}.
\end{aligned} \tag{2.8}$$

Obviously $m \leq n$. If $m < n$, then $r_i > 0$ and

$$\sum_{i=1}^{n} \frac{1}{r_i} = \sum_{i=1}^{n} \frac{p_i - m}{p_i(n-m)} = \frac{1}{n-m}\left[n - m\sum_{i=1}^{n} \frac{1}{p_i}\right] = 1,$$

that is, r_i are conjugate exponents. Hence, yet another application of Hölder's inequality implies that

$$\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/r_i}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|F_i^{1/r_i}\|_{r_i}} \leq 1.$$

Therefore, inequality (2.8) yields

$$\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|F_i^{1/n}\|_n} \leq \left[\frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}}\right]^{m/n},$$

that is,

$$\left[\frac{\int_{\mathbf{\Omega}} \prod_{i=1}^{n} F_i^{1/n}(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|F_i^{1/n}\|_n}\right]^{n/m} \leq \frac{\int_{\mathbf{\Omega}} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mu(\mathbf{x})}{\prod_{i=1}^{n} \|\phi_{ii}\omega_i f_i\|_{p_i}}.$$

Note also that the last inequality also holds for $m = n$. Finally, by substituting the functions F_i, defined by (2.2), in the last inequality, we get (2.6) as required. $\square$

Remark 2. Obviously, inequality (2.6) provides the converse of the Hilbert-type inequality (1.1) in the quotient form.

3. Homogeneous kernels and the best possible constant factors

In this section we apply our general results to homogeneous functions with the negative degree of homogeneity. Further, regarding the notations from the previous section, we assume that $\Omega_i = \mathbb{R}_+$, equipped with the non-negative Lebesgue measures $\mathrm{d}\mu_i(x_i) = \mathrm{d}x_i$, $i = 1, 2, \dots, n$. In addition, we have $\boldsymbol{\Omega} = \mathbb{R}_+^n$ and $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\,\mathrm{d}x_2 \dots \mathrm{d}x_n$.

We introduce the real parameters A_{ij}, $i, j = 1, 2, \dots, n$, such that $\sum_{i=1}^{n} A_{ij} = 0$, $j = 1, 2, \dots, n$, and denote $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $i = 1, 2, \dots, n$. Next, we consider the set of power functions $\phi_{ij} \colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\phi_{ij}(x_j) = {x_j}^{A_{ij}}. \tag{3.1}$$

Clearly, the set of the above defined power functions satisfy the condition

$$\prod_{i,j=1}^{n} \phi_{ij}(x_j) = \prod_{j=1}^{n}\prod_{i=1}^{n} {x_j}^{A_{ij}} = \prod_{j=1}^{n} {x_j}^{\sum_{i=1}^{n} A_{ij}} = 1,$$

since $\sum_{i=1}^{n} A_{ij} = 0$. Therefore, the functions ϕ_{ij}, $i, j = 1, 2, \dots, n$, satisfy the conditions as in Theorems 2.1 and 2.2.

Recall that the function $K \colon \mathbb{R}_+^n \to \mathbb{R}$ is said to be homogeneous of degree $-s$, $s > 0$, if $K(t\mathbf{x}) = t^{-s} K(\mathbf{x})$ for all $t > 0$. Furthermore, for $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1, j\neq i}^{n} u_j^{a_j}\, \hat{\mathrm{d}}^i\mathbf{u}, \qquad i = 1, 2, \dots, n, \tag{3.2}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$, $\hat{\mathrm{d}}^i\mathbf{u} = \mathrm{d}u_1 \dots \mathrm{d}u_{i-1}\,\mathrm{d}u_{i+1} \dots \mathrm{d}u_n$, and provided that the above integral converges. Note that the constant factor $k_i(\mathbf{a})$ does not depend on the component a_i. Thus, the component a_i can be replaced with an arbitrary real number. This fact will sometimes be used in the sequel, for the reason of simpler notation.

