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THE PASTING CONSTRUCTIONS FOR EFFECT ALGEBRAS

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Dedicated to Prof. A. Dvurecenskij on the occasion of his birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. The aim of this paper is to present several techniques of constructing a lattice-ordered effect algebra from a given family of lattice-ordered effect algebras, and to study the structure of finite lattice-ordered effect algebras. Firstly, we prove that any finite MV-effect algebra can be obtained by substituting the atoms of some Boolean algebra by linear MV-effect algebras. Then some conditions which can guarantee that the pasting of a family of effect algebras is an effect algebra are provided. At last, we prove that any finite lattice-ordered effect algebra E without atoms of type 2 can be obtained by substituting the atoms of some orthomodular lattice by linear MV-effect algebras. Furthermore, we give a way how to paste a lattice-ordered effect algebra from the family of MV-effect algebras.

1. Introduction

In a classical physical system, the measured events form a Boolean algebra. However, the event structure for quantum mechanics is not necessarily a Boolean algebra. Therefore, Birkhoff and von Neumann introduced orthomodular lattices as the event structure describing quantum mechanical experiments [1,10]. Later, orthomodular lattices were considered as the standard quantum logic [17]. Orthoalgebras as a generalization of orthomodular posets were introduced which

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were also considered as a "sharp" quantum logic [7,11,14]. In the Nineties of the last century, two equivalent quantum structures, D-posets and effect algebras, were introduced, which were considered as "unsharp" quantum logics [11,18]. They are generalizations of many structures which arise in quantum mechanics, in particular, of Boolean algebras, orthomodular lattices and MV-algebras etc. [4,7]. More about effect algebras see [7].

In 1958, Chang introduced the definition of MV-algebras to prove the completeness of Łukasiewicz propositional logic [4]. MV-algebras play an important role in many fields of mathematics [4,7]. Especially, with the development of quantum logic, MV-algebras have appeared in effect algebras in many ways: Mundici showed that starting from an AF C*-algebra we can obtain a countable MV-algebra, and conversely, any countable MV-algebra can be derived in such a way [4]. Ravindran proved that Φ-symmetric effect algebras are exactly MV-algebras [33], and also Boolean D-posets of Chovanec and Kôpka are MV-algebras [19,20].

In [34], it has been proved that every lattice-ordered effect algebra L is a union of its maximal subsets of pairwise compatible elements, called blocks. The blocks of a lattice-ordered effect algebra L are subeffect algebras and sublattices of L. Moreover, every block B of the lattice-ordered effect algebra L is an MV-algebra in its own right, which means that it can be organized into an MV-algebra [7,34]. A natural question is how to paste a lattice-ordered effect algebra from the given family of MV-algebras?

The pasting techniques and the Greechie diagrams were essentially used in the early studies of orthomodular lattices [5,12,17]. In [5,12,17,27,30], the authors gave some interesting ways how to paste a given family of Boolean algebras to obtain an orthomodular lattice. The pasting techniques gave a lot of interesting examples. On the one hand, the pasting techniques allowed us to start from simple structures and "paste" them to obtain more complex quantum structures and these ways were different from those ways in [26,28]. The pasting techniques also brought important results [12,22–24]. The first achievement was that of Greechie who used the pasting techniques for constructing "stateless logics" [12]. Then Shultz gave a characterization of the state spaces of orthomodular lattices [37]. Later, Navara and Weber independently constructed orthomodular lattices admitting no group-valued measure [22,38]. Furthermore, some descriptions of state spaces of orthomodular structures are given in [23,24].

Recently, in order to study the pasting constructions of MV-effect algebras and the structures of effect algebras, the Greechie diagrams of effect algebras were introduced by Navara in [29], and Chovanec introduced the generalized Greechie diagrams of the pasting of MV-algebras in [3]. In [39], the authors introduced the definitions of Greechie diagrams of MV-algebras and finite lattice-ordered effect algebras without of atoms type of 2.

The pasting techniques were also used to study the structure of effect algebras in [2,35]. In [2], the authors introduced a pasting technique from an "admissible system" of MV-algebras giving a D-poset. In [35], the author gave another method of constructing a lattice-ordered effect algebra E with the given family of MV-algebras. However, all the given MV-algebras must have the same nontrivial sub-MV-algebra. The present paper provides several ways of constructing a lattice-ordered effect algebra L from a given family of MV-algebras which are different from those used in [2] and [35]. The pasting techniques used in this paper allow us to show that the set of blocks of the lattice-ordered effect algebra E which is a pasting of a given family $\mathfrak{B}$ of MV-algebras coincides with the set $\mathfrak{B}$. Finally, the given family of MV-algebras $\mathfrak{B}$ used in this paper does not necessarily have the same nontrivial sub-MV-algebra.

This paper is organized as follows. In Section 2, we recall some basic definitions and facts on effect algebras. In Section 3, the definition of the substitution of atoms of Boolean algebras by linear MV-effect algebras is introduced. It is proved that any finite MV-effect algebra can be obtained by substituting the atoms of some Boolean algebra by linear MV-effect algebras. In Section 4, we introduce some conditions which will ensure that the pasting of (lattice-ordered) effect algebras is a (lattice-ordered) effect algebra. In Section 5, we introduce the definition of the substitution of sharp atoms of lattice-ordered effect algebras by lattice-ordered effect algebras. Then we prove that the finite effect algebra L which is a pasting of MV-effect algebras family $\mathfrak{B}$ and in which any atom is of type 1, is a lattice-ordered effect algebra if and only if the pasting of the Boolean algebras family $S(\mathfrak{B})$ which is the set $\{S(M) \mid M \in \mathfrak{B}\}$, is an orthomodular lattice, where $S(M)$ is the set of sharp elements of M.

2. Some basic definitions and facts

In this section, we will recall some basic facts on effect algebras.

Definition 2.1. ([17]) An *orthomodular lattice* is a bounded lattice $(L; \vee, \wedge, 0, 1)$ with a unary operation $'$ on L satisfying the following conditions for all $a, b \in L$:

(i) If $a \leqslant b$, then $b' \leqslant a'$;

(ii) $a'' = a$;

(iii) $a' \wedge a = 0$, $a' \vee a = 1$;

(iv) If $a \leqslant b$, then $a \vee (a' \wedge b) = b$.

Definition 2.2. ([7,11]) An *effect algebra* is a system $(E; \oplus, 0, 1)$ consisting of a set E with two special elements 0 and 1, called the zero and the unit, and with a partial binary operation $\oplus$ satisfying the following conditions for all $a, b, c \in E$:

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(E2) If $a \oplus b$ is defined and $(a \oplus b) \oplus c$ is defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(E3) For any $a \in E$, there exists a unique $b \in E$ such that $a \oplus b$ is defined and $a \oplus b = 1$;

(E4) If $a \oplus 1$ is defined, then $a = 0$.

Definition 2.3. ([7]) Let E be an effect algebra.

(i) Define a binary relation on E by $a \leqslant b$ if, for some $c \in E$, $c \oplus a = b$.

(ii) Define a binary relation $\perp$ on E by $a \perp b$ if and only if $a \oplus b$ exists in E.

Proposition 2.4. ([7]) *Let E be an effect algebra. The binary relation $\leqslant$ defined by* Definition 2.3(i) *is a partial ordering in E. Moreover,* 0 *and* 1 *are the least element and the greatest element of E, respectively.*

Definition 2.5. ([7, 11]) Let E be an effect algebra.

(i) An element p of an effect algebra E is *principal* if, for $x, y \in E$, $x, y \leqslant p$ and $x \perp y$, then $x \oplus y \leqslant p$.

(ii) An element a of an effect algebra E is *sharp* if the equality $a \wedge a' = 0$ holds.

(iii) An element a of an effect algebra E is called an *atom* if the interval $E[0, a] = \{x \in E \mid 0 \leqslant x \leqslant a\}$ equals the set $\{0, a\}$. Instead of $E[0, a]$, we write simply also $[0, a]$, where E is an effect algebra.

In this paper, we denote the sets of principal and sharp elements of an effect algebra E by $P(E)$ and $S(E)$, respectively. We denote the set of all atoms of E by $A(E)$.

Definition 2.6. ([7, 13]) Let E be an effect algebra. An element a of an effect algebra E is *central* if

(i) for all $p \in E$, there exist $q, r \in E$ such that $q \leqslant a$, $r \leqslant a'$, and $a = q \oplus r$;

(ii) a and a' are principal.

Definition 2.7. ([7]) Let E be an effect algebra. If the poset $(E; \leqslant)$ is a lattice, then E is called a *lattice-ordered effect algebra*.

