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A CATEGORICAL EQUIVALENCE FOR BOUNDED DISTRIBUTIVE QUASI LATTICES SATISFYING: $x \vee 0 = 0 \implies x = 0$

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ABSTRACT. In this work, we investigate a categorical equivalence between the class of bounded distributive quasi lattices that satisfy the quasiequation $x \vee 0 = 0 \implies x = 0$, and a category whose objects are *sheaves over Priestley spaces*.

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Introduction

Recently, several authors extended Priestley duality for distributive lattices [22] to other classes of algebras, such as, e.g. distributive lattices with operators [16], MV-algebras [21], MTL and $IMTL$ algebras [5]. Other interesting extensions of Priestley duality are those dealing with quasiordered structures, let us mention, for example, [4, 11, 12, 15, 23]. In the present article, we focus our attention on a quasivariety of bounded distributive quasi lattices (*bdq*-lattices) [6, 7, 10]. *bdq*-lattices form a variety that generalizes lattice ordered structures to quasiordered ones. They are interesting structures for several reasons. For example, they constitute the underlying quasiordered structure associated to algebraic systems from quantum computation [13, 17, 19]. Moreover, the variety of *bdq*-lattices plays a relevant part in the context of "normally presented varieties" [8, 9].

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In [2] a "Priestley-style" representation of *bdq*-lattices was proposed. In this paper we present an alternative form of dualization for a particular class of *bdq*-lattices, that satisfy the condition $x \vee 0 = 0 \implies x = 0$, via the notion of *Priestley* $\equiv$*-sheaf*. This class of sheaves determines a category, in which the notion of arrow is inspired by the idea of arrow of ringed spaces in algebraic geometry. A similar idea had led to a general representation theory for MV-algebras [14].

This paper is organized as follows. In Section 1, we provide all the basic notions. In Section 2, we recap several basic concepts from the theory of *bdq*-lattices. In Section 3, the category of Priestley $\equiv$-sheaf is presented. Finally, in Section 4, we provide a categorical equivalence between the category of Priestley $\equiv$-sheaf and the quasivariety of *bdq*-lattices defined by the condition $x \vee 0 = 0 \implies x = 0$.

1. Basic Notions

In this section we recall some basic facts about Priestley spaces. An *ordered topological space* is a triple $\langle X, \tau, \leq \rangle$ where τ is a topology on the poset $\langle X, \leq \rangle$. The set of *clopen* (closed and open) increasing sets of such space will be denoted $\mathrm{clp}(X)$. A *Priestley space* is an ordered topological space $\langle X, \leq, \tau \rangle$ such that:

(1) if $x \not\leq y$ then there exists $U \in \mathrm{clp}(X)$ such that $x \in U$ and $y \notin U$ (*totally order-separated space*).

(2) $\langle X, \tau \rangle$ is compact.

Clearly, Priestley space are Hausdorff spaces. Let $\langle X, \tau, \leq \rangle$ be a Priestley space. Then,

$$S = \mathrm{clp}(X) \cup \left\{ \complement Y : \ Y \in \mathrm{clp}(X) \right\}$$

is a subbase of τ, and $\complement Y$ is the complement of Y. We denote by $\mathcal{P}$ the category whose objects are Priestley spaces and whose arrows are continuous order preserving maps, that we shall call *Priestley-homomorphisms*. The following lemma is straightforward.

Lemma 1.1. *Let $f \colon X \to Y$ be a Priestley-homomorphisms. If $G \in \mathrm{clp}(Y)$ then $f^{-1}(G) \in \mathrm{clp}(X)$.*

In [22] it is proved that the category $\mathcal{D}$ of bounded distributive lattices and the category $\mathcal{P}$ of Priestley spaces are equivalent. In fact, on the one hand the functor $\mathcal{F}p \colon \mathcal{D} \to \mathcal{P}$ assigns a Priestley space $\langle \mathcal{F}p(L), \tau(L), \subseteq \rangle$ to each object $L \in \mathcal{D}$, where $\mathcal{F}p(L)$ is the set of prime filters in L, $\tau(L)$ is the topology generated by

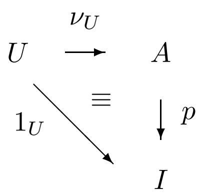
$$\{D_x\}_{x \in D} \cup \{\complement D_x\}_{x \in D}$$

with $D_x = \{F \in \mathcal{F}p(L) : \ x \in F\}$ for each $x \in L$, and $\subseteq$ is the inclusion between prime filters. Moreover, if $f\colon L_1 \to L_2$ is a $\mathcal{D}$-homomorphism, then $\mathcal{F}p(f)\colon \mathcal{F}p(L_2) \to \mathcal{F}p(L_1)$ is defined by $F \mapsto f^{-1}(F)$.

On the other hand the functor $\mathcal{E}\colon \mathcal{P} \to \mathcal{D}$ assigns the lattice $\langle \mathcal{E}(X), \cup, \cap, \emptyset, X\rangle$ to each Priestley space X, where $\mathcal{E}(X)$ is the set of all clopen increasing sets of X with set union and set intersection as lattice operations. Moreover, for any continuous order preserving function $g\colon X_1 \to X_2$, $\mathcal{E}(g)\colon \mathcal{E}(X_2) \to \mathcal{E}(X_1)$ is defined by $G \mapsto g^{-1}(G)$.

These functors give the mentioned dual equivalence, since $\mathcal{F}p\mathcal{E} = 1_{\mathcal{P}}$ and $\mathcal{E}\mathcal{F}p = 1_{\mathcal{D}}$ where, $1_{\mathcal{P}}$ and $1_{\mathcal{D}}$ are the identity functors in $\mathcal{P}$ and $\mathcal{D}$, respectively.

A *sheaf* over I is a triple (A, p, I) where A is a topological space and $p\colon A \to I$ is local homeomorphism. This means that each $a \in A$ has an open set G_a in A that is mapped homeomorphically by p onto $p(G_a) = \{p(x) : \ x \in G_a\}$, and the latter is open in I. It is clear that p is continuous and open map. If $p\colon A \to I$ is a sheaf over I, for each $i \in I$, the set $A_i = \{x \in A : \ p(x) = i\}$ is called the *fiber* over i. Each fiber has the discrete topology as subspace of A. *Local sections* of the sheaf (A, p, I) are continuous maps $\nu_U\colon U \to A$ such that U is an open set of I and the following diagram commute:



In particular we use the term *global section* only when $U = I$. Consequently if ν is a global section, $p\nu = 1_I$. Note that global sections are injective functions. If (A, p, I) admits a global section then p is a surjective map. Hence p results a closed map since for each $X \subseteq A$, $Cf(X) = f(CA)$.