Further, in the described setting we can find the explicit formula for the weight function (1.2) including the constant factor $k_i(\mathbf{a})$. More precisely, we use the substitution $x_j = u_j x_i$, $j \neq i$, that is, $\hat{\mathrm{d}}^i\mathbf{x} = x_i^{n-1}\hat{\mathrm{d}}^i\mathbf{u}$, while the homogeneity of the kernel K yields relation $K(\mathbf{x}) = x_i^{-s} K(\hat{\mathbf{u}}^i)$. Moreover, regarding definition

(3.2) we have

$$\begin{aligned}\omega_i(x_i) &= \left[\int_{\mathbb{R}_+^{n-1}} K(\mathbf{x}) \prod_{j=1, j\neq i}^{n} x_j^{\,p_i A_{ij}}\,\hat{\mathrm{d}}^i\mathbf{x}\right]^{1/p_i} \\ &= \left[x_i^{\,n-1-s+\sum_{j=1,j\neq i}^{n} p_i A_{ij}} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^i) \prod_{j=1,j\neq i}^{n} u_j^{p_i A_{ij}}\,\hat{\mathrm{d}}^i\mathbf{u}\right]^{1/p_i} \\ &= x_i^{(n-1-s)/p_i+\alpha_i-A_{ii}} k_i^{1/p_i}(p_i\mathbf{A_i}),\end{aligned} \tag{3.3}$$

where $\mathbf{A_i} = (A_{i1}, A_{i2}, \ldots, A_{in})$, $i = 1, 2, \ldots, n$.

Our next result is a simple consequence of Theorems 2.1 and 2.2 in the described setting with homogeneous kernels. Of course, inequalities (2.1) and (2.6) can be interpreted as the interpolating series of inequalities for the quotient between the left-hand side and the right-hand side of inequality (1.1).

Corollary 3.1. *Let $p_i > 1$, $i = 1, 2, \ldots, n$, be conjugate exponents and let A_{ij}, $i, j = 1, 2, \ldots, n$, be the real parameters such that $\sum_{i=1}^{n} A_{ij} = 0$, $j = 1, 2, \ldots, n$. If $K\colon \mathbb{R}_+^n \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, and $f_i\colon \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \ldots, n$, are non-negative measurable functions, then*

$$\begin{aligned} &\frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\sum_{j=1}^{n} p_j A_{ji}/n} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/\min_{1\le i\le n}\{p_i\}}}{\left[\prod_{i=1}^{n} k_i(p_i\mathbf{A_i})\right]^{1/\min_{1\le i\le n}\{p_i\}} \prod_{i=1}^{n} \|x_i^{(n-1-s)/p_i+\alpha_i} f_i\|_{p_i}^{p_i/\min_{1\le i\le n}\{p_i\}}} \\ &\le \frac{\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mathbf{x}}{\prod_{i=1}^{n} k_i^{1/p_i}(p_i\mathbf{A_i}) \prod_{i=1}^{n} \|x_i^{(n-1-s)/p_i+\alpha_i} f_i\|_{p_i}} \\ &\le \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\sum_{j=1}^{n} p_j A_{ji}/n} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/\max_{1\le i\le n}\{p_i\}}}{\left[\prod_{i=1}^{n} k_i(p_i\mathbf{A_i})\right]^{1/\max_{1\le i\le n}\{p_i\}} \prod_{i=1}^{n} \|x_i^{(n-1-s)/p_i+\alpha_i} f_i\|_{p_i}^{p_i/\max_{1\le i\le n}\{p_i\}}}, \end{aligned} \tag{3.4}$$

where $\alpha_i = \sum_{j=1}^{n} A_{ij}$, $i = 1, 2, \ldots, n$, and $k_i(\cdot)$, $i = 1, 2, \ldots, n$, is defined by (3.2).