Definition 2.8. ([4, 7]) An *MV-algebra* is an algebra $(M; +, ', 0, 1)$ consisting of a nonempty set M, two constant elements 0, 1 in M, a unary operation $'$ and a total binary operation $+$ on M satisfying the following equations for all $a, b, c \in M$:

(MVi) $a + b = b + a$;

(MVii) $(a + b) + c = a + (b + c)$;

(MViii) $a + 0 = 0 + a = a$;

(MViv) $1 + a = 1$;

(MVv) $(a')' = a$;

(MVvi) $0' = 1$;

(MVvii) $a + a' = 1$;

(MVviii) $(a' + b)' + b = (a + b')' + a$.

Definition 2.9. ([34]) Let E be an effect algebra, $a, b \in E$. The elements a and b are called *compatible* (abbreviation $a \leftrightarrow b$), if there are $a_1, b_1, c_1 \in E$ such that the following conditions hold:

(C1) $a = a_1 \oplus c_1$, $b = b_1 \oplus c_1$,

(C2) $a_1 \oplus b_1 \oplus c_1$ exists in E.

Definition 2.10. ([34]) Let E be a lattice-ordered effect algebra. A maximal subset M of mutually compatible elements of E is called a *block* of E.

Definition 2.11. ([34]) Let E be an effect algebra. Then $Q \subseteq E$ is called a *subeffect algebra* of E if

(i) $1 \in Q$,

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

If E is a lattice-ordered effect algebra and Q is a sublattice and a subeffect algebra of E, then Q is called a *lattice-ordered subeffect algebra* of E.

Theorem 2.12. ([7, 34])

(i) *Let $(E; \oplus, 0, 1)$ be a lattice-ordered effect algebra. If for every two elements $a, b \in E$, a and b are compatible, then $(E; +, ', 0, 1)$ is an MV-algebra, where the binary operation $+$ on E is defined as follows: for any $a, b \in E$, $a + b = a \oplus (a' \wedge b)$.*

(ii) *Let $(E; +, ', 0, 1)$ be an MV-algebra. If we put $a \oplus b := a + b$ if and only if $a \leqslant b'$ for any $a, b \in E$, then $(E; \oplus, 0, 1)$ is a lattice-ordered effect algebra.*

Remark 2.13. By Theorem 2.12, any MV-algebra $(E; +, ', 0, 1)$ can be organized into an effect algebra $(E; \oplus, 0, 1)$, which is called an *MV-effect algebra* [15].

Theorem 2.14. ([7, 34]) *Let E be a lattice-ordered effect algebra. Then every block of a lattice-ordered effect algebra E is an MV-effect algebra, and the lattice-ordered effect algebra E is the set-union of all its blocks.*

Whence, by Theorem 2.14, any lattice-ordered effect algebra is a set-union of MV-effect algebra . However, the question how to “paste” lattice-ordered effect algebras from a family of MV-effect algebras is open. In the sequel, we will give some ways how to construct a lattice-ordered effect algebra from a given family of MV-effect algebras.

Definition 2.15. ([7]) Let a be an element of an effect algebra E and $n \geqslant 0$ be an integer. We define $na = 0$ if $n = 0$, $1a = a$ if $n = 1$, and $na = (n-1)a \oplus a$ if $(n-1)a$ and $(n-1)a \oplus a$ are defined in E. We define the *isotropic index* $\imath(a)$ of the element a, as the maximal nonnegative number n such that na exists. If na exists for every integer n, we say that $\imath(a) = +\infty$.

3. Greechie diagrams of MV-effect algebras and the substitutions of MV-effect algebras

In this section, we recall the definition of the Greechie diagrams of MV-effect algebras, and introduce the definition of the substitutions of atoms in MV-effect algebras. Then we prove that every finite MV-effect algebra can be obtained by substituting atoms in a Boolean algebra by linear MV-effect algebras.

Theorem 3.1. ([8,39]) *Let M be a finite MV-effect algebra and $A(M)$ be the set of all atoms of M. For any nonzero element x, there exist a unique set of atoms $\{a_i \in A(M) \mid i \in I\}$ and positive integers k_i, $i \in I$, such that $x = \bigoplus_{i \in I} k_i a_i$.*

Remark 3.2. By Theorem 3.1, any nonzero element x in a finite MV-effect algebra M is determined by a unique set of atoms and a natural numbers sequence. Hence, the Greechie diagram of an MV-effect algebra can be defined based on both the atoms and isotropic indices of atoms.

Definition 3.3. A *hypergraph* is a couple $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a nonempty set and $\mathcal{E}$ is a covering of $\mathcal{V}$ by nonempty subsets of $\mathcal{V}$ (i.e., $\bigcup \mathcal{E} = \mathcal{V}$). The elements of $\mathcal{V}$, respectively, $\mathcal{E}$, are called *vertices*, respectively, *edges* of $\mathcal{H}$.

Definition 3.4. ([3,39]) Let M be a finite MV-effect algebra. The hypergraph $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{(\imath(a), a) \mid a \in A(M)\}$, $\mathcal{E} = \{\mathcal{V}\}$, is called the *Greechie diagram* of the MV-effect algebra M. Furthermore, the vertex $(1, x) \in \mathcal{V}$ is usually written as x.

***Example* 3.5.** Fig. 2 is the Greechie diagram of the MV-effect algebra (even a Boolean algebra) M_1 represented by Fig. 1.

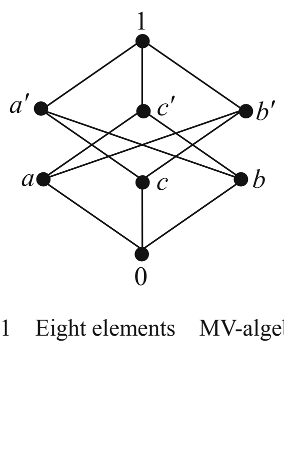

Fig. 1 Eight elements MV-algebra

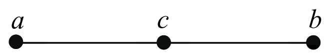

Fig.2 The Greechie diagram of MV-algebra M_1

***Example* 3.6.** Fig. 4 is the Greechie diagram of the MV-effect algebra M_2 represented by Fig. 3.

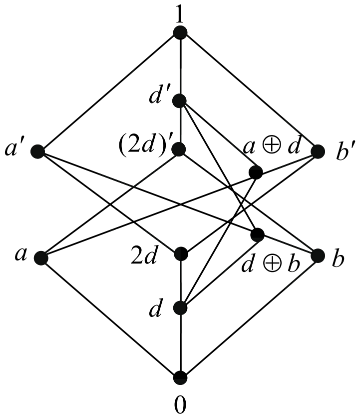

Fig. 3 Twelve elements MV-algebra M_2

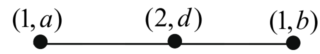

Fig.4 The Greechie diagram of MV-algebra M_2

Definition 3.7. Let L, M be finite MV-effect algebras, $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ be the Greechie diagrams of L and M, respectively. If there exists a bijection $f\colon \mathcal{V}_L \to \mathcal{V}_M$ such that for any $(\imath(a), a) \in \mathcal{V}_L$ and $(\imath(b), b) \in \mathcal{V}_M$, we have $\imath(a) = \imath(b)$ whenever $f((\imath(a), a)) = f((\imath(b), b))$, then the Greechie diagrams $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ are called *isomorphic*.

Theorem 3.8. ([39]) *Let L, M be finite MV-effect algebras, $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ be the Greechie diagrams of L and M, respectively. Then the Greechie diagram $(\mathcal{V}_L, \mathcal{E}_L)$ is isomorphic to $(\mathcal{V}_M, \mathcal{E}_M)$ if and only if L is isomorphic to M.*

Proposition 3.9. *Let M be an MV-effect algebra. For any sharp atom $a \in M$, $M = M[0, a'] \cup M[a, 1]$, $M[0, a'] \cap M[a, 1] = \emptyset$.*

Proof. Obviously, $M[0, a'] \cup M[a, 1] \subseteq M$. We have to prove that $M[0, a'] \cup M[a, 1] \supseteq M$. For any $x \in M$, we have that $a \wedge x = a$ or $a \wedge x = 0$. If $a \wedge x = a$, then $x \in M[a, 1]$. If $a \wedge x = 0$, then $x = x \wedge 1 = x \wedge (a \vee a') = (x \wedge a) \vee (x \wedge a') = x \wedge a'$, and so $x \in M[0, a']$. Hence, $M = M[0, a'] \cup M[a, 1]$.