2. Bounded distributive quasi lattices

Now we introduce some basic notions about quasi ordered sets and bounded distributive quasi lattices. A *quasiordered set* is a pair $\langle A, \leq\rangle$ such that $\leq$ is reflexive and transitive binary relation on A. The relation $\equiv$, defined in A as

$$x \equiv y \quad \text{iff} \quad x \leq y \quad \text{and} \quad y \leq x, \tag{1}$$

is an equivalence relation on A. Moreover, the quotient structure $A/_{\equiv}$ is a partially ordered by the relation $\leq_p$, naturally defined as follows:

$$[x] \leq_p [y] \quad \text{iff} \quad x \leq y. \tag{2}$$

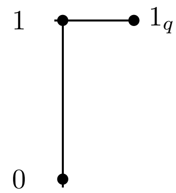
A *bounded distributive quasi lattice* [6], (*bdq*-lattice for short) is an algebra $\langle A, \vee, \wedge, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ satisfying the following equations:

(1)	$\langle A, \vee\rangle$ and $\langle A, \wedge\rangle$	are semigroups,
(2)	$x \vee (x \wedge y) = x \vee x, \;\; x \wedge (x \vee y) = x \wedge x$	(*weak absorption*)
(3)	$x \vee (y \vee y) = x \vee y, \;\; x \wedge (y \wedge y) = x \wedge y$	(*weak idempotence*)
(4)	$x \wedge x = x \vee x$	(*equalization*)
(5)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \wedge z)$	(*distributivity*)
(6)	$x \wedge 0 = 0 \quad x \vee 1 = 1$	

We denote by $\mathcal{BDQL}$ the variety of *bdq*-lattices. In general, the equations $x \wedge 1 = x$ and $x \vee 0 = x$ do not hold in $\mathcal{BDQL}$; however, it holds that $x \wedge 1 = x \vee 0 = x \wedge x = x \vee x$. An important example of bdq-lattice is $\mathbf{2}_q$ given by:

$\vee$	1	0	1_q
1	1	1	1
0	1	0	1
1_q	1	1	1

$\wedge$	0	1	1_q
1	0	1	1
0	0	0	0
1_q	0	1	1



Let A be a *bdq*-lattice and $a \in A$. We say that a is *regular* iff $a \vee a = a$ (and by equalization, also $a \wedge a = a$). We will denote this set by $\mathrm{Reg}(A)$. It can be verified that $\langle \mathrm{Reg}(A), \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. If $a \in \mathrm{Reg}(A)$ then we define the *cloud* of a as the set $\mathrm{cloud}(a) = \{x \in A \setminus \{a\} : \; x \vee 0 = a\}$. Note that if $x \in \mathrm{cloud}(a)$ then $x \notin \mathrm{Reg}(A)$. The following proposition can be easily proved:

Proposition 2.1. *Let A be a bdq-lattice. The binary relation $\leq$ on A defined by:*

$$x \leq y \quad \textit{iff} \quad x \vee y = y \vee 0 \quad (\textit{or equivalently} \quad x \wedge y = x \wedge 0)$$

is a quasiorder and the ordered set $\langle A/_{\equiv}, \leq\rangle$ is a lattice.

It can be seen that, $A/_{\equiv}$ is a bounded distributive lattice isomorphic to $\mathrm{Reg}(A)$ via the map $[x] \mapsto x \vee 0$ for each $x \in A$.

3. 0-proper *bdq*-lattices

In this section we introduce an interesting class of *bdq*-lattices called 0-proper. There are several important algebraic structures having an underlying 0-proper *bdq*-lattice structure. At first, we show two relevant examples and we introduce a particular type of sheaf over a Priestley space, whose local sections admit a natural 0-proper *bdq*-lattice structure.

Definition 3.1. A *bdq*-lattice A is said to be 0-*proper* iff $\operatorname{cloud}(0) = \emptyset$.

We denote by $\mathcal{BDQL}_0$ the class of 0-proper *bdq*-lattices. Note that $\mathcal{BDQL}_0$ is a quasivariety characterized by the quasiequation

$$x \vee 0 = 0 \implies x = 0.$$

One can easily verify that $\mathcal{BDQL}_0$ is a proper quasivariety, since it is not closed under quotients. Let us now propose some natural examples.

***Example* 3.2.** ([3]) A pseudocomplemented distributive lattice $\langle L, \vee, \wedge, \neg, 0, 1\rangle$ is a *Stone algebra* if it satisfies the equation

$$\neg x \vee \neg\neg x = 1.$$

By Glivenko's theorem, the mapping $x \mapsto \neg\neg x$, from L onto the Boolean subalgebra $R(L) = \{x \in L : \neg\neg x = x\}$ is an endomorphism. L can be naturally endowed with *bdq*-lattice operations $\vee^q$, $\wedge^q$ as follows:

$$\begin{aligned} x \vee^q y &= \neg\neg(x \vee y), \\ x \wedge^q y &= \neg\neg(x \wedge y). \end{aligned}$$

It is easy to prove that the structure $\langle L, \vee^q, \wedge^q, 0, 1\rangle$ is a *bdq*-lattice isomorphic to $\langle R(L), \vee, \wedge, 0, 1\rangle$. Moreover, for any $a \in L$, if $\neg\neg a = 0$, then $\neg a = 1$, and then $a = 0$. Therefore $\langle L, \vee^q, \wedge^q, 0, 1\rangle$ is in $\mathcal{BDQL}_0$.

***Example* 3.3.** Following [1], a *commutative Baer***-semigroup* is an algebra $\langle G, \cdot, ^*, ', 0\rangle$ of type $\langle 2, 1, 1, 0\rangle$ that satisfies the following equations:

(1) $\langle G, \cdot\rangle$ is a commutative semigroup,

(2) $0 \cdot x = 0$,

(3) $(x \cdot y)^* = x^* \cdot y^*$,

(4) $x^{**} = x$,

(5) $x' = (x')^* = x' \cdot x'$,

(6) $x' \cdot y \cdot (x \cdot y)' = y \cdot (x \cdot y)'$.

If we consider the set of closed projections $P_c(G) = \{x' : x \in G\}$, we can endow $P_c(G)$ with a Boolean algebra structure $\langle P_c(G), \vee, \wedge, ', 0, 1\rangle$ (see [20: Theorem 37.8]) where

$$\begin{aligned} x \wedge y &= x \cdot (y' \cdot x)' \\ x \vee y &= (x' \wedge y')' \\ 1 &= 0' \end{aligned}$$

In virtue of the definitions above, we can define in G the *bdq*-lattice operations $\vee^q$, $\wedge^q$, as follows

$$x \vee^q y = x'' \vee y''$$
$$x \wedge^q y = x'' \wedge y''$$

It is direct to verify that $\langle G, \vee^q, \wedge^q, 0, 1\rangle$ is a *bdq*-lattice. Since each Baer*-semigroup satisfies the equation $x = x \cdot x''$, if $x \vee^q 0 = 0$ then $x'' = 0$ and then $x = 0$, hence $\langle G, \vee^q, \wedge^q, 0, 1\rangle$ is in $\mathcal{BDQL}_0$.

We now introduce the notion of Priestley $\equiv$-sheaf and we study its relations with 0-proper *bdq*-lattices.

Definition 3.4. A $\equiv$*-topological space* is triple $\langle A, \equiv, \tau\rangle$ such that $\equiv$ is an equivalence in A and τ is a topology on A that satisfies: for each $x \in A$, the equivalence class $[x] \in \tau$.

Definition 3.5. Let $\langle A, \equiv, \tau\rangle$ be a $\equiv$-topological space. A *Priestley* $\equiv$*-sheaf* associated to $\langle A, \equiv\rangle$ is a function $p\colon A \to X$ where

(1) X is a Priestley space,

(2) for each $x \in A$, $p([x])$ is an clopen increasing set in X,

(3) the restriction $[x] \xrightarrow{p} p([x])$ is a homeomorphism.