P r o o f. The proof is a direct consequence of Theorems 2.1 and 2.2. Namely, if we substitute the functions ϕ_{ij} and ω_i, $i,j=1,2,\dots,n$, defined respectively by (3.1) and (3.3), in relations (2.1) and (2.6), we get the series of inequalities (3.4) after straightforward computation. □

Remark 3. The left-hand side inequality in (3.4) yields the converse, while the right-hand side inequality provides the refinement of the general Hilbert-type inequality from paper [8]. Moreover, by using the change of variables $x_i = u_i(t_i)$, where $u_i\colon \langle a_i, b_i\rangle \to \mathbb{R}$ are strictly increasing differentiable functions satisfying $u_i(a_i) = 0$, $u_i(b_i) = \infty$, the interpolating series (3.4) also yields refinement and converse of the corresponding multidimensional Hilbert-type inequality from paper [9].

In papers [1], [2], [5], [9], the authors investigated the conditions under which the constant factors involved in appropriate Hilbert-type inequalities are the best possible in the sense that they can not be replaced with the smaller constants.

In the sequel we consider the problem of the best possible constant factors involved in the interpolating series of inequalities (3.4). By the similar reasoning as in the above mentioned papers with the same problem area, the best possible constant factors can be obtained if they don't contain conjugate parameters p_i in the exponents. For that reason, we assume

$$k_1(p_1\mathbf{A_1}) = k_2(p_2\mathbf{A_2}) = \cdots = k_n(p_n\mathbf{A_n}). \tag{3.5}$$

If we use the change of variables $u_1 = 1/t_2$, $u_3 = t_3/t_2$, $u_4 = t_4/t_2$, $\dots$, $u_n = t_n/t_2$, which provides the Jacobian of the transformation

$$\left|\frac{\partial(u_1,u_3,\dots,u_n)}{\partial(t_2,t_3,\dots,t_n)}\right| = t_2^{-n},$$

we have

$$\begin{aligned} k_2(p_2\mathbf{A_2}) &= \int\limits_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{t}}^1) t_2^{s-n-p_2(\alpha_2-A_{22})} \prod_{j=3}^{n} t_j^{p_2A_{2j}}\, \hat{\mathrm{d}}^1\mathbf{t} \\ &= k_1\big(p_1A_{11}, s-n-p_2(\alpha_2-A_{22}), p_2A_{23},\dots,p_2A_{2n}\big). \end{aligned}$$

According to (3.5), we have $p_1A_{12} = s-n-p_2(\alpha_2-A_{22})$, $p_1A_{13} = p_2A_{23}$, $\dots$, $p_1A_{1n} = p_2A_{2n}$. In a similar manner we express $k_i(p_i\mathbf{A_i})$, $i=3,\dots,n$, in terms of $k_1(\cdot)$. In such a way we see that (3.5) is fulfilled if

$$p_jA_{ji} = s-n-p_i(\alpha_i - A_{ii}), \qquad i,j=1,2,\dots,n, \ \ i\neq j. \tag{3.6}$$

The above set of conditions also implies that $p_iA_{ik} = p_jA_{jk}$, when $k\neq i,j$. Hence, we use abbreviations $\widetilde{A}_1 = p_nA_{n1}$ and $\widetilde{A}_i = p_1A_{1i}$, $i\neq 1$. Since

$\sum\limits_{i=1}^{n} A_{ij} = 0$, one easily obtains that $p_j A_{jj} = \widetilde{A}_j(1-p_j)$ and $\sum\limits_{i=1}^{n} \widetilde{A}_i = s-n$ (see also paper [9]).

In order to obtain the best possible constant factors, we establish some more specific conditions about the convergence of the integral $k_1(\mathbf{a})$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, defined by (3.2). More precisely, we assume that $k_1(\mathbf{a}) < \infty$ for $a_2, \ldots, a_n > -1$, $\sum\limits_{i=2}^{n} a_i < s-n+1$, and $n \in \mathbb{N}$.

