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ON LEFT REGULAR AND INTRA-REGULAR ORDERED SEMIGROUPS

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ABSTRACT. We study the decomposition of left regular ordered semigroups into left regular components and the decomposition of intra-regular ordered semigroups into simple or intra-regular components, adding some additional information to the results considered in [KEHAYOPULU, N.: *On left regular ordered semigroups*, Math. Japon. **35** (1990), 1057–1060] and [KEHAYOPULU, N.: *On intra-regular ordered semigroups*, Semigroup Forum **46** (1993), 271–278]. We prove that an ordered semigroup S is left regular if and only if it is a semilattice (or a complete semilattice) of left regular semigroups, equivalently, it is a union of left regular subsemigroups of S. Moreover, S is left regular if and only if it is a union of pairwise disjoint left regular subsemigroups of S. The right analog also holds. The same result is true if we replace the words "left regular" by "intra-regular". Moreover, an ordered semigroup is intra-regular if and only if it is a semilattice (or a complete semilattice) of simple semigroups. On the other hand, if an ordered semigroup is a semilattice (or a complete semilattice) of left simple semigroups, then it is left regular, but the converse statement does not hold in general. Illustrative examples are given.

1. Introduction and prerequisites

Decomposition of left regular semigroups into left simple components and of intra-regular semigroups into simple components can be found in [1]. A *poe*-semigroup is an ordered semigroup (*po*-semigroup) S having a greatest element usually denoted by e (i.e. $e \geq a$ for all $a \in S$). We have seen in [2] that a *poe*-semigroup S is left regular if and only if it is a union of left simple

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subsemigroups of S. We have seen in [4] that an ordered semigroup S is intra-regular if and only if it is a semilattice of simple semigroups, equivalently, if S is a union of simple subsemigroups of S. In the present paper we first prove that if an ordered semigroup is a complete semilattice of left simple semigroups, then it is left regular, but the converse statement does not hold in general. We show that an ordered semigroup is left regular (resp. intra-regular) if and only if it is decomposable into left regular (resp. intra-regular) components. Some additional information are also obtained, adding additional characterizations of left regular and intra-regular ordered semigroups studied in [2] and [4].

An ordered semigroup S is called left duo if the left ideals of S are two-sided. Decomposition of ordered semigroups which are both left regular and left duo into simple components has been given in [5]. We have proved, among others, in [5] that an ordered semigroup S is left regular and left duo if and only if S is a semilattice of left simple semigroups (that is, there exists a semilattice congruence σ on S such that $(x)_\sigma$ is a left simple subsemigroup of S for every $x \in S$. We have also proved in [5] that S is left regular and left duo if and only if there exists a complete semilattice congruence ρ on S (different from σ, in general), such that $(x)_\rho$ is a left simple subsemigroup of S for every $x \in S$. In the present paper we characterize the left regular (not left duo, in general) ordered semigroups as semilattices (also complete semilattices) of left regular semigroups and the intra-regular ordered semigroups as semilattices or complete semilattices of intra-regular, as well as simple semigroups.

A *poe*-semigroup S is called *left regular* if $a \le ea^2$ for every $a \in S$ and this is equivalent to saying that for each $a \in S$ there exists $x \in S$ such that $a \le xa^2$. Let $(S, \cdot, \le)$ be an ordered semigroup. A subset T of S is called *semiprime* if for each $a \in S$ such that $a^2 \in T$, we have $a \in T$. For a subsemigroup T of S and a subset H of T, we denote by $(H]_T$ the subset of T defined by

$$(H]_T := \{t \in T \mid t \le h \text{ for some } h \in H\}.$$

In particular, for $T = S$, we write $(H]$ instead of $(H]_S$. So, for $T = S$, we have

$$(H] := \{t \in S \mid t \le h \text{ for some } h \in H\}.$$

A subsemigroup T of S is called *left regular* if for every $a \in T$ there exists $x \in T$ such that $a \le xa^2$, that is, if $a \in (Ta^2]_T$ for every $a \in T$. It is called *right regular* if for every $a \in T$ there exists $x \in T$ such that $a \le a^2x$, that is, if $a \in (a^2T]_T$ for every $a \in T$. A subsemigroup T of S is called *intra-regular* if for every $a \in T$ there exist $x, y \in T$ such that $a \le xa^2y$, that is, if $a \in (Ta^2T]_T$ for every $a \in T$. A nonempty subset T of S is called a *left ideal* (resp. *right ideal*) of S if

(1) $ST \subseteq T$ (resp. $TS \subseteq T$) and
(2) if $a \in T$ and $S \ni b \le a$, then $b \in T$ (that is, $(T] = T$).