Now, we prove that $M[0, a'] \cap M[a, 1] = \emptyset$. Suppose that $M[0, a'] \cap M[a, 1] \neq \emptyset$. If $x \in M[0, a'] \cap M[a, 1]$, then $a \leqslant x \leqslant a'$ and $a \wedge a' = a \neq 0$, which is a contradiction. Hence, $M[0, a'] \cap M[a, 1] = \emptyset$. □

Proposition 3.10. *Let $(M; \oplus, 0, 1)$ be an MV-effect algebra and C_m be the linear MV-effect algebra $\{0, \frac{1}{m}, \dots, \frac{m-1}{m}, 1\}$, where m is a natural number with $m \geqslant 1$. Assume that a is an atom of M such that $a \wedge a' = 0$. Then $(M^*; \oplus^*, (0, 0), (1, a'))$ is an MV-effect algebra, where M^* equals $C_m \times M[0, a']$, and for any $(b, c), (d, e) \in M^*, (b, c) \oplus^* (d, e)$ exists and equals $(b \oplus_{C_m} d, c \oplus e)$ whenever both $b \oplus_{C_m} d$ and $c \oplus e$ exist in C_m and M, respectively.*

P r o o f. It is easy to see that $M[0, a']$ is also an MV-effect algebra when we restrict the partial operation $\oplus$ of M to $M[0, a']$. Then the result is obvious. $\square$

Remark 3.11. In Proposition 3.10, since M equals $M[0, a'] \cup M[a, 1]$ by Proposition 3.9, we can define an MV-effect algebra embedding $f\colon M \to M^*$ as follows: for any $x \in M$,

$$f(x) := \begin{cases} (0, x), & \text{if } x \in M[0, a'], \\ (1, a_x), & \text{if } x \in M[a, 1] \text{ and } x = a \oplus a_x. \end{cases}$$

Hence, the MV-effect algebra M can be considered as a sub MV-effect algebra of M^*. Furthermore, by [13: Thm 42] and [6: Thm 3.2], the MV-effect algebra M is isomorphic to the direct product $M[0, a] \times M[0, a']$ for a sharp element $a \in M$. Hence, in this paper, we identify the MV-effect algebra M with the MV-effect algebra $M[0, a] \times M[0, a']$, whenever a is a sharp element of MV-effect algebra M.

By the above Proposition 3.10 and Remark 3.11, we can introduce the following definition.

Definition 3.12.

(i) Assume that M is an MV-effect algebra and a is a sharp atom of M. Let M^* be the MV-effect algebra $C_m \times M[0, a']$. Then we say that the MV-effect algebra M^* is *originated* by *substituting* the atom a in M by the MV-effect algebra C_m, or *substituting* the interval $M[0, a]$ by the MV-effect algebra C_m.

(ii) Let M_0 be a finite MV-effect algebra and $a_0, a_1, \dots, a_{n-1}$ be atoms of M_0, and $C_{m_1}, C_{m_2}, \dots, C_{m_n}$ be finite chain MV-effect algebras. If a_0 is a sharp atom of M_0, then we set $M_1 = M_0[0, a'_0] \times C_{m_1}$. If a_1 is a sharp atom of M_1, then we set $M_2 = M_1[0, a'_1] \times C_{m_2}$. If we have M_{n-1}, and a_{n-1} is a sharp atom of M_{n-1}, then we set $M_n = M_{n-1}[0, a'_{n-1}] \times C_{m_n}$. Thus, we have an MV-effect algebra sequence $M_0, M_1, \dots, M_n$, where MV-effect algebra M_i is originated by substituting atom a_{i-1} of M_{i-1} by MV-effect algebras C_{m_i}, for $i = 1, 2, \dots, n$. We say that MV-effect algebra M_n is *originated* by substituting atoms $a_0, a_1, \dots, a_{n-1}$ of M_0 by MV-effect algebras $C_{m_1}, C_{m_2}, \dots, C_{m_n}$, respectively, n-times.

***Example* 3.13.** The MV-effect algebra L_2 (Fig. 3) in Example 3.6 is originated by substituting the atom c of the MV-effect algebra L_1 (Fig. 1) in Example 3.5 by the linear MV-effect algebra $\{0, d, 2d\}$.

Lemma 3.14. *Let a mapping $f\colon M_1 \to M_2$ be an isomorphism between MV-effect algebras and let a be a sharp atom of M_1. If M_1^* and M_2^* are originated by substituting the atom a of M_1 and the atom $f(a)$ of M_2 by C_m, respectively, then the MV-effect algebras M_1^* and M_2^* are isomorphic.*

Proof. Note that M_1^* and M_2^* are isomorphic to $C_m \times M_1[0,a']$ and $C_m \times M_2[0,(f(a))']$, respectively, and $f\colon M_1[0,a'] \to M_2[0,(f(a))']$ is an isomorphism by assumptions, we have that M_1^* and M_2^* are isomorphic. $\square$

Lemma 3.15. *Let M be the MV-effect algebra $C_m \times 2^{n-1}$. Then M is isomorphic to the MV-effect algebra M^*, originated by substituting the atom $(1,\underbrace{0,\dots,0}_{n-1})$ of B $(=2^n)$ consisting of the sharp elements of M by the MV-effect algebra C_m.*

Proof. Let $a=(1,\underbrace{0,\dots,0}_{n-1})$. Then a is an atom of B with $a \wedge a' = 0$. Since B is isomorphic to the direct product $B[0,a] \times B[0,a']$, we have that the MV-effect algebra $M^* = C_m \times B[0,a']$ is isomorphic to the MV-effect algebra $C_m \times 2^{n-1}$. $\square$

Theorem 3.16. *Let M be the MV-effect algebra $C_{m_1} \times \dots \times C_{m_n}$, where C_{m_i} is $\left\{0, \frac{1}{m_i}, \dots, \frac{m_i-1}{m_i}, 1\right\}$, $m_i \geqslant 1$, $i=1,2,\dots,n$. Then the MV-effect algebra M^*, which is originated by substituting atoms $a_1,a_2,\dots,a_n$ of B by MV-effect algebras $C_{m_1},\dots,C_{m_n}$, respectively, n-times, is isomorphic to the MV-effect algebra M, where $a_1=(1,\underbrace{0,\dots,0}_{n-1})$, ..., $a_n=(\underbrace{0,\dots,0}_{n-1},1)$.*

Proof. Note that a_1 is a sharp atom of B, assume that M_1 is originated by substituting the atom a_1 by the MV-effect algebra C_{m_1}, then $M_1 = C_{m_1} \times B[0,a_1']$ $= M_{m_1} \times \{0\} \times \underbrace{\{0,1\} \times \dots \times \{0,1\}}_{n-1}$, which is isomorphic to the MV-effect algebra $M_{m_1} \times \underbrace{\{0,1\} \times \dots \times \{0,1\}}_{n-1}$, and so, we can set $M_1 = C_{m_1} \times \underbrace{\{0,1\} \times \dots \times \{0,1\}}_{n-1}$.

Note that a_2 is also a sharp atom of M_1, assume that M_2 is originated by substituting the atom a_2 by the MV-effect algebra M_{m_2}, then $M_2 = C_{m_2} \times M_1[0,a_2']$ $= C_{m_2} \times C_{m_1} \times \{0\} \times \underbrace{\{0,1\} \times \dots \times \{0,1\}}_{n-2}$. Again, we can set $M_2 = C_{m_2} \times M_1[0,a_1']$ $= C_{m_2} \times C_{m_1} \times \underbrace{\{0,1\} \times \dots \times \{0,1\}}_{n-2}$.

Repeating the above process, we get that $M_n = C_{m_n} \times C_{m_{(n-1)}} \times \dots \times C_{m_1}$.

Setting $M_0 = B$, $M^* = M_n$, we have a sequence of MV-effect algebras $M_0, M_1, \dots, M_n$, where M_n is originated by substituting atoms $a_1,a_2,\dots,a_n$ of B by MV-effect algebras $C_{m_1},\dots,C_{m_n}$, respectively, n -times. Then M_n is isomorphic to the MV-algebra M. $\square$

Definition 3.17. Let M be a finite MV-effect algebra and let a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be the Greechie diagram of M, $(1, a) \in \mathcal{V}$, $\{(n, b)\} \cap \mathcal{V} = \emptyset$, $\mathcal{E}_1 = \{\mathcal{V}_1\}$, where $\mathcal{V}_1 = \big(\mathcal{V} - \{(1, a)\}\big) \cup \{(n, b)\}$ and $n \in \mathbf{N}$, $n \geqslant 1$. Writing $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, we say that $\mathcal{H}_1$ is originated by *substituting* the vertex $(1, a)$ of $\mathcal{H}$ by (n, b).