Hence, in the described setting, the interpolating series of inequalities (3.4) can be rewritten as

$$\begin{aligned}
&k_1^{1-n/m}(\widetilde{\mathbf{A}}) \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/m}}{\prod_{i=1}^{n} \|x_i^{-\widetilde{A}_i-1/p_i} f_i\|_{p_i}^{p_i/m}} \\
&\le \frac{\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mathbf{x}}{\prod_{i=1}^{n} \|x_i^{-\widetilde{A}_i-1/p_i} f_i\|_{p_i}} \\
&\le k_1^{1-n/M}(\widetilde{\mathbf{A}}) \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/M}}{\prod_{i=1}^{n} \|x_i^{-\widetilde{A}_i-1/p_i} f_i\|_{p_i}^{p_i/M}},
\end{aligned} \tag{3.7}$$

where $m = \min\limits_{1\le i\le n}\{p_i\}$, $M = \max\limits_{1\le i\le n}\{p_i\}$, and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$.

In the sequel, we show that the constant factors involved in the series of inequalities (3.7) are the best possible under certain assumptions on the homogeneous kernel.

Theorem 3.1. *Let $K\colon \mathbb{R}_+^n \to \mathbb{R}$ be a non-negative measurable homogeneous function of degree $-s$, $s > 0$, such that for every $i = 2, 3, \ldots, n$,*

$$K(1, t_2, \ldots, t_i, \ldots, t_n) \le CK(1, t_2, \ldots, 0, \ldots, t_n), \qquad 0 \le t_i \le 1, \tag{3.8}$$

where C is a positive constant. Then, the constant factors $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ and $k_1^{1-n/m}(\widetilde{\mathbf{A}})$ are the best possible in the series of inequalities (3.7).

Proof. Suppose $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ is not the best possible constant factor in (3.7), that is, there exist a positive constant $\alpha < k_1^{1-n/M}(\widetilde{\mathbf{A}})$ such that the right-hand side inequality in (3.7) holds if we replace $k_1^{1-n/M}(\widetilde{\mathbf{A}})$ with α. In other words,

$$\begin{aligned}
&\frac{\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)\, \mathrm{d}\mathbf{x}}{\prod_{i=1}^n \|x_i^{-\widetilde{A}_i-1/p_i} f_i\|_{p_i}} \\
&\le \alpha \frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)\, \mathrm{d}\mathbf{x}\right]^{n/M}}{\prod_{i=1}^n \|x_i^{-\widetilde{A}_i-1/p_i} f_i\|_{p_i}^{p_i/M}}
\end{aligned} \tag{3.9}$$

holds for all non-negative measurable functions $f_i\colon \mathbb{R}_+ \to \mathbb{R}$, provided that all the integrals in the inequality converge. For this purpose, let's substitute the functions

$$\widetilde{f}_i(x_i) = \begin{cases} 0, & 0 < x < 1, \\ x_i^{\widetilde{A}_i - \varepsilon/p_i}, & x \ge 1, \end{cases} \tag{3.10}$$

where $0 < \varepsilon < \min_{1\le i\le n} \{p_i + p_i \widetilde{A}_i\}$, in the previous inequality.

Since $\|x_i^{-\widetilde{A}_i-1/p_i} \widetilde{f}_i\|_{p_i} = \|x_i^{-(1+\varepsilon)/p_i}\|_{p_i} = \varepsilon^{-1/p_i}$, the left-hand side of inequality (3.9) becomes

$$I = \varepsilon \int\limits_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \varepsilon/p_i}\, \mathrm{d}\mathbf{x},$$

while the right-hand side becomes

$$I_M = \alpha \left[\varepsilon \int\limits_{[1,\infty\rangle^n} K(\mathbf{x}) \prod_{i=1}^n x_i^{\widetilde{A}_i - \varepsilon/n}\, \mathrm{d}\mathbf{x}\right]^{n/M}.$$

Obviously, by using the variable changes $u_i = x_i/x_1$, $i = 2, \dots, n$, and the homogeneity of the kernel K, the left-hand side I can be rewritten as

$$I = \varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\int\limits_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \hat{\mathrm{d}}^1\mathbf{u}\right] \mathrm{d}x_1,$$

providing the inequality

$$\begin{aligned} I &\geq \varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\widetilde{A}_i - \varepsilon/p_i} \, \hat{\mathrm{d}}^1 \mathbf{u} \right] \mathrm{d}x_1 \\ &\quad - \varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u} \right] \mathrm{d}x_1 \\ &= k_1 \left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p} \right) - \varepsilon \int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u} \right] \mathrm{d}x_1, \end{aligned} \tag{3.11}$$