It is called an *ideal* of S is it is both a left and right ideal of S. For an element a of S, we denote by $L(a)$ (resp. $R(a)$) the left (resp. right) ideal of S generated by a and by $I(a)$ the ideal of S generated by a. We have $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup aSa]$ for every $a \in S$. A subsemigroup T of S is called *left simple* (resp. *right simple*), if T is the only left ideal (resp. right ideal) of T; it is called *simple* if it is the only ideal of T. A subsemigroup T of S is left simple if and only if $(Ta]_T = T$ for every $a \in T$, that is, for every $a, b \in T$ there exists $x \in T$ such that $b \le xa$. T is right simple if and only if $(aT]_T = T$ for every $a \in T$, T is simple if and only if $(TaT]_T = T$ for every $a \in T$. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. If σ is a semilattice congruence on S, then $(x)_\sigma$ is a subsemigroup of S for every $x \in S$. A semilattice congruence σ on S is called *complete* if $a \le b$ implies $(a, ab) \in \sigma$. A subsemigroup F of S is called a *filter* of S if

(1) $a, b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$ and
(2) if $a \in F$ and $S \ni b \ge a$, then $b \in F$.

For an element x of S, we denote by $N(x)$ the filter of S generated by x and by $\mathcal{N}$ the equivalence relation on S defined by $\mathcal{N} := \big\{(x, y) \mid N(x) = N(y)\big\}$. The relation $\mathcal{N}$ is the least complete semilattice congruence on S. An ordered semigroup S is called a *semilattice* (resp. *complete semilattice*) *of left regular semigroups* if there exists a semilattice (resp. complete semilattice) congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left regular subsemigroup of S for every $x \in S$. S is a semilattice of left regular semigroups if and only if there exists a semilattice Y and a family $\{S_\alpha \mid \alpha \in Y\}$ of pairwise disjoint left regular subsemigroups of S whose union is S and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for every $\alpha, \beta \in Y$. If in addition, $S_\alpha \cap (S_\beta] \ne \emptyset$ implies $\alpha = \alpha\beta$ ($= \beta\alpha$), then S is a complete semilattice of left regular semigroups. The semilattice and complete semilattices of intra-regular semigroups are defined in a similar way. Finally, if S is an ordered semigroup and A, B subsets of S, then we have

(1) $A \subseteq (A]$.
(2) If $A \subseteq B$, then $(A] \subseteq (B]$.
(3) $(A](B] \subseteq (AB]$.
(4) $((A]] = (A]$.
(5) $((A]B] = (A(B]] = ((A](B]] = (AB]$.

For further information we refer to [5].

2. Main results

Lemma 1. *An ordered semigroup S is left regular if and only if every left ideal of S is semiprime.*

Proof.

$\Longrightarrow$. Let T be a left ideal of S and $a \in S$, $a^2 \in T$. Then, since S is left regular, we have $a \in (Sa^2] \subseteq (ST] \subseteq (T] = T$, and $a \in T$. Thus T is semiprime.

$\Longleftarrow$. Let $a \in S$. Since $a^2 \in L(a^2)$ and $L(a^2)$ is a left ideal of S, by hypothesis, we have $a \in L(a^2) = (a^2 \cup Sa^2]$. Then we have

$$a^2 \in (a](a^2 \cup Sa^2] \subseteq (a(a^2 \cup Sa^2)] = (a^3 \cup aSa^2] \subseteq (Sa^2],$$

and $a \in ((Sa^2] \cup Sa^2] = ((Sa^2]] = (Sa^2]$. So S is left regular. $\square$

Theorem 2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. We consider the statements:*