Let us remark that the local inverse $[x] \xleftarrow{p^{-1}} p([x])$ defines a local (eventually global) section u of p, since the restriction $p \restriction [x]$ is a bijective open map. We shall refer to these sections as $\equiv$*-sections*. Suppose that the Priestley $\equiv$-sheaf $p\colon A \to X$ admits a $\equiv$-section $A \xleftarrow{u} X$ which is global. For each $U \in \mathrm{clp}(X)$ consider the restriction $u_U = u \restriction U$. Since u is continuous, u_U is a continuous map, and then it is a local section of p. In particular we denote by $u_\emptyset$ the empty local section of p. Consider the following sets:

$$\mathcal{Q}_R(p) = \left\{u(U) \xleftarrow{u_U} U : \ U \in \mathrm{clp}(X)\right\} \cup \{u_\emptyset\}$$
$$\mathcal{Q}(p) = \mathcal{Q}_R(p) \cup \{\sigma : \ \sigma \text{ is a } \equiv\text{-section of } p\}$$

Proposition 3.6. *Let $p\colon A \to X$ be a Priestley $\equiv$-sheaf and u be a global $\equiv$-section of p. Consider $\mathcal{Q}(p)$ endowed with the following operations: for $\sigma_1(U_1) \xleftarrow{\sigma_1} U_1$ and $\sigma_2(U_2) \xleftarrow{\sigma_2} U_2$ in $\mathcal{Q}(p)$*

$$\sigma_1 \vee \sigma_2 = u_{U_1 \cup U_2} \qquad 1 = u$$
$$\sigma_1 \wedge \sigma_2 = u_{U_1 \cap U_2} \qquad 0 = u_\emptyset$$

Then:

(1) $\langle \mathcal{Q}(p), \vee, \wedge, 0, 1\rangle$ *is a 0-proper bdq-lattice,*

(2) $\sigma_1 \leq \sigma_2$ *iff* $U_1 \subseteq U_2$ *is the associated quasi order relation,*

(3) $\mathrm{Reg}(\mathcal{Q}(p)) = \mathcal{Q}_R(p)$.

Proof.

1) First, it is straightforward to verify that cloud(0) $= \emptyset$. As regards the axioms, we confine ourselves in proving axiom 2: $\sigma_1 \vee (\sigma_2 \wedge \sigma_1) = u_{U_1 \cup (U_2 \cap U_1)} = u_{U_1 \cup U_1} = \sigma_1 \vee \sigma_1$. The other axioms are left as an easy exercise for the interested reader.

2) Straightforward.

3) $\sigma \in \mathrm{Reg}(\mathcal{Q}(p))$ iff $\sigma = \sigma \vee 0 = u_{U \cup \emptyset} = u_U$ iff $\sigma \in \mathcal{Q}_R(p)$. □

Priestley $\equiv$-sheaves with a distinguished global $\equiv$-section u are denoted by

$$\langle A, p, X, u \rangle,$$

where $p\colon A \to X$ is a Priestley $\equiv$-sheaf. Moreover by $\mathcal{Q}(p)$ we mean the 0-proper *bdq*-lattice structure $\langle \mathcal{Q}(p), \vee, \wedge, 0, 1 \rangle$ from Proposition 3.6.

We now introduce the category $\mathcal{PS}$ whose objects are Priestley $\equiv$-sheaves with a distinguished global section and whose arrows are

$$\langle A, p, X, u \rangle \xrightarrow{(f,\varphi)} \langle B, q, Y, v \rangle$$

such that $f\colon X \to Y$ is a continuous order preserving function and $\varphi = (\varphi_x)_{x \in X}$, where φ_x is a function between fibers $A_x \xleftarrow{\varphi_x} B_{f(x)}$ satisfying the following:

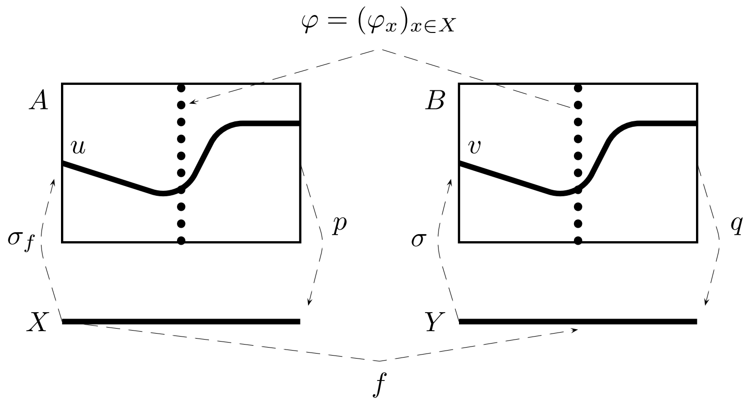
(1) for each $\sigma\colon U \to B \in \mathcal{Q}(q)$, the map $\sigma_f\colon f^{-1}(U) \to A$ defined by

$$\sigma_f(x) = \varphi_x(\sigma f(x))$$

belongs to $\mathcal{Q}(p)$.

(2) $v_f = u$.

This notion of arrow is inspired by the notion of arrow of ringed spaces in algebraic geometry (see for example [18: p. 72]). The next diagram provides some graphical intuitions.



Remark 3.7. Consider two arrows

$$\langle A, p, X, u\rangle \xrightarrow{(f,\varphi)} \langle B, q, Y, v\rangle \xrightarrow{(g,\gamma)} \langle C, r, Z, w\rangle$$

in $\mathcal{PS}$. Apparently, the composition

$$\langle A, p, X, u\rangle \xrightarrow{(g,\gamma)(f,\varphi)} \langle C, r, Z, w\rangle$$

is defined as $(g, \gamma)(f, \varphi) = (gf, \varphi\gamma)$ in which gf is the usual composition, and $\varphi\gamma$ is the family $(\varphi\gamma_x)_{x\in X}$, where $\varphi\gamma_x$ is the composition function between fibers $A_x \xleftarrow{\varphi_x} B_{f(x)} \xleftarrow{\gamma_{f(x)}} C_{gf(x)}$.

The identity arrow $\langle A, p, X, u\rangle \xrightarrow{(1_A,\mathbf{1})} \langle A, p, X, u\rangle$ is given by the pair where 1_A is the identity in A and $\mathbf{1}$ is the family of identity functions 1_{A_x} in the fiber A_x for each $x \in X$.

Thus, it can be verified that an arrow $\langle A, p, X, u\rangle \xrightarrow{(f,\varphi)} \langle B, q, Y, v\rangle$ is a *$\mathcal{PS}$-isomorphism* iff f is an order preserving homeomorphism, and for each $\varphi_x \in \varphi$, φ_x is a bijective map.

Proposition 3.8. *Let $\langle A, p, X, u\rangle \xrightarrow{(f,\varphi)} \langle B, q, Y, v\rangle$ be an arrow in $\mathcal{PS}$. Consider the map $\mathcal{Q}(f, \varphi)\colon \mathcal{Q}(q) \to \mathcal{Q}(p)$ such that $\mathcal{Q}(q) \ni \sigma \mapsto \sigma_f$. Then:*

(1) *If $v_U \in \mathcal{Q}_R(q)$ then, $[\mathcal{Q}(f, \varphi)](v_U) = u_{f^{-1}(U)}$; and then $[\mathcal{Q}(f, \varphi)](v_U) \in \mathcal{Q}_R(p)$.*
(2) *$\mathcal{Q}(f, \varphi)$ is a bdq-lattice homomorphism.*

Proof.