where $\mathbb{D}_i = \left\{ (u_2, u_3, \ldots, u_n) : \;\; 0 < u_i \leq 1/x_1, \;\; u_j > 0, \;\; j \neq i \right\}$ and $\mathbf{1}/\mathbf{p} = (1/p_1, \ldots, 1/p_n)$. Without loss of generality, it is enough to find the upper bound for the integral $\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u}$. Regarding (3.8), we have

$$\begin{aligned} &\int_{\mathbb{D}_2} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u} \\ &\leq C \left[\int_{\mathbb{R}_+^{n-2}} K(1, 0, u_3, \ldots, u_n) \prod_{j=3}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \mathrm{d}u_3 \ldots \mathrm{d}u_n \right] \int_0^{1/x_1} u_2^{\widetilde{A}_2 - \varepsilon/p_2} \, \mathrm{d}u_2 \\ &= C(1 - \varepsilon/p_2 + \widetilde{A}_2)^{-1} x_1^{\varepsilon/p_2 - \widetilde{A}_2 - 1} k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \ldots, \widetilde{A}_n - \varepsilon/p_n), \end{aligned}$$

where $k_1(\widetilde{A}_1 - \varepsilon/p_1, \widetilde{A}_3 - \varepsilon/p_3, \ldots, \widetilde{A}_n - \varepsilon/p_n)$ is well defined since obviously $\sum_{i=3}^n \widetilde{A}_i < s - n + 2$. Hence, we have

$$\int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u} = x_1^{\varepsilon/p_i - \widetilde{A}_i - 1} O(1), \qquad i = 2, 3, \ldots, n,$$

and consequently

$$\int_1^\infty x_1^{-1-\varepsilon} \left[\sum_{i=2}^n \int_{\mathbb{D}_i} K(\hat{\mathbf{u}}^1) \prod_{j=2}^n u_j^{\widetilde{A}_j - \varepsilon/p_j} \, \hat{\mathrm{d}}^1 \mathbf{u} \right] \mathrm{d}x_1 = O(1).$$

Thus, by using (3.11), we have

$$I \geq k_1 \left(\widetilde{\mathbf{A}} - \varepsilon \mathbf{1}/\mathbf{p} \right) - o(1), \qquad \text{when} \quad \varepsilon \to 0^+. \tag{3.12}$$

On the other hand, by using the fact that $\sum_{i=1}^{n} \widetilde{A}_i = s-n$, the expression I_M can be bounded from above in the following way:

$$\begin{aligned} I_M &= \alpha\left[\varepsilon\int_1^{\infty} x_1^{-1-\varepsilon}\left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i-\varepsilon/n}\,\hat{\mathrm{d}}^1\mathbf{u}\right]\mathrm{d}x_1\right]^{n/M} \\ &\le \alpha\left[\varepsilon\int_1^{\infty} x_1^{-1-\varepsilon}\left[\int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i-\varepsilon/n}\,\hat{\mathrm{d}}^1\mathbf{u}\right]\mathrm{d}x_1\right]^{n/M} \\ &= \alpha k_1^{n/M}\left(\widetilde{\mathbf{A}}-\varepsilon/n\mathbf{1}\right), \qquad \text{when} \quad \varepsilon\to 0^+. \end{aligned} \tag{3.13}$$

Here, $\mathbf{1}$ denotes the constant n-tuple $(1,1,\ldots,1)$. Finally, relations (3.12) and (3.13) yield inequality

$$k_1\left(\widetilde{\mathbf{A}}-\varepsilon\mathbf{1}/\mathbf{p}\right)-o(1)\le \alpha k_1^{n/M}\left(\widetilde{\mathbf{A}}-\varepsilon/n\mathbf{1}\right), \qquad \text{when} \quad \varepsilon\to 0^+,$$

i.e., $k_1^{1-n/M}(\widetilde{\mathbf{A}})\le\alpha$, which is obviously opposite to our assumption.