(1) *S is a complete semilattice of left simple semigroups.*

(2) *S is a semilattice of left simple semigroups.*

(3) *S is a union of pairwise disjoint left simple subsemigroups of S.*

(4) *S is a union of left simple subsemigroups of S.*

(5) *S is left regular.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). *But the property* (5) *does not imply* (1) *or* (2), *in general.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (5). Let $S = \bigcup\limits_{\alpha \in Y} S_\alpha$, where S_α is a left simple subsemigroup of S for every $\alpha \in Y$. According to Lemma 1, we prove that every left ideal of S is semiprime. Let T be a left ideal of S and $a \in S$, $a^2 \in T$. Then $a \in T$. In fact: We have $a \in S_\alpha$ for some $\alpha \in Y$. The set $T \cap S_\alpha$ is a left ideal of S_α. This is because the set $T \cap S_\alpha$ is a nonempty subset of S_α (since $a^2 \in T$, $a^2 \in S_\alpha$), $S_\alpha(T \cap S_\alpha) \subseteq S_\alpha T \cap S_\alpha^2 \subseteq ST \cap S_\alpha \subseteq T \cap S_\alpha$, and if $x \in T \cap S_\alpha$ and $S_\alpha \ni y \leq x$ then, since $S \ni y \leq x \in T$ and T is a left ideal of S, we have $y \in T$, so $y \in T \cap S_\alpha$. Since S_α is left simple, we have $T \cap S_\alpha = S_\alpha$, and $a \in T$.

We prove the rest of the theorem by the following example.

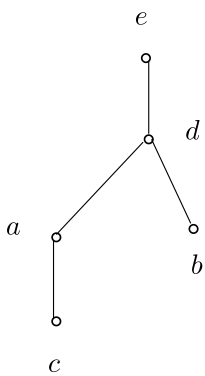
***Example* 3.** Let $S = \{a, b, c, d, e\}$ be the ordered semigroup given by the multiplication and the order below:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	a	d	d
d	a	d	a	d	d
e	a	d	a	d	e

$$\begin{aligned}\leq := \big\{&(a,a),(a,d),(a,e),(b,b),(b,d),(b,e),\\ &(c,a),(c,c),(c,d),(c,e),(d,d),(d,e),(e,e)\big\}.\end{aligned}$$

We give the covering relation and the figure of S.

$$\prec\ = \big\{(a,d),(b,d),(c,a),(d,e)\big\}.$$



For an easy way to check that this is an ordered semigroup we refer to [3].

This is a left regular ordered semigroup but it is not a complete semilattice of left simple semigroups. In fact: We give all the semilattice congruences on S. They are four and they are the following:

$$\sigma_1 = \big\{(a,a),(a,c),(a,d),(b,b),(c,a),(c,c),(c,d),(d,a),(d,c),(d,d),(e,e)\big\}.$$

$$\begin{aligned}\sigma_2 = \big\{&(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),\\ &(d,a),(d,b),(d,c),(d,d),(e,e)\big\}.\end{aligned}$$

$$\begin{aligned}\sigma_3 = \big\{&(a,a),(a,c),(a,d),(a,e),(b,b),(c,a),(c,c),(c,d),(c,e),\\ &(d,a),(d,c),(d,d),(d,e),(e,a),(e,c),(e,d),(e,e)\big\}.\end{aligned}$$

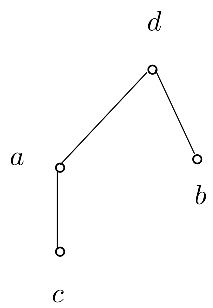
$$\sigma_4 = S \times S.$$

The semilattice congruences σ_1 and σ_3 are not complete since $b \leq d$, but $(b, bd) \notin \sigma_1$ and $(b, bd) \notin \sigma_3$. The relations σ_2 and σ_4 ($= S \times S$) are complete semilattice congruences on S. Besides, $\sigma_2 = \mathcal{N}$. S is not a complete semilattice of left simple semigroups. Indeed: For the complete semilattice congruence σ_2 of S, we have

$$(a)_{\sigma_2} = (b)_{\sigma_2} = (c)_{\sigma_2} = (d)_{\sigma_2} = \{a, b, c, d\} \qquad \text{and} \qquad (e)_{\sigma_2} = \{e\}.$$

The subsemigroup $\{a, b, c, d\}$ of S is not a left simple subsemigroup of S. This is the ordered semigroup given by the multiplication and the figure below:

.	a	b	c	d
a	a	d	a	d
b	a	b	a	d
c	a	d	a	d
d	a	d	a	d



As the last table and picture show, the set $\{a, c\}$ is a left ideal of $\{a, b, c, d\}$ different than $\{a, b, c, d\}$. As far as the complete semilattice congruence σ_4 is concerned, we have $(x)_{\sigma_4} = S$ for every $x \in S$ and the set $\{a, c\}$ is a proper left ideal of S.

For the semilattice congruence σ_1, we have

$$(a)_{\sigma_1} = (c)_{\sigma_1} = (d)_{\sigma_1} = \{a, c, d\}, \qquad (b)_{\sigma_1} = \{b\}, \qquad (e)_{\sigma_1} = \{e\}$$

and the set $\{a, c, d\}$ is not a left simple subsemigroup of S as the set $\{a, c\}$ is a proper left ideal of $\{a, c, d\}$. Thus S is not a semilattice of left simple semigroups.