***Example* 3.18.** The Greechie diagram in Fig. 4 is originated by substituting the vertex $(1, c)$ of the Greechie diagram in Fig. 2 by $(2, d)$.

Theorem 3.19. *Let M be a finite MV-effect algebra. If $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ are the Greechie diagrams of M and $S(M)$, respectively, then the Greechie diagram $\mathcal{H}$ is originated by substituting every vertex $(1, \imath(x)x)$ of $\mathcal{H}_1$ by the vertex $(\imath(x), x)$.*

Proof. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{H}_1$=($\mathcal{V}_1$, $\mathcal{E}_1$) be the Greechie diagrams of M and $S(M)$, respectively. Then $\mathcal{V}$ equals $\big\{(\imath(x), x) \mid x \text{ is an atom of } M\big\}$ and $\mathcal{V}_1$ equals $\big\{(1, \imath(x)x) \mid \imath(x)x \text{ is an atom of } S(M)\big\}$, hence, $\mathcal{H}$ is originated by substituting every vertex $(1, \imath(x)x)$ by $(\imath(x), x)$ of $\mathcal{H}_1$. $\square$

Theorem 3.20. *Let M and N be two finite MV-effect algebras. Assume that $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are the Greechie diagrams of M and N, respectively. Then the Greechie diagram of $\mathcal{H}_1$ is originated by substituting the vertex $(1, a)$ of $\mathcal{H}_2$ by the vertex (n, b) if and only if M is originated by substituting the sharp atom a of N by the MV-effect algebra C_b, where C_b equals the linear MV-effect algebra $\{0, b, \dots, nb\}$ up to isomorphisms.*

Proof.

(Sufficiency) Assume that $\mathcal{V}_2 = \{(1, a), (\imath(a_1), a_1), \dots, (\imath(a_m), a_m)\}$. Then N is isomorphic to $[0, a] \times [0, \imath(a_1)a_1] \times \dots \times [0, \imath(a_m)a_m]$. The MV-effect algebra $[0, nb] \times [0, \imath(a_1)a_1] \times \dots \times [0, \imath(a_m)a_m]$ is originated by substituting the atom a of N by $C_b = \{0, b, \dots, nb\}$. Thus M is isomorphic to $[0, nb] \times [0, \imath(a_1)a_1] \times \cdots \times [0, \imath(a_m)a_m]$. Hence, $\mathcal{V}_1 = \{(n, b), (\imath(a_1), a_1), \dots, (\imath(a_m), a_m)\}$, $\mathcal{H}_2$ is originated by substituting the vertex $(1, a)$ of $\mathcal{H}_1$ by (n, b).

(Necessity) By the definitions of $\mathcal{H}_1$ and $\mathcal{H}_2$, M and N are isomorphic to $[0, nb] \times [0, \imath(a_1)a_1] \times \dots \times [0, \imath(a_m)a_m]$ and $[0, a] \times [0, \imath(a_1)a_1] \times \dots \times [0, \imath(a_m)a_m]$, respectively. Hence, M is originated by substituting the atom a of N by $C_b = \{0, b, \dots, nb\}$. $\square$

Remark 3.21. By Theorem 3.19 and 3.20, any finite MV-effect algebra M is originated by substituting the atoms of the Boolean algebra $S(M)$ by linear MV-effect algebras.

4. The pasting of lattice-ordered effect algebras

In this section, we will present some conditions to construct an effect algebra from a given family of effect algebras.

Firstly, we recall the following two kinds of basic techniques of constructing a (lattice-ordered) effect algebra with a given family of (lattice-ordered) effect algebras [7].

If $\mathcal{M}$ is a family of (lattice-ordered) effect algebras, then the cartesian product $\prod\limits_{M\in\mathcal{M}} M$ is organized into a (lattice-ordered) effect algebra in the obvious way using coordinatewise operations and relations.

If $\mathcal{M}$ is a family of (lattice-ordered) effect algebras such that $A\cap B=\{0,1\}$ for any two distinct effect algebras $A,B\in\mathcal{M}$, then the horizontal sum of all is defined in such a way that, for any $x,y\in\bigcup\limits_{M\in\mathcal{M}} M$, $x\oplus y$ is defined if and only if there exists an effect algebra $M\in\mathcal{M}$ such that $x\oplus_M y$ exists in M, in which case $x\oplus y=x\oplus_M y$. Then the algebraic system $(\bigcup\limits_{M\in\mathcal{M}} M;\oplus,0,1)$ is called the *horizontal sum* of $\mathcal{M}$, or the $\{0,1\}-$pasting and it is a (lattice-ordered) effect algebra.

Now we give the general definitions of the pasting of the family of effect algebras.

Definition 4.1. Let $\mathcal{M}$ be a family of effect algebras such that $\mathcal{M}$ satisfies the following conditions:

(i) for any $A,B\in\mathcal{M}$ with $A\neq B$, $A\nsubseteq B$,

(ii) for any $A,B\in\mathcal{M}$, $A\cap B$ is a subeffect algebra of both A and B on which the induced structures coincide, and $0=0_A=0_B$, $1=1_A=1_B$.

Let $L=\bigcup\limits_{M\in\mathcal{M}} M$. The partial addition operation $\oplus$ is defined in the following way: for any $x,y\in L$, $x\oplus y$ exists in L if and only if there exists $M\in\mathcal{M}$ such that $x\oplus_M y$ exists in M, in which case, $x\oplus y=x\oplus_M y$. For any $A,B\in\mathcal{M}$, $A\cap B$ is a subeffect algebra of both A and B on which the induced structures coincide, then both A and B have same zero element $0=0_A=0_B$ and the unit element $1=1_A=1_B$. Then the algebraic system $(L;\oplus,0,1)$ is called the *pasting* of the effect algebras family $\mathcal{M}$.

If all elements $M\in\mathcal{M}$ are MV-effect algebras, then $(L;\oplus,0,1)$ is called the *pasting* of the MV-effect algebras family $\mathcal{M}$.

Remark 4.2.

(i) We define a binary relation $\leqslant$ on L as follows, for any $x,y\in L$, $x\leqslant y$ if and only if there exists $M\in\mathcal{M}$ such that $x\leqslant_M y$, where $\leqslant_M$ is the partial order in M. Then $\leqslant$ is reflexive and antisymmetric.

(ii) We define a binary relation $\perp$ on L as follows, for any $x, y \in L$, $x \perp y$ if and only if there exists $M \in \mathcal{M}$ such that $x \perp_M y$, where $\perp_M$ is the orthogonality relation on M. For $x, y \in L$, if $x \perp y$, then we say that x and y are *orthogonal*.

(iii) We define a unary operation $'$ on L as follows, for any $x, y \in L$, if $x \oplus y$ exists and equals 1, then we write that $x' = y$. In fact, if there exist two effect algebras A, B such that $x \oplus_A y = x \oplus_B z = 1$ (where $\oplus_A$ and $\oplus_B$ are the partial additions on A and B, respectively), then $x \in A \cap B$. Since $A \cap B$ is a subeffect algebra of both A and B, then we have that $y = z = x'$.

Remark 4.3. Let $(L; \oplus, 0, 1)$ be the pasting of a family $\mathcal{M}$ of effect algebras. For any $A, B \in \mathcal{M}$, if $A \cap B = \{0, 1\}$, then $(L; \oplus, 0, 1)$ is the horizontal sum of the effect algebras family $\mathcal{M}$.

In the following part of this section, we will discuss the pasting of effect algebras.

Proposition 4.4. *Let $\mathcal{M}$ be a family of effect algebras and $(L; \oplus, 0, 1)$ be a pasting of $\mathcal{M}$. Suppose that $\mathcal{M}$ satisfies the following condition:*

(E_0) *for any $a, b, c \in L$, if $a \oplus b$ and $(a \oplus b) \oplus c$ exist in L, then there exists an $M \in \mathcal{M}$ such that $a, b, c \in M$.*

Then $(L; \oplus, 0, 1)$ is an effect algebra.