It remains to prove that $k_1^{1-n/m}(\widetilde{\mathbf{A}})$ is the best possible constant factor in the left-hand side inequality in (3.7). Suppose, on the contrary, that there exist a constant $\beta > k_1^{1-n/m}(\widetilde{\mathbf{A}})$, such that the inequality

$$\begin{aligned} &\frac{\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} f_i(x_i)\,\mathrm{d}\mathbf{x}}{\prod_{i=1}^{n}\|x_i^{-\widetilde{A}_i-1/p_i}f_i\|_{p_i}} \\ &\ge \beta\frac{\left[\int_{\mathbb{R}_+^n} K(\mathbf{x})\prod_{i=1}^{n} x_i^{\widetilde{A}_i(1-p_i/n)} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/m}}{\prod_{i=1}^{n}\|x_i^{-\widetilde{A}_i-1/p_i}f_i\|_{p_i}^{p_i/m}} \end{aligned} \tag{3.14}$$

holds for all non-negative measurable functions $f_i\colon \mathbb{R}_+\to\mathbb{R}$, provided that all the integrals in the inequality converge. For the above choice of functions $\widetilde{f}_i$ defined by (3.10), the left-hand side of inequality (3.14) becomes I as before, while the right-hand side, denoted here by I_m, can be rewritten as

$$I_m = \beta\left[\varepsilon\int_1^{\infty} x_1^{-1-\varepsilon}\left[\int_{[1/x_1,\infty\rangle^{n-1}} K(\hat{\mathbf{u}}^1)\prod_{i=2}^{n} u_i^{\widetilde{A}_i-\varepsilon/n}\,\hat{\mathrm{d}}^1\mathbf{u}\right]\mathrm{d}x_1\right]^{n/m}.$$

Now, similarly as in the first part of the proof, we get the estimates

$$\begin{aligned} I &\le k_1\left(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p}\right) \\ I_m &\ge \beta k_1^{\frac{n}{m}}\left(\widetilde{\mathbf{A}} - \varepsilon/n\mathbf{1}\right) - o(1), \end{aligned} \tag{3.15}$$

i.e., $k_1(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p}) \ge \beta k_1^{\frac{n}{m}}(\widetilde{\mathbf{A}} - \varepsilon/n\mathbf{1}) - o(1)$, when $\varepsilon \to 0^+$. Finally, by letting $\varepsilon \to 0^+$ we get $k_1^{1-n/m}(\widetilde{\mathbf{A}}) \ge \beta$, which is a contradiction. The proof is now completed. $\square$

4. Two examples and concluding remarks

This section is devoted to the results from previous section in some particular settings. In such a way we shall obtain the refinements and converses of some previously known results from the literature.

First example

Let

$$A_{ii} = \frac{(n-s)(p_i-1)}{p_i^2} \quad \text{and} \quad A_{ij} = \frac{s-n}{p_ip_j}, \qquad i,j = 1,2,\dots,n, \ \ i \ne j. \tag{4.1}$$

These parameters are symmetric and

$$\sum_{i=1}^n A_{ij} = \sum_{j=1}^n A_{ij} = \frac{(n-s)(p_i-1)}{p_i^2} + \sum_{j=1, j\ne i}^n \frac{s-n}{p_ip_j} = \frac{n-s}{p_i}\left(1 - \sum_{j=1}^n \frac{1}{p_j}\right) = 0.$$

Moreover, the above defined parameters satisfy conditions (3.6), so the resulting relations will include the best possible constant factors. More precisely, in the described setting, the interpolating series of inequalities (3.7) reads