It might be noted that for the semilattice congruence σ_3 on S, we have

$$(a)_{\sigma_3} = (c)_{\sigma_3} = (d)_{\sigma_3} = (e)_{\sigma_3} = \{a, c, d, e\}, \qquad \text{and} \qquad (b)_{\sigma_3} = \{b\}$$

and the subsemigroup $\{a, c, d, e\}$ is not left simple as the set $\{a, c\}$ is a proper left ideal of $\{a, c, d, e\}$. Thus again S is not a semilattice of left simple semigroups.

□

A characterization of left regular ordered semigroups is given in the following theorem.

Theorem 4. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is a complete semilattice of left regular semigroups.*

(2) *S is a semilattice of left regular semigroups.*

(3) *S is a union of pairwise disjoint left regular subsemigroups of S.*

(4) *S is a union of left regular subsemigroups of S.*

(5) *S is left regular.*

(6) *If σ is a complete semilattice congruence on S, then $(a)_\sigma$ is a left regular subsemigroup of S for every $a \in S$.*

(7) *$(a)_{\mathcal{N}}$ is a left regular subsemigroup of S for every $a \in S$.*

P r o o f. The first three implications are obvious.

(4) $\Longrightarrow$ (5). Let $S = \bigcup_{\alpha \in Y} S_\alpha$, where S_α is a left regular subsemigroup of S for every $\alpha \in Y$ and let $a \in S$. Suppose $a \in S_\alpha$ for some $\alpha \in Y$. Since S_α is left regular, there exist $x \in S_\alpha$ ($\subseteq S$) such that $a \le xa^2$. Thus S is left regular.

(5) $\Longrightarrow$ (6). Let σ be a complete semilattice congruence on S, $a \in S$ and $b \in (a)_\sigma$. Then there exists $z \in (a)_\sigma$ such that $b \le zb^2$. In fact: Since $b \in S$ and S is left regular, we have $b \le xb^2$ for some $x \in S$. Then we have

$$b \le xb(xb^2) = (xbx)b^2.$$

On the other hand, $xbx \in (a)_\sigma$. Indeed: Since σ is complete, we have $(b, bxb^2) \in \sigma$. Since σ is a semilattice congruence, we have $(b^2, b) \in \sigma$, $(bxb^2, bxb) \in \sigma$, $(b(xb), xb^2) \in \sigma$, $(xb^2, xb) \in \sigma$. Then we have $(b, xb) \in \sigma$. Besides, $(xb, bx) \in \sigma$, so we get $(b, bx) \in \sigma$, and $(xb, xbx) \in \sigma$. Since $(b, xb) \in \sigma$ and $(xb, xbx) \in \sigma$, we obtain $(b, xbx) \in \sigma$, so $xbx \in (b)_\sigma = (a)_\sigma$.

(6) $\Longrightarrow$ (7) $\Longrightarrow$ (1) since $\mathcal{N}$ is a complete semilattice congruence on S. □

Remark 5. The right analog of our results also hold.

Lemma 6. *An ordered semigroup S is intra-regular if and only if every ideal of S is semiprime.*

P r o o f.

$\Longrightarrow$. Let T be an ideal of S, $a \in S$, $a^2 \in T$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (STS] \subseteq (T] = T$, and T is semiprime.

$\Longleftarrow$. Let $a \in S$. Since $a^2 \in I(a^2)$ and $I(a^2)$ is an ideal of S, by hypothesis, we have $a \in I(a^2)$, by easy calculation, we have $a \in (Sa^2S]$, and S is intra-regular. □

The following theorem holds true:

Theorem 7. (cf. also [4: Theorem 1]) *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is a complete semilattice of simple semigroups.*

(2) *S is a semilattice of simple semigroups.*

(3) *S is a union of pairwise disjoint simple subsemigroups of S.*

(4) *S is a union of simple subsemigroups of S.*

(5) *S is intra-regular.*

Theorem 8. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is a complete semilattice of intra-regular semigroups.*

(2) *S is a semilattice of intra-regular semigroups.*

(3) *S is a union of pairwise disjoint intra-regular subsemigroups of S.*

(4) *S is a union of intra-regular subsemigroups of S.*

(5) *S is intra-regular.*

(6) *If σ is a complete semilattice congruence on S, then $(a)_\sigma$ is an intra-regular subsemigroup of S for every $a \in S$.*

(7) *$(a)_{\mathcal{N}}$ is an intra-regular subsemigroup of S for every $a \in S$.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (5). Let $S = \bigcup\limits_{\alpha \in Y} S_\alpha$, where S_α is an intra-regular subsemigroup of S for every $\alpha \in Y$ and let $a \in S$. Suppose $a \in S_\alpha$ for some $\alpha \in Y$. Since S_α is intra-regular, there exist $x, y \in S_\alpha$ ($\subseteq S$) such that $a \le xa^2y$. Thus S is intra-regular.