Proof. Let $a, b, c \in L$. If $a \oplus b$ exists, then there exists an effect algebra $M \in \mathcal{M}$ such that $a \oplus b = a \oplus_M b = b \oplus_M a$. Hence, $b \oplus a$ exists and equals $a \oplus b$. If $a \oplus b$ and $(a \oplus b) \oplus c$ exists, then by the assumptions there exists an effect algebra $M \in \mathcal{M}$ such that $a, b, c \in M$. Hence, we have that $(a \oplus b) \oplus c = (a \oplus_M b) \oplus_M c = a \oplus_M (b \oplus_M c) = a \oplus (b \oplus c)$. For any $a \in L$, by Remark 4.2(iii), there exists a unique element b such that $a \oplus b = 1$. Assume that $a \oplus 1$ exists, then $a \oplus_M 1$ exists in some $M \in \mathcal{M}$, and so $a = 0$. Thus, we have proved that $(L; \oplus, 0, 1)$ is an effect algebra. □

Proposition 4.5. *Let $\mathcal{M}$ be a family of effect algebras and $(L; \oplus, 0, 1)$ be a pasting of $\mathcal{M}$. Suppose that $\mathcal{M}$ satisfies the following condition:*

(E_1) *for any $A, B \in \mathcal{M}$, if $a \in A \cap B$, then there exists an effect algebra $C \in \mathcal{M}$ such that $A[0, a] \cup B[0, a'] \subseteq C$.*

Then $(L; \oplus, 0, 1)$ is an effect algebra.

Proof. By Proposition 4.4, it suffices to prove (E_0). Let us assume that $a, b, c \in L$ and $(a \oplus b) \oplus c$ exists. Then there exist two effect algebras $A, B \in \mathcal{M}$ such that $a \oplus b = a \oplus_A b$, $(a \oplus b) \oplus c = (a \oplus_A b) \oplus_B c$, and so $a \oplus b \in A \cap B$. By the condition (E_1), there exists an effect algebra $C \in \mathcal{M}$ such that $A[0, a \oplus b] \cup B[0, (a \oplus b)'] \subseteq C$. Hence, $a, b, c \in C$. □

Remark 4.6. There is an example ([27: Example 3.6]) which shows that condition (E_0) does not imply the condition (E_1).

Definition 4.7. ([7]) Let $(E;\oplus,0,1)$ be an effect algebra.

(i) A non empty subset I of E is called an *ideal* of E if
  (1) $p\oplus q\in I$ whenever $p,q\in I$ and $p\perp q$, and
  (2) for any $p\in E$ and $q\in I$, $p\leqslant q$ implies $p\in I$.

(ii) The ideal I_e generated by a single element e is called a *principal ideal*.

Remark 4.8. Let E be an effect algebra, then the interval $E[0,e]$ is a principal ideal if and only if e is a principal element of E [7]. In fact, let e be a principal element of an effect algebra, then it is easy to see that $E[0,e]$ is an ideal, and so $E[0,e]=I_e$. However, if e is not a principal element of an effect algebra, then the principal ideal I_e does not always equal the interval $E[0,x]$ for some $x\in E$.

Proposition 4.9. *Let $\mathcal{M}$ be a family of effect algebras and $(L;\oplus,0,1)$ be a pasting of $\mathcal{M}$. Suppose that $\mathcal{M}$ satisfies the following condition:*

(E_2) *for any $A,B\in\mathcal{M}$, we have $A\cap B=I\cup I'$, where I is an ideal of both A and B and the set I' equals $\{i'\mid i\in I\}$.*

Then $(L;\oplus,0,1)$ is an effect algebra.

Proof. By Proposition 4.5, it is sufficient to prove (E_1). Let us assume that $a\in A\cap B$ for some $A,B\in\mathcal{M}$. If $A=B$, then (E_1) is clearly satisfied. Assume now that $A\neq B$ and $A\cap B=I\cup I'$. If $a\in I$, then $A[0,a]=B[0,a]$, we have that $A[0,a]\cup B[0,a']\subseteq B$. If $a\in I'$, then $A[0,a']=B[0,a']$, we have that $A[0,a]\cup B[0,a']\subseteq A$. □

Remark 4.10. The condition (E_1) does not imply the condition (E_2). Let D be the diamond effect algebra $\{0,a,b,1\}$, where $a\oplus a=b\oplus b=1$, and for any $x\in D$, $x\oplus 0=x$. Suppose that A is the horizontal sum of D and the linear effect algebra $\{0,c,1\}$ and B is the horizontal sum of D and the linear effect algebra $\{0,d,1\}$. Let $\mathcal{M}=\{A,B\}$, then $\mathcal{M}$ satisfies the condition (E_1) but not (E_2).

Proposition 4.11. *Let $\mathcal{M}$ be a family of effect algebras and $(L;\oplus,0,1)$ be a pasting of $\mathcal{M}$. Suppose that $\mathcal{M}$ satisfies the following condition:*

(E_3) *there exists an $M\in\mathcal{M}$ such that, for any $A,B\in\mathcal{M}$ and for any $a\in(A\cap B)-\{0,1\}$, we have $A[0,a]\cup B[0,a']\subseteq M$.*

Then $(L;\oplus,0,1)$ is an effect algebra.

Proof. It is easy to see that the condition (E_3) implies the condition (E_1). □

Remark 4.12. Example 5.12 (Fig. 9 and Fig. 10) shows that the condition (E$_2$) does not imply the condition (E$_3$). By Remark 4.10, we can also conclude that the condition (E$_3$) does not imply the condition (E$_2$).

Now, we give a sufficient condition which can guarantee that the pasting of the family of lattice-ordered effect algebras is also a lattice-ordered effect algebra.

Proposition 4.13. *Let $\mathcal{M}$ be a family of lattice-ordered effect algebras and $(L; \oplus, 0, 1)$ be a pasting of $\mathcal{M}$. Suppose that $\mathcal{M}$ satisfies the following condition:*

(E$_4$) *there exists a lattice-ordered effect algebra $M \in \mathcal{M}$ such that, for any $A, B \in \mathcal{M}$ with $A \neq B$, we have $A \cap B = I \cup I'$, where $I = M[0, m]$ for some $m \in M$ is a principal ideal of A, B and M.*

Then $(L; \oplus, 0, 1)$ is a lattice-ordered effect algebra.

Proof. Noticing that the condition (E$_4$) implies the condition (E$_3$), then $(L; \oplus, 0, 1)$ is an effect algebra.

Let $a, b \in L$. Then there are the following different cases.

(i) If $a, b \in M$, then $a \vee_M b = a \vee_L b$. Let $c \in L$ and $a \leqslant c$, $b \leqslant c$. We assert that there exists $x \in M$ such that $a, b \leqslant x \leqslant c$. There exist $A, B \in \mathcal{M}$ such that $a \leqslant_A c$, $b \leqslant_B c$. If $A \neq B$, then $c \in A \cap B \subseteq M$. Setting $x = c$, we have that $x \leqslant c$. If $A = B$, then $a, b \in A \cap M = I \cup I' = M[0, m] \cup M[m', 1]$. If $a, b \in M[0, m]$, then we set $x = c \wedge_A m$, and so $a \leqslant_A c \wedge_A m \leqslant_A c$ and $b \leqslant_A c \wedge_A m \leqslant_A c$. When $M[m', 1] \cap \{a, b\} \neq \emptyset$, we assume that $a \in M[m', 1]$, then $a \leqslant_A c \in M$, set $x = c$.

(ii) When there exists only one element in $M \cap \{a, b\}$, we assume that $a \in M \cap \{a, b\}$. If there exists a lattice-ordered effect algebra $A \in \mathcal{M}$ such that $a, b \in A$, then upper bounds of a, b belong to A, and so $a \vee_L b = a \vee_A b$. Otherwise, if $\{a, b\}$ is not contained in any lattice-ordered effect algebra of $\mathcal{M}$, then the upper bounds of a, b belong to M. In fact, if x is an upper bound of a and b, then there exist two different lattice-ordered effect algebras $A, B \in \mathcal{M}$ such that $a, x \in A, b, x \in B$ and $a \leqslant_A x$, $b \leqslant_B x$. Noticing the condition (E$_4$) and $A \neq B$, we have that $x \in A \cap B \subseteq M$. By $b \notin M$, we have that $B \neq M$, thus we assume that $B \cap M = M[0, m_B] \cup M[m'_B, 1]$. Then we have that the upper bounds of b in M belong to $M[b \vee_B m'_B, 1]$, and so $a \vee_L b = a \vee_M (b \vee_B m'_B)$.