$$\begin{aligned} &\left[k_1((s-n)\mathbf{1}/\mathbf{p})\right]^{1-n/m} \frac{\left[\int\limits_{\mathbb{R}_+^n} K(\mathbf{x}) \prod\limits_{i=1}^n x_i^{(s-n)(n-p_i)/(np_i)} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/m}}{\prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i} f_i\|_{p_i}^{p_i/m}} \\ &\le \frac{\int\limits_{\mathbb{R}_+^n} K(\mathbf{x}) \prod\limits_{i=1}^n f_i(x_i)\,\mathrm{d}\mathbf{x}}{\prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i} f_i\|_{p_i}} \\ &\le \left[k_1((s-n)\mathbf{1}/\mathbf{p})\right]^{1-n/M} \frac{\left[\int\limits_{\mathbb{R}_+^n} K(\mathbf{x}) \prod\limits_{i=1}^n x_i^{(s-n)(n-p_i)/(np_i)} f_i^{p_i/n}(x_i)\,\mathrm{d}\mathbf{x}\right]^{n/M}}{\prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i} f_i\|_{p_i}^{p_i/M}}, \end{aligned} \tag{4.2}$$

where $m = \min_{1\le i\le n}\{p_i\}$, $M = \max_{1\le i\le n}\{p_i\}$. However, under assumption (3.8), the constant factors $\left[k_1((s-n)\mathbf{1}/\mathbf{p})\right]^{1-n/m}$ and $\left[k_1((s-n)\mathbf{1}/\mathbf{p})\right]^{1-n/M}$ are the best possible in the interpolating series (4.2).

A typical example of a homogeneous kernel with the negative degree of homogeneity is the function $K\colon \mathbb{R}_+^n \to \mathbb{R}$, defined by

$$K(x) = \frac{1}{\left(\sum_{i=1}^n x_i\right)^s}, \qquad s > 0. \tag{4.3}$$

Clearly, K is a homogeneous function of degree $-s$ and the constant (3.2) can be expressed in terms of the usual Gamma function Γ. For that reason, we use the well-known formula

$$\int_{\mathbb{R}_+^n} \frac{\prod_{i=1}^{n-1} u_i^{a_i-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n a_i}}\, \hat{\mathrm{d}}^n\mathbf{u} = \frac{\prod_{i=1}^n \Gamma(a_i)}{\Gamma\left(\sum_{i=1}^n a_i\right)}, \tag{4.4}$$

which holds for $a_i > 0$, $i = 1, 2, \ldots n$ (see, e.g. [2]). In such a way, the constant factors $k_i(p_i\mathbf{A_i})$, $i = 1, 2, \ldots, n$, involved in the series of inequalities (3.4) become

$$k_i(p_i\mathbf{A_i}) = \frac{\Gamma\left(s-n+1-p_i\alpha_i+p_iA_{ii}\right)}{\Gamma(s)} \prod_{j=1, j\ne i}^n \Gamma(1+p_iA_{ij}), \qquad i = 1, 2, \ldots, n,$$

provided that $A_{ij} > -1/p_i$, $i \ne j$, and $A_{ii} - \alpha_i > (n-s-1)/p_i$.

It is easy to see that the kernel (4.3) satisfies the relation (3.8). Hence, according to Theorem 3.1, the interpolating series of inequalities (3.4), equipped with the kernel (4.3) and the parameters A_{ij} satisfying conditions (3.6), contains the best possible constant factors.

Remark 4. If $K\colon \mathbb{R}_+^n \to \mathbb{R}$ is defined by (4.3), then, regarding (4.4), we easily compute the constant factor $k_1((s-n)\mathbf{1}/\mathbf{p})$ included in the interpolating series of inequalities in (4.2). Namely, we have

$$k_1((s-n)\mathbf{1}/\mathbf{p}) = \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma\left(\frac{p_i+s-n}{p_i}\right),$$

provided that $s > n - m$. This constant factor appears in paper [2] as the best possible in the Hilbert-type inequality determined with the middle quotient in the interpolating series (4.2). Hence, relations as in (4.2) represent the refinement and converse of the corresponding Hilbert-type inequality from paper [2].