1) Suppose that $Y \xleftarrow{v_U} U$ belong to $\mathcal{Q}_R(q)$. By definition of arrow in $\mathcal{PS}$ we have the following two facts:

i. $[\mathcal{Q}(f, \varphi)](v_U) = v_{Uf}$ where v_{Uf} is the local section in $\mathcal{Q}(p)$ defined as $f^{-1}(U) \ni x \mapsto v_{Uf}(x) = \varphi_x(v_U f(x))$.
ii. $[\mathcal{Q}(f, \varphi)](v) = v_f = u$ and then $u(x) = \varphi_x(vf(x))$ for each $x \in X$.

Thus, for each $x \in f^{-1}(U)$, $u_{f^{-1}(U)}(x) = \varphi_x(vf(x)) = v_{Uf} = [\mathcal{Q}(f, \varphi)](v_U)$ and $[\mathcal{Q}(f, \varphi)](v_U) \in \mathcal{Q}_R(p)$.

2) By definition of arrow in $\mathcal{PS}$, $[\mathcal{Q}(f, \varphi)](1) = [\mathcal{Q}(f, \varphi)](v) = u = 1$ and $[\mathcal{Q}(f, \varphi)](0) = [\mathcal{Q}(f, \varphi)](v_\emptyset) = u_\emptyset = 0$. Let $Y \xleftarrow{\sigma} U$ and $Y \xleftarrow{\rho} V$ two local sections in $\mathcal{Q}(q)$. We prove that $[\mathcal{Q}(f, \varphi)](\sigma \vee \rho) = [\mathcal{Q}(f, \varphi)](\sigma) \vee [\mathcal{Q}(f, \varphi)](\rho)$. On the one hand, by Proposition 3.6 and item 1, $[\mathcal{Q}(f, \varphi)](\sigma \vee \rho) = [\mathcal{Q}(f, \varphi)](v_{U\cup V})$ $= u_{f^{-1}(U\cup V)}$. On the other hand, $[\mathcal{Q}(f, \varphi)](\sigma)$ is a local section in $\mathcal{Q}(p)$ with domains $f^{-1}(U)$ and $[\mathcal{Q}(f, \varphi)](\rho)$ is a local section in $\mathcal{Q}(p)$ with domains $f^{-1}(V)$. Thus, by Proposition 3.6, $[\mathcal{Q}(f, \varphi)](\sigma) \vee [\mathcal{Q}(f, \varphi)](\rho) = u_{f^{-1}(U)\cup f^{-1}(V)}$ $= u_{f^{-1}(U\cup V)}$. It proves that $[\mathcal{Q}(f, \varphi)](\sigma \vee \rho) = [\mathcal{Q}(f, \varphi)](\sigma) \vee [\mathcal{Q}(f, \varphi)](\rho)$. With an analogous argumentation we can prove that $[\mathcal{Q}(f, \varphi)](\sigma \wedge \rho) = [\mathcal{Q}(f, \varphi)](\sigma) \wedge [\mathcal{Q}(f, \varphi)](\rho)$. Hence $\mathcal{Q}(f, \varphi)$ is a *bdq*-lattice homomorphisms. □

By Proposition 3.6 and Proposition 3.8, the following result can be easily proved.

Proposition 3.9. *$\mathcal{Q}\colon \mathcal{PS} \to \mathcal{BDQL}_0$ given by*

$\langle A, p, X, u\rangle \mapsto \mathcal{Q}(p)$ *for each object* $\langle A, p, X, u\rangle$ *in* $\mathcal{PS}$

$(f, \varphi) \mapsto \mathcal{Q}(f, \varphi)$ *for each arrow* (f, φ) *in* $\mathcal{PS}$

defines a contravariant functor from the category of Priestley $\equiv$-sheaves in the category of 0-proper bdq-lattices.

4. Inverting the functor $\mathcal{Q}$

Definition 4.1. Let A be bdq-lattice and F be a subset of A. Then F is a *pointed filter* iff:

(1) $F_R = F \cap \mathrm{Reg}(A)$ is a prime filter in $\mathrm{Reg}(A)$,

(2) there exists at most one $x \notin \mathrm{Reg}(A)$ such that $x \in F$ and $x \vee 0 \in F$.

We denote by $\mathbb{E}(A)$ the set of pointed filters. In $\mathbb{E}(A)$ we define the following relation:

$$F \equiv_A G \quad \text{iff} \quad \begin{cases} F = F_R \cup \{x\}, \quad G = G_R \cup \{x\} \quad \text{or} \\ F, G \subseteq \mathrm{Reg}(A). \end{cases} \tag{3}$$

It is immediate that $\equiv_A$ defines an equivalence relation in $\mathbb{E}(A)$. Moreover the equivalence classes of the partition are necessarily of the following form:

$$\mathbb{E}_R(A) = \big\{F \in \mathbb{E}(A) : \ F \subseteq \mathrm{Reg}(A)\big\},$$
$$D^q_x = \big\{F \in \mathbb{E}(A) : \ F = F_R \cup \{x\}\big\} \qquad \text{for each} \quad x \in A - \mathrm{Reg}(A).$$

Note that $\mathbb{E}_R(A)$ is the set of prime filters of the lattice $\langle \mathrm{Reg}(A), \vee, \wedge, 0, 1\rangle$. Thus we consider $\mathbb{E}_R(A)$ endowed with the usual Priestley topology where clopen increasing sets are of the form

$$D_x = \big\{F \in \mathbb{E}_R(A) : \ x \in F\big\},$$

for each $x \in \mathrm{Reg}(A)$. Let us remark that $\mathbb{E}_R(A)$ can be regarded as a subset of $\mathbb{E}(A)$.

If $x \in \mathrm{Reg}(A)$, for simplicity, when no danger of confusion is possible, we agree in the following notation $D^q_x := D_x$. Thus $D^q_1 = D_1 = \mathbb{E}_R(A)$.

Proposition 4.2. *Let A be a 0-proper bdq-lattice and consider the relational structure $\langle \mathbb{E}(A), \equiv_A\rangle$. Define the function*

$$p\colon \mathbb{E}(A) \to \mathbb{E}_R(A) \quad s.t. \quad F \mapsto p(F) = F_R$$

and taking into account the Priestley topology in $\mathbb{E}_R(A)$ *we consider in* $\mathbb{E}(A)$ *the topology* $\tau_{\mathbb{E}(A)}$ *generated by the set*

$$\left\{p^{-1}(G):\ G \text{ is open in } \mathbb{E}_R(A)\right\} \cup \left\{\mathbb{E}_R(A)\right\} \cup \left\{D_x^q:\ x \in A - \mathrm{Reg}(A)\right\}$$

Then:

(1) *For each* $x \in A$*, the restriction* $p \restriction_{D_x^q}$ *is an open bijection from* D_x^q *onto* $D_{x\vee 0}$*.*

(2) *The inclusion map* $\mathbb{E}(A) \overset{u}{\hookleftarrow} \mathbb{E}_R(A)$ *is a global* $\equiv_A$*-section.*

(3) $\mathcal{B}(A) = (\mathbb{E}(A), p, \mathbb{E}_R(A), u)$ *is an object in the category* $\mathcal{PS}$ *where*

$$\mathcal{Q}_R(p) = \left\{D_x \overset{u_{D_x}}{\longleftarrow} D_x:\ x \in \mathrm{Reg}(A)\right\} \cup \{u_\emptyset\}$$

$$\mathcal{Q}(p) = \mathcal{Q}_R(p) \cup \left\{D_x^q \overset{p\restriction_{D_x^q}^{-1}}{\longleftarrow} D_{x\vee 0}:\ x \in A \setminus \mathrm{Reg}(A)\right\}$$

Proof.