(iii) When $M \cap \{a, b\}$ is the empty set, there are the following two cases. If there exists a lattice-ordered effect algebra $A \in \mathcal{M}$ such that $a, b \in A$, then the upper bounds of a, b belong to A, and so $a \vee_L b = a \vee_A b$. Otherwise, if $\{a, b\}$ is not contained in any lattice-ordered effect algebra of $\mathcal{M}$, then the upper bounds of a, b belong to M. In fact, if x is an upper bound of a and b, then there exist two different lattice-ordered effect algebras $A, B \in \mathcal{M}$ such that $a, x \in A$, $b, x \in B$ and $a \leqslant_A x$, $b \leqslant_B x$. Noticing the condition (E_4) and $A \neq B$, we have that $x \in A \cap B \subseteq M$. By $a, b \notin M$, we assume that $A \cap M = M[0, m_A] \cup M[m'_A, 1]$, $B \cap M = M[0, m_B] \cup M[m'_B, 1]$. Since $M \cap \{a, b\} = \emptyset$, the upper bounds of a and b belong to $M[a \vee_A m'_A, 1]$ and $M[b \vee_B m'_B, 1]$, and so $a \vee_L b = (a \vee_A m'_A) \vee_M (b \vee_B m'_B)$. □

Remark 4.14. It is easy to see that the condition (E_4) implies the conditions (E_0), (E_1), (E_2), (E_3), however, the reverse implications do not hold by Remarks 4.6, 4.10, 4.12.

5. MV-effect algebras pasting

The substitution techniques have been firstly introduced in [27] by Navara and Rogalewicz to explore the pasting techniques of orthomodular lattices. In this section, we will give the definitions of the substitution of effect algebras. As the applications of the definitions, the structures of the finite lattice-ordered effect algebras with only atoms of type 1 are discussed. Furthermore, one kind of techniques of pasting a lattice-ordered effect algebra from the given family of MV-effect algebras is presented.

5.1. The substitution of effect algebras

In the paper [16: Lem 3.1], the authors proved that an element a is sharp in a lattice-ordered effect algebra L, if and only if a is central in some block of L, if and only if a is central in every block containing a. Similarly, in the following theorem we will prove that an element a is sharp if and only if a is principal in a lattice-ordered effect algebra L.

Theorem 5.1. *If $(L; \oplus, 0, 1)$ is a lattice-ordered effect algebra, then the set $P(L)$ of principal elements coincides with the set $S(L)$ of sharp elements.*

Proof. It suffices to prove that $P(L) \supseteq S(L)$. Assume that $a \in S(L)$, $x, y \in L$ and $x \leqslant y'$, $x \leqslant a$, $y \leqslant a$. Then there exists a block M of L such that $a, x, y \in M$, and so a is a principal element of the MV-effect algebra M and $x \oplus y \leqslant_M a$ in M. Hence, $x \oplus y \leqslant a$ holds in L and $a \in P(L)$. □

By [16: Cor 3.2], the authors proved that for any lattice-ordered effect algebra E the set $S(E)$ of sharp elements is an orthomodular lattice. By [16: Thm 5.1, Cor 3.2], we have the following result.

Corollary 5.2. *If $(L; \oplus, 0, 1)$ is a lattice-ordered effect algebra, then the algebraic system $(P(L); \oplus_{P(L)}, 0, 1)$ is an orthomodular lattice, where the operation $\oplus_{P(L)}$ is the restriction of $\oplus$ of L to the set $P(L)$.*

Theorem 5.3. *Let M and L be two lattice-ordered effect algebras. Suppose that a is an atom of L with $a \wedge a' = 0$. Then the algebraic system $(L^*; \oplus_{L^*}, (0,0), (a',1))$ is a lattice-ordered effect algebra, where L^* equals $L[0,a'] \times M$ and the partial operation $\oplus_{L^*}$ is defined as follows: for any $(b,c),(d,e) \in L^*$, $(b,c) \oplus_{L^*} (d,e)$ is defined if and only if $b \oplus_{a'} d$ and $c \oplus_M e$ are defined in $L[0,a']$ and M, respectively, in which a case, $(b,c) \oplus_{L^*} (d,e) = (b \oplus_{a'} d, c \oplus_M e)$, where the operation $\oplus_{a'}$ is the restriction of the operation $\oplus$ of L on the set $L[0,a']$ and the operation $\oplus_M$ is the partial addition on M.*

Proof. By Theorem 5.1, $S(L)$ equals $P(L)$, and $a' \in P(L)$. Then the algebraic system $(L[0,a']; \oplus_{a'}, 0, a')$ is a lattice-ordered effect algebra. The rest is easy to prove. □

Remark 5.4. Since $S(L)$ equals $P(L)$, we have that $L[0,a'] \cup L[a,1]$ is a lattice-ordered effect algebra. By Theorem 5.3, we can define an embedding $u\colon L[0,a'] \cup L[a,1] \to L^*$ as follows: for any $x \in L[0,a'] \cup L[a,1]$,

$$u(x) := \begin{cases} (x,0), & \text{if } x \in L[0,a'], \\ (a_x,1), & \text{if } x \in L[a,1] \text{ and } x = a \oplus a_x. \end{cases}$$

Hence, the lattice-ordered effect algebra $(L[0,a'] \cup L[a,1]; \oplus, 0, 1)$, which is a subeffect algebra of L, can be considered as a subeffect algebra of L^*. Hence, we identify any element $x \in L[0,a'] \cup L[a,1]$ with the element $u(x) \in L^*$ in the union $L \cup L^*$. Under this identification, we see that the family of lattice-ordered effect algebras L, L^* satisfies the condition (E_4). Therefore, we have the following substitution theorem.

Theorem 5.5 (Substitution Theorem). *Let M and L be two lattice-ordered effect algebras. Suppose that a is an atom of L with $a \wedge a' = 0$. Let U be the set $L \cup L^*$, where the lattice-ordered effect algebra L^* equals $L[0,a'] \times M$ and it is similar to the one in* Theorem 5.3. *We define a partial operation $\oplus$ on U as follows: for any $a, b \in U$,*

$$a \oplus b := \begin{cases} a \oplus_L b, & \text{if } a,b \in L, \\ a \oplus_{L^*} b, & \text{if } a,b \in L^* - L. \end{cases}$$

Then the algebraic system $(U; \oplus, 0, 1)$ is a lattice-ordered effect algebra.

Proof. By Remark 5.4, there exists an isomorphism between the subeffect algebra $L[0,a'] \cup L[a,1]$ of L and the subeffect algebra $L[0,a'] \times \{0,1\}$ of L^*. Thus, both the effect algebras $L[0,a'] \cup L[a,1]$ and $L[0,a'] \times \{0,1\}$ can be considered as the intersection of lattice-ordered effect algebras L and L^*. Hence, the partial operation $\oplus$ on U is well defined and the algebraic system $(U; \oplus, 0, 1)$ is the pasting of lattice-ordered effect algebras L and L^*. Since the pasting U satisfies the condition (E_4), we have that U is a lattice-ordered effect algebra by Proposition 4.13. □

Definition 5.6. We say that the lattice-ordered effect algebra U is originated by *substituting* the atom a in L by the lattice-ordered effect algebra M, or *substituting* the interval $L[0, a]$ by the MV-effect algebra M, where the meaning of U, L, and M is same as those in Theorem 5.3 and Theorem 5.5.

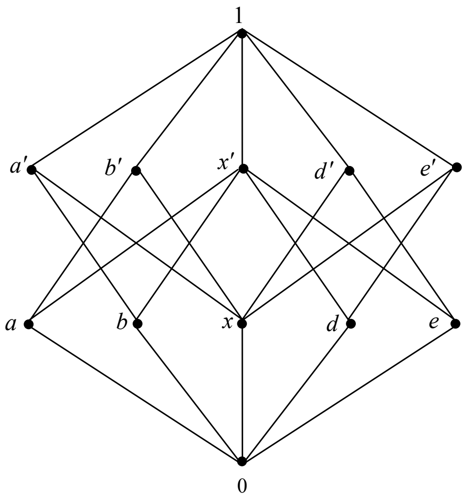

Fig. 5 A lattice-ordered effect algebra

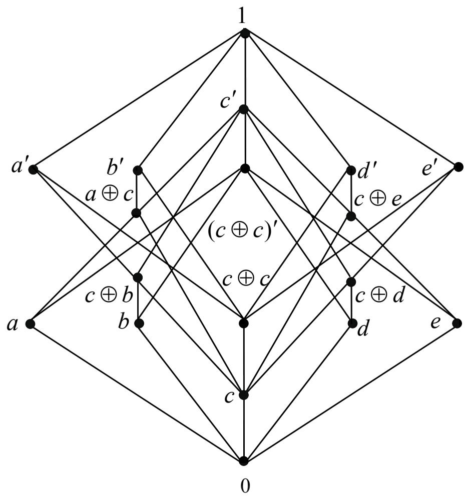

Fig.6 The lattice-ordered effect algebra U which originated by substituting the atom x of a lattice-ordered effect algebra L

***Example* 5.7.** Fig. 5 and Fig. 6 are the Hasse diagrams of lattice-ordered effect algebras L and U. It is easy to see that effect algebra U is originated by substituting the atom x of L by the effect algebra $\{0, c, 2c\}$.