(5) $\Longrightarrow$ (6). Let σ be a complete semilattice congruence on S, $a \in S$ and $b \in (a)_\sigma$. Then there exist $z, w \in (a)_\sigma$ such that $b \le zb^2w$. In fact: Since S is intra-regular, we have $b \le xb^2y$ for some $x, y \in S$. Then

$$\begin{aligned} b &\le x(xb^2y)(xb^2y)y \le x^2b^2yx(xb^2y)(xb^2y)y^2 \\ &= (x^2b^2yx^2)b^2(yxb^2y^3) \end{aligned}$$

Moreover, $x^2b^2yx^2 \in (a)_\sigma$ and $yxb^2y^3 \in (a)_\sigma$. In fact: Since σ is a complete semilattice congruence on S, we have $(b, bxb^2y) \in \sigma$. On the other hand, since σ is a semilattice congruence on S, we have $(bxb^2y, x^2b^2yx^2) \in \sigma$. Indeed: Since $(bx, b^2x) \in \sigma$, $(b^2x, b^2x^2) \in \sigma$, $(b^2x^2, x^2b^2) \in \sigma$, we have $(bx, x^2b^2) \in \sigma$, $(bxb^2, x^2b^4) \in \sigma$. Then, since $(x^2b^4, x^2b^2) \in \sigma$, we get $(bxb^2, x^2b^2) \in \sigma$, and $(bxb^2y, x^2b^2y) \in \sigma$. Moreover, $(x^2b^2y, x^4b^2y) \in \sigma$. Since $(x^2b^2y, b^2yx^2) \in \sigma$, we have $(x^4b^2y, x^2b^2yx^2) \in \sigma$. Thus we have $(bxb^2y, x^2b^2yx^2) \in \sigma$. Since $(b, bxb^2y) \in \sigma$ and $(bxb^2y, x^2b^2yx^2) \in \sigma$, we obtain $(b, x^2b^2yx^2) \in \sigma$, so $x^2b^2yx^2 \in (b)_\sigma = (a)_\sigma$. Finally, $yxb^2y^3 \in (a)_\sigma$. In fact: As we have already seen $(b, bxb^2y) \in \sigma$. On the other hand, since σ is a semilattice congruence on S, we have $(bxb^2y, yxb^2y^3) \in \sigma$. Indeed: We have $(b^2y, yb^2) \in \sigma$, $(bxb^2y, bxyb^2) \in \sigma$, $(bxyb^2, byxb^2) \in \sigma$, $(byxb^2, yxb^3) \in \sigma$, $(yxb^3, yxb^2) \in \sigma$, $(yxb^2, y^4xb^2) \in \sigma$, $(y^4xb^2, yxb^2y^3) \in \sigma$, and then $(bxb^2y, yxb^2y^3) \in \sigma$. Hence we obtain $(b, yxb^2y^3) \in \sigma$, and $yxb^2y^3 \in (b)_\sigma = (a)_\sigma$.

(6) $\Longrightarrow$ (7) $\Longrightarrow$ (1) since $\mathcal{N}$ is a complete semilattice congruence on S. $\square$

Remark 9. The left (resp. right) regular ordered semigroups are intra-regular. In fact: Let S be a left regular ordered semigroup and $a \in S$. Then we have

$$a \in (Sa^2] \subseteq (S(Sa^2]a] = (S(Sa^2)a] \subseteq (Sa^2S],$$

and S is intra-regular. Thus the left, also the right regular ordered semigroups, are decomposable into simple components, as well.

As an illustrative example of the Theorems we give the following:

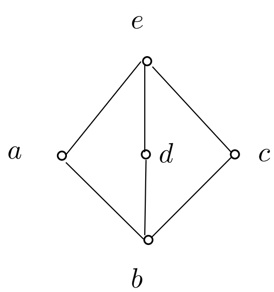
***Example* 10.** The ordered semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication and the order below is left regular and so intra-regular as well. By the Theorems 4, 7 and 8, it is a semilattice also a complete semilattice of left regular, simple, and intra-regular semigroups.