1) Assume that $x \notin \mathrm{Reg}(A)$. If $F = F_R \cup \{x\}$ then $x \vee 0 \in F_R$ and then $p(F) \in D_{x\vee 0}$. Thus, $\mathrm{Imag}(p \restriction_{D_x^q}) \subseteq D_{x\vee 0}$. Let $F \in D_{x\vee 0}$. If we consider $F_1 = F \cup \{x\}$ then, $F_1 \in D_x^q$ and $p(F_1) = F_{1R} = F$. Hence p is surjective and $\mathrm{Imag}(p \restriction_{D_x^q}) = D_{x\vee 0}$. Suppose that $F \neq G$ in D_x^q. Since $F = F_R \cup \{x\}$ and $G = G_R \cup \{x\}$ then $F_R \neq G_R$ and the function is injective. Hence $p \restriction_{D_x^q}$ is a bijection from D_x^q onto $D_{x\vee 0}$.

Now we prove that $p \restriction_{D_x^q}$ is an open map. Let G be an open set in $\mathbb{E}_R(A)$. We first prove that $p \restriction_{D_x^q} (D_x^q \cap p^{-1}(G)) = D_{x\vee 0} \cap G$. On the one hand $p \restriction_{D_x^q} (D_x^q \cap p^{-1}(G)) \subseteq p \restriction_{D_x^q} (D_x^q) \cap p \restriction_{D_x^q} (p^{-1}(G)) \subseteq D_{x\vee 0} \cap G$. On the other hand let $F \in D_{x\vee 0} \cap G$. Consider the pointed filter $F_1 = F \cup \{x\}$. It is clear that $F_1 \in D_x^q$ and $F_1 \in p^{-1}(G)$ i.e. $F_1 \in D_x^q \cap p^{-1}(G)$. Hence $F = p \restriction_{D_x^q} (F_1) \in p \restriction_{D_x^q} (D_x^q \cap p^{-1}(G))$ and $D_{x\vee 0} \cap G \subseteq p \restriction_{D_x^q} (D_x^q \cap p^{-1}(G))$. Let B be a basic open set in $\mathbb{E}(A)$. Since D_x^q is an equivalence class in $\mathbb{E}(A)$, there exists a finite family $G_1, \ldots, G_n$ of open sets in the topology of $\mathbb{E}_R(A)$ such that $D_x^q \cap B = D_x^q \cap \bigcap_{i=1}^{n} p^{-1}(G_i)$. By the foregoing argument we have: $p \restriction_{D_x^q} (D_x^q \cap B) = p \restriction_{D_x^q} \left(D_x^q \cap \bigcap_{i=1}^{n} p^{-1}(G_i)\right) = p \restriction_{D_x^q} \left(D_x^q \cap p^{-1}\left(\bigcap_{i=1}^{n} (G_i)\right)\right) = D_{x\vee 0} \cap \bigcap_{i=1}^{n} (G_i)$ and it is an open in $\mathbb{E}_R(A)$. Since $p \restriction_{D_x^q}$ send open basics of D_x^q onto open sets in $\mathbb{E}_R(A)$, $p \restriction_{D_x^q}$ is an open map. Assume that $x \in \mathrm{Reg}(A)$. Then $D_x^q = \mathbb{E}_R(A) \cap p^{-1}(D_{x\vee 0})$ and D_x^q is an open set in $\mathbb{E}(A)$. Note that $p \restriction_{D_x^q}: D_x^q \to D_x$ is the identity. Let B be a basic open set in $\mathbb{E}(A)$. Since for each $y \notin \mathrm{Reg}(A)$, $D_x^q \cap D_y^q = \emptyset$ there exists a finite family $G_1, \ldots, G_n$ of open sets in the topology of $\mathbb{E}_R(A)$ such that $D^q_x \cap B = D^q_x \cap \bigcap_{i=1}^{n} p^{-1}(G_i)$. Since $p\restriction_{D_x}$ is the identity, $p\restriction_{D^q_x}(D^q_x \cap B) = p\restriction_{D^q_x}\left(D^q_x \cap \bigcap_{i=1}^{n} p^{-1}(G_i)\right) = D_x \cap \bigcap_{i=1}^{n} G_i$ and $p\restriction_{D^q_x}$ is an open map.

2) By the same argument used in item 1, $p\restriction_{\mathbb{E}_R(A)}$ is an open bijection from $\mathbb{E}_R(A)$ onto $\mathbb{E}_R(A)$. Hence u is a global $\equiv_A$-section of p.

3) Immediate from items 1 and 2. □

Proposition 4.3. *Let A, B be two 0-proper bdq-lattices, $f\colon A \to B$ be a bdq-lattice homomorphism and consider the objects $\mathcal{B}(A) = \langle \mathbb{E}(A), p, \mathbb{E}_R(A), u\rangle$ and $\mathcal{B}(B) = \langle \mathbb{E}(B), q, \mathbb{E}_R(B), v\rangle$ in $\mathcal{PS}$ described in* Proposition 4.2. *Define the following functions:*

- *$g\colon \mathbb{E}_R(B) \to \mathbb{E}_R(A)$ s.t. $\mathbb{E}_R(B) \ni G \mapsto g(F) = f^{-1}(F)$*
- *For each $F \in \mathbb{E}_R(B)$, and for each $g(F) \cup \{y\}$ belonging to the fiber $\mathbb{E}(A)_{g(F)}$ we define*

$$\varphi_F\big(g(F) \cup \{y\}\big) = F \cup \{f(y)\}$$

Then:

(1) *φ_F defines a map $\mathbb{E}(B)_F \xleftarrow{\varphi_F} \mathbb{E}(A)_{g(F)}$ between fibers.*

(2) *For each $x \in A$, $g^{-1}(D_{x\vee 0}) = D_{f(x\vee 0)}$.*

(3) *If $\sigma \in \mathcal{Q}(p)$ has the form $D_{x\vee 0} \xrightarrow{p\restriction^{-1}_{D^q_x}} D^q_x$, then $\sigma_g \in \mathcal{Q}(q)$ is the local section $D_{f(x\vee 0)} \xrightarrow{q\restriction^{-1}_{D^q_{f(x)}}} D^q_{f(x)}$ in $\mathcal{Q}(q)$.*

(4) *If we consider the family $\varphi = (\varphi_F)_{F\in\mathbb{E}_R(B)}$, then $\mathcal{B}(f) = (g, \varphi)$ defines an arrow $\mathcal{B}(B) \xrightarrow{\mathcal{B}(f)} \mathcal{B}(A)$ in the category $\mathcal{PS}$.*

Proof.