5.2. The structure of lattice-ordered effect algebra with only atoms of type 1

In this section, we will study the structure of finite lattice-ordered effect algebras with only atoms of type 1. Using the definition of the substitution of effect algebras, we will prove that any finite lattice-ordered effect algebra with only atoms of type 1 can be gotten by some orthomodular lattice. Furthermore, we present some ways how to construct a lattice-ordered effect algebra with a family of the MV-effect algebras.

Definition 5.8. ([7]) Let L be an effect algebra and $a \in L$ be an atom. The atom a is of *type* **1** if whenever m is a positive integer and ma is defined, the interval $L[0, ma] = \{0, a, \dots, ma\}$. Otherwise it is of *type* **2**.

Proposition 5.9. *Let L be a finite lattice-ordered effect algebra and let all atoms of L be of type 1. If A and B are different blocks of L, then the sets of sharp elements $S(A)$ and $S(B)$ are also different.*

Proof. For any atom x of A, we assert that $\imath(x)x$ is an atom of $S(A)$. In fact, for any $y \in L$, if $y \leqslant \imath(x)x$ and $y \leqslant (\imath(x)x)'$, then there exists a natural number n such that $y = nx$ since the atom x is of type 1. And, $y = nx \leqslant (\imath(x)x)'$, which implies that $n = 0$, and so $y = 0$. Hence, $\imath(x)x \in S(A)$. Furthermore, for any $y \in L$, if $y < \imath(x)x$ and $y \wedge y' = 0$, then there exists a natural number $n < \imath(x)$ such that $y = nx$, and so $(nx) \wedge (nx)' = 0$. Notice that if $n \geqslant 1$, then $x \leqslant (nx) \wedge (nx)'$. Consequently, we have that $n = 0$, which implies that $y = 0$. Hence, $\imath(x)x$ is an atom of $S(A)$.

Assume that $S(A)$ equals $S(B)$. By the assumption $S(A) = S(B)$, $\imath(x)x$ is also an atom of $S(B)$, then there exists an atom y of B such that $\imath(y)y = \imath(x)x$. Since L does not contain the atom of type 2, we get that the atom x equals the atom y. Thus the blocks A and B are the same MV-effect algebras, which is a contradiction. Hence, the fact that the sets $S(A)$ and $S(B)$ are different holds. $\square$

Definition 5.10. Let L be a finite effect algebra which is a pasting of a family $\mathfrak{B}$ of MV-effect algebras, and all atoms of L be of type 1. The hypergraph $(\mathcal{V}, \mathcal{E})$, where

$$\begin{aligned} &\mathcal{V} = \big\{(\imath(a), a) \mid a \in A(L)\big\},\\ &\mathcal{E} = \{E \subseteq \mathcal{V} \mid \text{there exists an MV-effect algebra } B \in \mathfrak{B} \text{ such that}\\ &\qquad\qquad E \text{ is an the edge of the Greechie diagram of } B\}, \end{aligned}$$

is a hypergraph called the *Greechie diagram* of effect algebra L.

***Example* 5.11.** Fig. 7 and Fig. 8 are the Greechie diagrams of the lattice-ordered effect algebras L (Fig. 5) and U (Fig. 6), respectively.

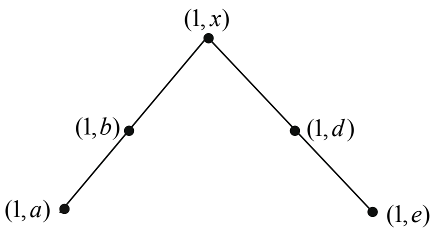

Fig. 7 The Greechie diagram of a lattice-ordered effect algebra L

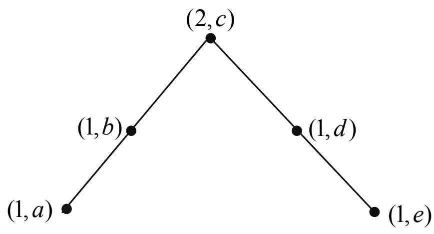

Fig. 8 The Greechie diagram of a lattice-ordered effect algebra U

***Example* 5.12.** ([39]) Fig. 9 and Fig. 10 are the Greechie diagrams of the effect algebras, where none of them is a lattice-ordered effect algebra.

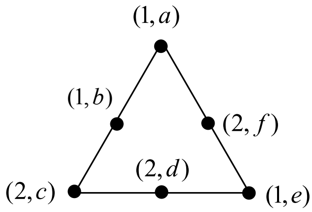

Fig. 9 The Greechie diagram of a non-lattice effect algebra

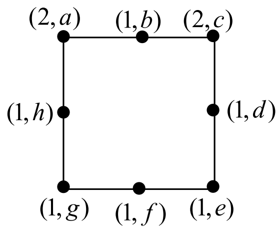

Fig. 10 The Greechie diagram of a non-lattice effect algebra

In the following, we suppose that the atoms of effect algebras are all of type 1.

Definition 5.13. ([39]) Let L, M be finite effect algebras which are the pastings of two families of MV-effect algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ be the Greechie diagrams of L and M, respectively. If there exist bijections $f\colon \mathcal{V}_L \to \mathcal{V}_M$ and $g\colon \mathcal{E}_L \to \mathcal{E}_M$ such that

(i) for any $(\imath(a), a) \in \mathcal{V}_L$, if $f((\imath(a), a)) = (\imath(b), b)$, then $\imath(a) = \imath(b)$,

(ii) for any $E \in \mathcal{E}_L$, $g(E) = \big\{f((\imath(a), a)) \mid (\imath(a), a) \in E\big\}$,

then the Greechie diagrams $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ are called *isomorphic.*

Theorem 5.14. ([39]) *Let L, M be finite lattice-ordered effect algebras which are the pastings of two families $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and $(\mathcal{V}_L, \mathcal{E}_L)$ and $(\mathcal{V}_M, \mathcal{E}_M)$ be the Greechie diagrams of L and M, respectively. Then L is isomorphic to M if and only if $(\mathcal{V}_L, \mathcal{E}_L)$ is isomorphic to $(\mathcal{V}_M, \mathcal{E}_M)$.*

Definition 5.15. Let L be a finite effect algebra which is a pasting of a family $\mathcal{M}$ of MV-effect algebras and $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be the Greechie diagram of L. The hypergraph $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is called the *substitution* of the hypergraph $\mathcal{H}$, if $(1, a) \in \mathcal{V}$, $(n, b) \in \mathcal{V}_1$, and the hypergraph $\mathcal{H}_1$ satisfies the following conditions:

(i) $\mathcal{V}_1 = (\mathcal{V} - \{(1, a)\}) \cup \{(n, b)\}$, where $\{(n, b)\} \cap \mathcal{V} = \emptyset$ and n is a natural number with $n \geqslant 1$.

(ii) $\mathcal{E}_1 = \mathcal{E}_{21} \cup \mathcal{E}_{12}$, where, $\mathcal{E}_{11} = \big\{E \mid (1, a) \in E,\ \ E \in \mathcal{E}\big\}$, $\mathcal{E}_{12} = \mathcal{E} - \mathcal{E}_{11}$, and $\mathcal{E}_{21} = \big\{E_1 \mid E_1 = \big(E - \{(1, a)\}\big) \cup \big\{(n, b)\big\},\ \ E \in \mathcal{E}_{11}\big\}$.

Theorem 5.16. *Let L, P be two finite lattice-ordered effect algebras without atoms of type 2, and $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ be the Greechie diagrams of L, P, respectively. Assume that a is an atom of L such that $a \notin P$ with $a \wedge a' = 0$, and b is an atom of P such that $b \notin L$ with $\imath(b) = n$. Then the effect algebra P is originated by substituting the sharp atom a of L by $C_b^n = \{0, b, \dots, nb\}$ if and only if the hypergraph $\mathcal{H}_2$ is originated by substituting the vertex $(1, a)$ of $\mathcal{H}_1$ by the vertex (n, b).*

Proof.