Second example

We conclude this paper with yet another interesting example. Suppose A_i, $i = 1, 2, \dots, n$, are the real parameters satisfying relations $(n - s - 1)/p_{i-1} < A_i < 1/p_{i-1}$, provided that $s > n - 2$. Of course, we use convention $p_0 = p_n$. Now, we define parameters A_{ij}, $i, j = 1, 2, \dots, n$, by

$$A_{ij} = \begin{cases} A_i, & j = i, \\ -A_{i+1}, & j = i+1, \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

where the indices are taken modulo n from the set $\{1, 2, \dots, n\}$. Now, if the kernel $K\colon \mathbb{R}_+^n \to \mathbb{R}$ is defined by (4.3), then the series of inequalities (3.4) becomes

$$\begin{aligned} &\frac{\left\{ \int\limits_{\mathbb{R}_+^n} \big(\sum\limits_{i=1}^n x_i\big)^{-s} \prod\limits_{i=1}^n x_i^{(p_i - p_{i-1})A_i/n} f_i^{p_i/n}(x_i)\, \mathrm{d}\mathbf{x} \right\}^{n/\min\limits_{1\le i\le n}\{p_i\}}}{\mathcal{R}_m \prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i + A_i - A_{i+1}} f_i\|_{p_i}^{p_i/\min\limits_{1\le i\le n}\{p_i\}}} \\ &\le \frac{\int\limits_{\mathbb{R}_+^n} \big(\sum\limits_{i=1}^n x_i\big)^{-s} \prod\limits_{i=1}^n f_i(x_i)\, \mathrm{d}\mathbf{x}}{\mathcal{R} \prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i + A_i - A_{i+1}} f_i\|_{p_i}} \\ &\le \frac{\left\{ \int\limits_{\mathbb{R}_+^n} \big(\sum\limits_{i=1}^n x_i\big)^{-s} \prod\limits_{i=1}^n x_i^{(p_i - p_{i-1})A_i/n} f_i^{p_i/n}(x_i)\, \mathrm{d}\mathbf{x} \right\}^{n/\max\limits_{1\le i\le n}\{p_i\}}}{\mathcal{R}_M \prod\limits_{i=1}^n \|x_i^{(n-1-s)/p_i + A_i - A_{i+1}} f_i\|_{p_i}^{p_i/\max\limits_{1\le i\le n}\{p_i\}}}, \end{aligned} \tag{4.6}$$

where the constant factors $\mathcal{R}_m$, $\mathcal{R}$, and $\mathcal{R}_M$ are given by

$$\mathcal{R}_m = \left[\frac{\prod\limits_{i=1}^n \Gamma\left(s - n + 1 + p_i A_{i+1}\right) \Gamma(1 - p_i A_{i+1})}{\Gamma(s)} \right]^{1/\min\limits_{1\le i\le n}\{p_i\}}$$

$$\mathcal{R} = \frac{\prod\limits_{i=1}^n \Gamma\left(s - n + 1 + p_i A_{i+1}\right)^{1/p_i} \Gamma(1 - p_i A_{i+1})^{1/p_i}}{\Gamma(s)}$$

$$\mathcal{R}_M = \left[\frac{\prod\limits_{i=1}^n \Gamma\left(s - n + 1 + p_i A_{i+1}\right) \Gamma(1 - p_i A_{i+1})}{\Gamma(s)} \right]^{1/\max\limits_{1\le i\le n}\{p_i\}}.$$

Remark 5. The interpolating series of inequalities (4.6) provides the refinement and converse of the multidimensional Hilbert-type inequality from paper [3] (see also paper [6]). Moreover, the parameters A_{ij} defined by (4.5), can satisfy the set of conditions as in (3.6) only for $n = 2$. In this case, the set of conditions (3.6) reduces to the relation $p_1A_2 + p_2A_1 = 2 - s$, providing the best possible constant factors

$$\left[\frac{\Gamma(1-p_1A_2)\Gamma(1-p_2A_1)}{\Gamma(s)}\right]^{1-2/\min\{p_1,p_2\}}$$

and

$$\left[\frac{\Gamma(1-p_1A_2)\Gamma(1-p_2A_1)}{\Gamma(s)}\right]^{1-2/\max\{p_1,p_2\}}$$

in (4.6) for $n = 2$ (see also paper [5]).

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University of Zagreb
Faculty of Electrical Engineering
and Computing
Unska 3
HR–10000 Zagreb
CROATIA
E-mail: mario.krnic@fer.hr