1) By the usual Priestley duality, $g\colon \mathbb{E}_R(B) \to \mathbb{E}_R(A)$ is a continuous order-preserving function. Let $g(F) \cup \{y\}$ in the fiber $\mathbb{E}(A)_{g(F)}$. Then $y \vee 0 \in g(F) = f^{-1}(F)$ and $f(y)\vee 0 = f(y)\vee f(0) = f(y\vee 0) \in f(f^{-1}(F)) = F$. Hence $F \cup \{f(y)\}$ is an element of the fiber $\mathbb{E}(B)_F$ and φ_F defines a map $\mathbb{E}(B)_F \xleftarrow{\varphi_F} \mathbb{E}(A)_{g(F)}$.

2) Let $x \in A$. Since g maps prime filters from $\mathbb{E}_R(B)$ to prime filters of $\mathbb{E}_R(A)$ we have:

$$\begin{aligned} F \in g^{-1}(D_{x\vee 0}) \quad &\text{iff} \quad f^{-1}(F) = g(F) \in D_{x\vee 0} \\ &\text{iff} \quad x \vee 0 \in f^{-1}(F) \\ &\text{iff} \quad f(x \vee 0) \in F \\ &\text{iff} \quad F \in D_{f(x\vee 0)} \end{aligned}$$

3) Let $\sigma \in \mathcal{Q}(p)$. By Proposition 4.2-3, σ has the form $D_{x\vee 0} \xrightarrow{p\restriction^{-1}_{D^q_x}} D^q_x$ for some $x \in A$. The domain of σ_g is $g^{-1}(D_{x\vee 0})$. Thus by item 2, σ_g is the function $\sigma_g\colon D_{f(x\vee 0)} \to \mathbb{E}(B)$ such that $\sigma_g(F) = \varphi_F(\sigma g(F))$. The following diagram provides some intuition:

Let $F \in D_{f(x\vee 0)}$. Then $g(F) \in D_{x\vee 0}$ and $\sigma g(F) = g(F) \cup \{x\}$. By definition of φ_F, $\varphi_F(\sigma g(F)) = F \cup \{f(x)\} \in D^q_{f(x)}$. Thus the codomain of σ_g is $D^q_{f(x)}$ and σ_g is the local section $q \restriction^{-1}_{D^q_{f(x)}}$ in σ_g. If $\sigma = u$, by the same argument, $u_f = v$.

4) Immediate from the foregoing items. $\square$

Let A be a 0-proper *bdq*-lattice then $\mathcal{B}(A) = \langle \mathbb{E}(A), p, \mathbb{E}_R(A), u\rangle$ will stand for the object in $\mathcal{PS}$ described in Proposition 4.2; and if $f\colon A \to B$ is a 0-proper *bdq*-lattice homomorphism, then $\mathcal{B}(f)$ will denote the arrow in $\mathcal{PS}$ described in Proposition 4.3. By the mentioned propositions, the following result can be easily proved.

Corollary 4.4. $\mathcal{B}\colon \mathcal{BDQL}_0 \to \mathcal{PS}$ *given by*

$A \mapsto \mathcal{B}(A) = \langle \mathbb{E}(A), p, \mathbb{E}_R(A), u\rangle$ *each object* A *in* $\mathcal{BDQL}_0$,

$f \mapsto \mathcal{B}(f)$ *for each homomorphisms* f *in* $\mathcal{BDQL}_0$,

defines a contravariant functor from the category of 0*-proper bdq-lattices in the category of Priestley* $\equiv$*-sheaves.*

Proposition 4.5. *The composite functor* $\mathcal{QB}\colon \mathcal{DBQL}_0 \to \mathcal{DBQL}_0$ *is naturally equivalent to the identity functor* $1_{\mathcal{DBQL}_0}$.

P r o o f. We need to prove two facts:

1) If A is a *bdq*-lattice in $\mathcal{DBQL}_0$, we show that A is *bdq*-isomorphic to $\mathcal{QB}(A)$. Let $\mathcal{B}(A) = \langle \mathbb{E}(A), p, \mathbb{E}_R(A), u\rangle$. By Proposition 4.2-3, $\sigma \in \mathcal{Q}(\mathbb{E}(A))$ has the form $D_{x\vee 0} \xrightarrow{p\restriction^{-1}_{D^q_x}} D^q_x$ for some $x \in A$. Consider the mapping $i\colon A \to \mathcal{Q}(\mathbb{E}(A))$ such that

$$A \ni x \mapsto i(x) = p \restriction^{-1}_{D^q_x}$$

i) i is bijective. The surjectivity of i is immediate. Let x, y two diverse elements in A. By definition, $D^q_x \neq D^q_y$ and $p \restriction^{-1}_{D^q_x} \neq p \restriction^{-1}_{D^q_y}$. Hence i is injective.

ii) i is a *bdq*-lattice homomorphism. If $x \in \mathrm{Reg}(A)$ then $i(x) = u_{D_x}$ i.e. the identity $D_x \xrightarrow{u} D_x$. In particular $i(1) = u_{D_1} = u$ and $i(0) = u_\emptyset$. Since $x \vee y \in \mathrm{Reg}(A)$, $i(x \vee y) = u_{D_{x\vee y}}$, by Proposition 3.6, $i(x) \vee i(y) = p \restriction^{-1}_{D^q_x} \vee p \restriction^{-1}_{D^q_y} = u_{D_{x\vee 0} \cup D_{y\vee 0}} = u_{D_{x\vee y}}$. Therefore $i(x \vee y) = i(x) \vee i(y)$. With the same argument we can show that $i(x \wedge y) = i(x) \wedge i(y)$. Hence i is a *bdq*-lattice isomorphism from A onto $\mathcal{Q}(\mathbb{E}(A))$.

2) Let $f\colon A \to B$ be a *bdq*-lattice homomorphisms. Consider the following objects $\mathcal{B}(A) = \langle \mathbb{E}(A), p, \mathbb{E}_R(A), u\rangle$ and $\mathcal{B}(B) = \langle \mathbb{E}(B), q, \mathbb{E}_R(B), v\rangle$ in $\mathcal{PS}$. We shall prove that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ \mathcal{QB}(A) & \xrightarrow[\mathcal{QB}(f)]{} & \mathcal{QB}(B) \end{array}$$

By Proposition 4.3, $\mathcal{B}(f)$ is the $\mathcal{PS}$-arrow $\mathcal{B}(B) \xrightarrow{(g,\gamma)} \mathcal{B}(A)$ where $g\colon \mathbb{E}_R(B) \to \mathbb{E}_R(A)$ is the mapping $\mathbb{E}_R(B) \ni G \mapsto g(F) = f^{-1}(F) \cap \mathrm{Reg}(A)$ and $\gamma = (\gamma_F)_{F\in\mathbb{E}_R(B)}$ in which, for each $g(F) \cup \{y\}$ belonging to the fiber $\mathbb{E}(A)_{g(F)}$, $\gamma_F(g(F) \cup \{y\}) = F \cup \{f(y)\}$.

Let $x \in A$. On the one hand, by the first part of the proof, $i(x)$ is the local section $D_{x\vee 0} \xrightarrow{p\restriction^{-1}_{D^q_x}} D^q_x$. Thus $(\mathcal{QB}(f))(i(x)) = i(x)_g$ and by Proposition 4.3 (item 2 and item 3), $i(x)_g$ is the local section $D_{f(x\vee 0)} \xrightarrow{q\restriction^{-1}_{D^q_{f(x)}}} D^q_{f(x)}$. On the other hand, using again the first part of the proof, $jf(x)$ is the local section $D_{f(x)\vee 0} \xrightarrow{q\restriction^{-1}_{D^q_{f(x)}}} D^q_{f(x)}$. Hence $(\mathcal{QB}(f))i = jf$.