$\cdot$	a	b	c	d	e
a	e	b	a	d	e
b	b	b	b	b	b
c	a	b	c	d	e
d	d	b	d	d	d
e	e	b	e	d	e

$$\leq := \big\{(a,a),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),(d.d),(d,e),(e,e)\big\}.$$

We give the covering relation and the figure of S.

$$\prec\ = \big\{(a,e),(b,a),(b,c),(b,d),(c,e),(d,e)\big\}.$$



We give all the semilattice congruences on S. They are eight and they are the following:

$$\sigma_1 = \big\{(a,a),(a,b),(a,d),(a,e),(b,a),(b,b),(b,d),(b,e),(c,c),\\ (d,a),(d,b),(d,d),(d,e),(e,a),(e,b),(e,d),(e,e)\big\}.$$
$$\sigma_2 = \big\{(a,a),(a,c),(a,d),(a,e),(b,b),(c,a),(c,c),(c,d),(c,e),\\ (d,a),(d,c),(d,d),(d,e),(e,a),(e,c),(e,d),(e,e)\big\}.$$
$$\sigma_3 = \big\{(a,a),(a,c),(a,e),(b,b),(b,d),(c,a),(c,c),(c,e),(d,b),(d,d),\\ (e,a),(e,c),(e,e)\big\}.$$
$$\sigma_4 = \big\{(a,a),(a,c),(a,e),(b,b),(c,a),(c,c),(c,e),(d,d),(e,a),(e,c),(e,e)\big\}.$$
$$\sigma_5 = \big\{(a,a),(a,d),(a,e),(b,b),(c,c),(d,a),(d,d),(d,e),(e,a),(e,d),(e,e)\big\}.$$
$$\sigma_6 = \big\{(a,a),(a,e),(b,b),(c,c),(d,d),(e,a),(e,e)\big\}.$$
$$\sigma_7 = \big\{(a,a),(a,e),(b,b),(b,d),(c,c),(d,b),(d,d),(e,a),(e,e)\big\}.$$
$$\sigma_8 = S \times S.$$

The semilattice congruences σ_1, σ_5, σ_6, σ_7 are not complete ($c \le e$ but $(c, ce) \notin \sigma_i$ for $i = 1,5,6,7$), the relations σ_2, σ_3, σ_4 and σ_8 are complete semilattice congruences on S and the relation σ_4 is equal to $\mathcal{N}$. By the theorems of this paper, the class $(x)_{\mathcal{N}}$ is a left regular, simple and intra-regular subsemigroup of S for every $x \in S$, so S is a complete semilattice (also a semilattice) of left regular, simple, and intra-regular semigroups. The class $(x)_{\sigma_6}$ is also a left regular, simple, and intra-regular subsemigroup of S for every $x \in S$. So σ_6 is another semilattice congruence on S (different from $\mathcal{N}$) according to which S is again a semilattice of left regular, simple, and intra-regular semigroups.

Remark 11. The example 10 shows that the left regular (resp. the right regular) ordered semigroups are not left simple (resp. right simple) semigroups, in general. As we can easily see from the table and figure, the set $\{b\}$, for example, is a proper left ideal (and a proper right ideal) of S.

We finally give an example of an intra-regular ordered semigroup which is not left regular.

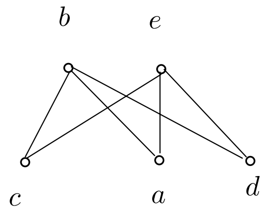
***Example* 12.** Let $S = \{a,b,c,d,e\}$ be the ordered semigroup with the multiplication and the order below:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leq: \ = \big\{(a,a),(a,b),(a,e),(b,b),(c,b),(c,c),(c,e),(d,b),(d,d),(d,e),(e,e)\big\}.$$

We give the covering relation and the figure of S.

$$\prec \ = \big\{(a,b),(a,e),(c,b),(c,e),(d,b),(d,e)\big\}.$$



This is an intra-regular but not a left regular ordered semigroup. There are two semilattice congruences on S and they are the following:

$$\sigma_1 = \big\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),$$
$$(d,a),(d,b),(d,c),(d,d),(e,e)\big\}.$$

$$\sigma_2 = S \times S.$$

The relation σ_1 is a complete semilattice congruence on S, it is equal to $\mathcal{N}$, and the class $(x)_{\sigma_1}$ is an intra-regular and simple subsemigroup of S for every $x \in S$.

For the examples of the paper we used computer programs.

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