Necessity. Let the block set of L be $\mathcal{B}$, and $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ be the sets $\big\{B \mid (1,a) \in B,\ B \in \mathcal{B}\big\}$ and $\{B \mid B \notin \mathcal{B}_{11},\ B \in \mathcal{B}\}$, respectively. Assume that $\mathcal{B}_{21}$ is the set $\big\{(B \cap L[0,a']) \times C_b^n \mid B \in \mathcal{B}_{11}\big\}$. Then the block set of P is $\mathcal{B}_{21} \cup \mathcal{B}_{12}$. Noticing that for any $B \in \mathcal{B}_{21}$, it is originated by substituting the atom a of B by C_b^n. Hence, $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ is the Greechie diagram of P.

Sufficiency. Let the block set of L be $\mathcal{B}$, and $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ be $\{B \mid a \in B, B \in \mathcal{B}\}$ and $\{B \mid B \notin \mathcal{B}_{11},\ B \in \mathcal{B}\}$, respectively, then $\mathcal{B}$ equals $\mathcal{B}_{11} \cup \mathcal{B}_{12}$. Assume that $\mathcal{E}_{11}$ is the set $\{E \mid E \text{ is an edge of the Greechie diagram of } B, B \in \mathcal{B}_{11}\}$, and $\mathcal{E}_{12}$ is the set $\{E \mid E \text{ is an edge of the Greechie diagram of } B, B \in \mathcal{B}_{12}\}$.

Since $\mathcal{V}_2$ equals $(\mathcal{V}_1 - \{(1,a)\}) \cup \{(n,b)\}$ and $\mathcal{E}_2$ equals $\mathcal{E}_{12} \cup \mathcal{E}_{21}$, where $\mathcal{E}_{21}$ equals $\big\{E_1 \mid E_1 = \big(E - \{(1,a)\}\big) \cup \big\{(n,b)\big\},\ E \in \mathcal{E}_{11}\big\}$. Assume L_1 is originated by substituting the atom a of L by $C_b^n = \{0, b, \dots, nb\}$, then the hypergraph $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ is the Greechie diagram of L_1 by necessity. Hence, L_1 and P have the same Greechie diagram, and so $L_1 = P$. □

THEOREM 5.17. *Let L be a finite lattice-ordered effect algebra without atoms of type 2. Then the effect algebra L is originated by the substitution the atoms of $S(L)$.*

Proof. Let $\mathcal{H}_0 = (\mathcal{V}_0,\ \mathcal{E}_0)$ be the Greechie diagram of $S(L)$, where $\mathcal{V}_0$ equals $\big\{(1, \imath(x)x) \mid x \in A(L)\big\}$ and $\mathcal{E}_0$ equals $\{E \mid \text{there exists a block } B \text{ of } S(L) \text{ such that } E \text{ is the edge of Greechie diagram of } B\}$. Since any atom a of L is of type 1, for any atoms a, b of L, we have that $\imath(a)a$ equals $\imath(b)b$ if and only if a equals b. Substituting every vertex $(1, \imath(x)x)$ of $\mathcal{H}_0$ by $(\imath(x), x)$, we can get the diagram $\mathcal{H}_1 = (\mathcal{V}_1,\ \mathcal{E}_1)$, where $\mathcal{V}_1$ equals the set $\big\{(\imath(x), x) \mid x \in A(L)\big\}$, $\mathcal{E}_1$ equals the set $\big\{E \mid \text{there exists an edge } E_0 \text{ of } \mathcal{H}_0 \text{ such that } E = \{(\imath(x), x) \mid (1, \imath(x)x) \in E_0\}\big\}$. By Theorem 5.16, $\mathcal{H}_1$ is the Greechie diagram of L, and L is originated by the substitution the atoms of $S(L)$. □

Remark 5.18. By Theorem 5.17, any finite lattice-ordered effect algebra L without atoms of type 2 can be obtained by substituting the atoms of orthomodular lattice $S(L)$ by linear MV-effect algebras.

***Example* 5.19.** The Greechie diagram Fig. 8 is originated by substituting the vertex $(1, x)$ of the Greechie diagram Fig. 7 by the vertex $(2, c)$, and the effect algebra U (Fig. 6) is originated by substituting the sharp atom x of the elements L (Fig. 5) by the linear MV-effect algebra $\{0, c, 2c\}$.

Theorem 5.20. *Let L be a finite effect algebra which is a pasting of a family $\mathcal{M}$ of MV-effect algebras. Assume that for any $M_1, M_2 \in \mathcal{M}$, for any $a, b \in A(M_1) \cup A(M_2)$, we have that a equals b, whenever $a \leqslant mb$ for some natural number $m \leqslant \imath(b)$. Then the following statements hold.*

(i) *Let L_0 be the set $\bigcup\{S(M) \mid M \in \mathcal{M}\}$, then L_0 is a pasting of the family of Boolean algebras $\{S(M) \mid M \in \mathcal{M}\}$.*

(ii) *L is a lattice-ordered effect algebra, whenever L_0 is an orthomodular lattice.*

Proof.

(i) Assume that $B_1, B_2 \in \{S(M) \mid M \in \mathcal{M}\}$ with $B_1 \subseteq B_2$, then there exist two MV-algebras $M_1, M_2 \in \mathcal{M}$ such that B_1 equals $S(M_1)$ and B_2 equals $S(M_2)$. For any $x \in A(M_1)$, $\imath(x)x \in B_1$, we have that $\imath(x)x \in B_2 \subseteq M_2$. Then there exists an atom of y of M_2 such that $y \leqslant \imath(x)x$, and so $y = x \in M_2$ by the assumption, which implies $A(M_1) \subseteq A(M_2)$. Hence, $M_1 \subseteq M_2$. Since L is a pasting of the family of MV-algebras $\mathcal{M}$, we have that $M_1 = M_2$, and so $B_1 = B_2$. Furthermore, since $B_1 \cap B_2$ equals $S(M_1) \cap S(M_2)$ and $M_1 \cap M_2$ is the subalgebra of M_1 and M_2, we have that $B_1 \cap B_2$ is a subalgebra of B_1 and B_2. Hence, L_0 is the pasting of a family of Boolean algebras $\{S(M) \mid M \in \mathcal{M}\}$.

(ii) Assume that the Greechie diagram of L_0 is $\mathcal{H}_0$ which is the pair $(\mathcal{V}_0, \mathcal{E}_0)$, where $\mathcal{V}_0$ equals $\{(1, \imath(x)x) \mid x \in A(L)\}$, and $\mathcal{E}_0$ is the set $\{E \mid$ there exists a block B of L_0 such that E is an edge of the Greechie diagram of $B\}$. Since any atom of L is of type 1, then $\imath(a)a$ equals $\imath(b)b$ if and only if a equals b for atoms $a, b \in A(L)$.

Let $\mathcal{V}_1$ be the set $\{(\imath(x), x) \mid x \in A(L)\}$. Assume that E_B is the set $\{(\imath(x), x) \mid (1, \imath(x)x)$ is a vertex of the Greechie diagram of the block B of $L_0\}$, that is, E_B is the edge is originated by substituting every vertex $(1, \imath(x)x)$ of the Greechie diagram of B by the vertex $(\imath(x), x)$. Let $\mathcal{E}_1$ be the set $\{E_B \mid B$ is a block of $L_0\}$. Substituting every vertex $(1, \imath(x)x)$ of the Greechie diagram $\mathcal{H}_0$ by $(\imath(x), x)$, we can get the hypergraph $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$. By the definition of hypergraph $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, the hypergraph $\mathcal{H}_1$ is the Greechie diagram of L. Hence, L is originated by substituting the atoms of L_0 and it is a lattice-ordered effect algebra by Theorem 5.5. □

6. Conclusions and discussion

In this paper, we gave the definitions of substituting the atoms of MV-effect algebras. Using this definition, we proved that any finite MV-effect algebra is originated by substituting of the atoms of the Boolean algebra by linear MV-effect algebras. Then we gave some conditions such that the pasting of effect algebras is also an effect algebra. Similarly, we introduce the definition of the substitution of sharp atoms of lattice-ordered effect algebras by lattice-ordered effect algebras. All these definitions and results generalize those for orthomodular posets in [27]. At last, we proved that the finite effect algebra L without atoms of type 2, which is a pasting of MV-effect algebras family $\mathfrak{B}$, is a lattice-ordered effect algebra if and only if the pasting of the Boolean algebras family $S(\mathfrak{B})$ determined by the given MV-effect algebras $\mathfrak{B}$, is an orthomodular lattice. The last result (Theorem 5.20) provides one kind of technique of pasting a lattice-ordered effect algebra by a family of MV-effect algebras, however, any atom of the pasting lattice-ordered effect algebra must be of type 1. Hence, the question how to paste a lattice-ordered effect algebra by a family of MV-effect algebras is still open.

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