By the steps 1 and 2, the composite functor $\mathcal{QB}\colon \mathcal{DBQL}_0 \to \mathcal{DBQL}_0$ is naturally equivalent to the identity functor $1_{\mathcal{DBQL}_0}$. □

Proposition 4.6. *The composite functor $\mathcal{BQ}\colon \mathcal{PS} \to \mathcal{PS}$ is naturally equivalent to the identity functor $1_{\mathcal{PS}}$.*

Proof.

1) Let $\langle A, p, X, u\rangle$ be an object in $\mathcal{PS}$ and $\equiv$ the equivalence in A that defines the $\equiv$-sections. We first prove that $\langle A, p, X, u\rangle$ is $\mathcal{PS}$-isomorphic to $\mathcal{BQ}(p)$. First, we build $\mathcal{BQ}(p)$.

Consider the set $\mathcal{Q}_R(p) = \big\{u(U) \xleftarrow{u_U} U :\ U \in \mathrm{clp}(X)\big\} \cup \{u_\emptyset\}$ and the usual *bdq*-lattice structure in

$$\mathcal{Q}(p) = \mathcal{Q}_R(p) \cup \{\sigma :\ \sigma \quad \text{is a } \equiv\text{-section of } p\}$$

given by $\sigma_1 \vee \sigma_2 = u_{U_1\cup U_2}$, $\sigma_1 \wedge \sigma_2 = u_{U_1\cap U_2}$, $1 = u$, $0 = u_\emptyset$. By Proposition 3.6-3, $\mathrm{Reg}(\mathcal{Q}(p)) = \mathcal{Q}_R(p)$. Let $\mathbb{E}_R(\mathcal{Q}(p))$ be the set of prime filters in $\mathrm{Reg}(\mathcal{Q}(p))$ endowed with the usual Priestley topology. By the Priestley duality, the map $i\colon X \to \mathbb{E}_R(\mathcal{Q}(p))$ such that

$$X \ni x \mapsto i(x) = \big\{u(U) \xleftarrow{u_U} U :\ x \in U \ \text{ and } \ U \in \mathrm{clp}(X)\big\} \tag{4}$$

is a Priestley isomorphisms. Moreover the clopen increasing sets in $\mathbb{E}_R(\mathcal{Q}(p))$ are given by

$$\begin{aligned} D_{u_U} &= \big\{i(x) \in \mathbb{E}_R(\mathcal{Q}(p)) :\ u_U \in i(x)\big\} \\ &= \big\{i(x) \in \mathbb{E}_R(\mathcal{Q}(p)) :\ x \in U\big\} \end{aligned}$$

for each $u_U \in \mathcal{Q}_R(p)$.

Let $\mathbb{E}(\mathcal{Q}(p))$ be the set of pointed filters in $\mathcal{Q}(p)$. We can characterize the elements of $\mathbb{E}(\mathcal{Q}(p))$, in fact:

$$F \in \mathbb{E}(\mathcal{Q}(p)) \quad \text{iff} \quad \exists x \in X \text{ s.t. } F = i(x) \cup \{\sigma\} \quad \text{and} \quad x \in \mathrm{dom}(\sigma)$$

In this context $F_R = i(x)$. Hence the usual equivalence relation between pointed filters in $\mathbb{E}(\mathcal{Q}(p))$, given in (3), is defined as follows:

$$F \equiv_p G \quad \text{iff} \quad \begin{cases} F = i(x) \cup \{\sigma\}, \quad G = i(y) \cup \{\sigma\}, \quad \text{or} \\ F, G \subseteq \mathbb{E}_R(\mathcal{Q}(p)) \end{cases}$$

and the equivalence classes of the partition in $\mathbb{E}(\mathcal{Q}(p))$ are necessarily of the following form:

$$\begin{aligned} &\mathbb{E}_R(\mathcal{Q}(p)) = \big\{i(x) :\ x \in X\big\} \\ &\qquad D^q_\sigma = \big\{F = i(x) \cup \{\sigma\} :\ x \in \mathrm{dom}(\sigma) \subseteq X\big\}, \qquad \text{for each} \quad \sigma \notin \mathcal{Q}_R(p). \end{aligned}$$

By Proposition 4.2, if we define the function $\hat{p} \colon \mathbb{E}(\mathcal{Q}(p)) \to \mathbb{E}_R(\mathcal{Q}(p))$ such that

$$F = i(x) \cup \{\sigma\} \mapsto \hat{p}(i(x) \cup \{\sigma\}) = i(x),$$

and the global section

$$\mathbb{E}(\mathcal{Q}(p)) \supseteq \mathbb{E}_R(\mathcal{Q}(p)) \xleftarrow{\hat{u}} \mathbb{E}_R(\mathcal{Q}(p))$$

then, $\mathcal{BQ}(p) = \langle \mathbb{E}(\mathcal{Q}(p)), \hat{p}, \mathbb{E}_R(\mathcal{Q}(p)), \hat{u} \rangle$ is an object in the category $\mathcal{PB}$.

For each $x \in X$ we define the map $A_x \xleftarrow{\xi_x} \mathbb{E}(\mathcal{Q}(p))_{i(x)}$ such that

$$\xi_x(F) = \begin{cases} \sigma(x), & \text{if } F = i(x) \cup \{\sigma\} \\ u(x), & \text{if } F = i(x) \end{cases}$$

We show that ξ_x is bijective. Suppose that $F \neq G$ in the fiber $\mathbb{E}(\mathcal{Q}(p))_{i(x)}$. We can assume that $F = i(x) \cup \{\sigma\}$. Then $G = i(x) \cup \{\rho\}$ for $\rho \neq \sigma$ or $G = i(x)$. Hence $\xi_x(F) \neq \xi_x(G)$ and ξ_x is injective. Let $t \in A_x$, i.e. $p(t) = x$. Let σ be the local section in $\mathcal{Q}(p)$ given by $[t] \xleftarrow{p^{-1}} p([t])$. Then $i(x) \cup \{\sigma\} \in \mathbb{E}(\mathcal{Q}(p))_{i(x)}$ and by definition $\xi_x(i(x) \cup \{\sigma\}) = t$, i.e the pointed filter $i(x) \cup \{\sigma\}$ is the preimage under ξ_x of t. Thus ξ_x is surjective.

Let $\xi = (\xi_x)_{x\in X}$. We have to prove that

$$\langle A, p, X, u\rangle \xrightarrow{(i,\xi)} \big\langle \mathbb{E}(\mathcal{Q}(p)), \hat{p}, \mathbb{E}_R(\mathcal{Q}(p)), \hat{u}\big\rangle$$

is a $\mathcal{PS}$-isomorphisms.

We show that (i, ξ) is an arrow in $\mathcal{PS}$. Let $\hat{\theta}$ be a local section in $\mathcal{Q}(\mathbb{E}(\mathcal{Q}(p)))$. By Proposition 4.2-3, $\hat{\theta}$ is of the form $D^q_\sigma \overset{\hat{\theta}}{\leftarrow} D_{u_U}$. Since i is a Priestley isomorphisms (see the mapping defined in (4)) $i^{-1}(D_{u_U}) = U$. Thus we define

$$\hat{\theta}_i \colon U \to A \quad \text{s.t.} \quad \hat{\theta}_i(x) = \xi_x(\hat{\theta}i(x)).$$

We prove that $\sigma = \hat{\theta}_i$. Since the codomain of $\hat{\theta}$ is D^q_σ, $\hat{\theta}i(x) = i(x) \cup \{\sigma\}$ and $\xi_x(\hat{\theta}i(x)) = \xi_x(i(x) \cup \{\sigma\}) = \sigma(x)$. Thus $\hat{\theta}_i = \sigma \in \mathcal{Q}(p)$. If $\hat{\theta} = \hat{u}$, by the same argument, $\hat{u}_i = u$. Hence (i, ξ) is an arrow in $\mathcal{PS}$. Since i is a Priestley homomorphisms and for each $x \in x$, ξ_x is a bijective map, (i, ξ) is $\mathcal{PS}$-isomorphisms.

2) Now we prove that, if $\langle A, p, X, u\rangle \xrightarrow{(f,\varphi)} \langle B, q, Y, n\rangle$ is a $\mathcal{PS}$-arrow then the following diagram is commutative:

$$\begin{array}{ccc}
\langle A, p, X, u\rangle & \xrightarrow{(f,\varphi)} & \langle B, q, Y, n\rangle \\
{\scriptstyle (i,\xi)}\big\downarrow & & \big\downarrow{\scriptstyle (j,\rho)} \\
\mathcal{BQ}(p) & \xrightarrow[\mathcal{DQ}(f,\varphi)]{} & \mathcal{BQ}(q)
\end{array}$$

Here, (i, ξ) and (j, ρ) are the $\mathcal{PS}$-isomorphisms we have seen in the first part of the proof. We first characterize the arrow $\mathcal{DQ}(f, \varphi)$. By Proposition 3.8, $\mathcal{Q}(f, \varphi)$ is the *bdq*-homomorphisms defined as:

$$\mathcal{Q}(f, \varphi) \colon \mathcal{Q}(q) \to \mathcal{Q}(p) \quad \text{s.t.} \quad \sigma \mapsto \mathcal{Q}(f, \varphi)(\sigma) = \sigma_f.$$

By Proposition 4.3, and the first part of the proof, $\mathcal{DQ}(f, \varphi)$ is the $\mathcal{PS}$-arrow (g, γ) defined as follows:

- $g \colon \mathbb{E}_R(\mathcal{Q}(p)) \to \mathbb{E}_R(\mathcal{Q}(q))$ such that for each $F \in \mathbb{E}_R(\mathcal{Q}(p))$, $g(F) = \mathcal{Q}(f, \varphi)^{-1}(F)$.
- for each $F \in \mathbb{E}_R(\mathcal{Q}(p))$ and for each $g(F) \cup \{\sigma\}$ belonging to the fiber $\mathbb{E}_R(\mathcal{Q}(q))_{g(F)}$ we define:

$$\gamma_F(g(F) \cup \{\sigma\}) = F \cup \{\mathcal{Q}(f, \varphi)(\sigma)\} = F \cup \{\sigma_f\}$$

In order to show the commutativity of the diagram, taking into account Remark 3.7, we need to prove the following two facts:

i) $gi = jf$. By definition of i, for each $x \in X$ we have:

$$i(x) = \left\{ u(U) \xleftarrow{u_U} U : \ x \in U \ \text{ and } \ U \in \mathrm{clp}(X) \right\}$$

$gi(x) = \mathcal{Q}(f,\varphi)^{-1}(i(x))$. Elements of $\mathcal{Q}(f,\varphi)^{-1}(i(x))$ are local sections $v(V) \xleftarrow{v_V} V$ in $\mathcal{Q}_R(q)$ such that $\mathcal{Q}(f,\varphi)(v_V) \in i(x)$. By Proposition 3.8, $\mathcal{Q}(f,\varphi)(v_V)$ is of the form $v(f^{-1}(V)) \xleftarrow{v_{f^{-1}(V)}} f^{-1}(V)$ and then, $\mathcal{Q}(f,\varphi)(v_V) \in i(x)$ iff $x \in f^{-1}(V)$ iff $f(x) \in V$. Thus

$$\begin{aligned} gi(x) &= \mathcal{Q}(f,\varphi)^{-1}(i(x)) \\ &= \left\{ v(V) \xleftarrow{v_V} V \in \mathcal{Q}_R(q) : \ f(x) \in V \right\} \\ &= jf(x) \end{aligned}$$

and $gi = jf$.

ii) For each $x \in A$, we consider the following composition of functions between fibers:

$$A_x \xleftarrow{\xi_x} \mathbb{E}(\mathcal{Q}(q))_{i(x)} \xleftarrow{\gamma_{i(x)}} \mathbb{E}(\mathcal{Q}(q))_{gi(x)}$$

$$A_x \xleftarrow{\varphi_x} B_{f(x)} \xleftarrow{\rho_{f(x)}} \mathbb{E}(\mathcal{Q}(q))_{jf(x)}$$

Since $gi = jf$, $\mathbb{E}(\mathcal{Q}(q))_{gi(x)} = \mathbb{E}(\mathcal{Q}(q))_{jf(x)}$. Then we have to prove that

$$\varphi_x \circ \rho_{f(x)} = \xi_x \circ \gamma_{i(x)}$$

Let $G \in \mathbb{E}(\mathcal{Q}(q))_{gi(x)} = \mathbb{E}(\mathcal{Q}(q))_{jf(x)}$. Assume that $G \notin \mathbb{E}_R(\mathcal{Q}(q))$ otherwise the commutativity follows by item i).

From the fact that $G \in \mathbb{E}(\mathcal{Q}(q))_{gi(x)}$, $G = gi(x) \cup \{\sigma\}$, where $i(x) \in \mathbb{E}_R(\mathcal{Q}(p))$. Then by definition of $\gamma_{i(x)}$,

$$\gamma_{i(x)}(G) = \gamma_{i(x)}(gi(x) \cup \{\sigma\}) = i(x) \cup \sigma_f$$

and hence

$$\xi_x(\gamma_{i(x)}(G)) = \begin{cases} \sigma_f(x), & \text{if } \ \sigma_f \notin i(x) \\ u(x), & \text{if } \ \sigma_f \in i(x) \end{cases}$$

Since $G \in \mathbb{E}(\mathcal{Q}(q))_{jf(x)}$, $G = jf(x) \cup \{\sigma\}$ and $\rho_{f(x)}(G) = \sigma(f(x))$. Thus $\varphi_x(\rho_{f(x)}(G)) = \varphi_x \sigma(f(x)) = \sigma_f(x)$ and $\varphi_x \circ \rho_{f(x)} = \xi_x \circ \gamma_{i(x)}$.

Hence, the composite functor $\mathcal{BQ}$: $\mathcal{PS} \to \mathcal{PS}$ is naturally equivalent to the identity functor $1_{\mathcal{PS}}$. □

Now by Propositions 4.5 and 4.6 we conclude:

Theorem 4.7. *$\mathcal{BDQL}_0$ is categorically equivalent to $\mathcal{PS}$.*

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