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THE HOFFMANN-JØRGENSEN INEQUALITY OF NA RANDOM VARIABLES AND ITS APPLICATIONS TO THE LOGARITHM LAW

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ABSTRACT. In this paper, the Hoffmann-Jørgensen inequality for negatively associated (NA) random variables is derived. As an important tool, it will be applied to the establishing for the logarithm law of NA arrays, and the results of Su, Hu and Liang [15] are extended.

1. Introduction and the results

Hoffmann-Jørgensen established an inequality for independent and identically distributed (i.i.d) random variables in 1974 (see [3] for detail), then, as a principal tool, the inequality was used in the construction of the results on complete convergence. Later on, the different versions of Hoffmann-Jørgensen's approach were gradually developed, mainly on dependent random variable, see Ghosal and Chandra [2] for instant.

The random variables $X_1, X_2, \dots, X_n$ are said to be *negatively associated* (NA) if for every pair of disjoint subset A_1, A_2 of $\{1, 2, \dots, n\}$,

$$\mathsf{Cov}\big(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\big) \le 0,$$

whenever f_1 and f_2 are increasing. NA is one qualitative version of negative dependence among random variables, and a great many basic theoretical properties have been derived for NA random variables. In this paper, we give a

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Hoffmann-Jørgensen inequality of NA random variables with applications to the law of logarithm.

The law of the iterated logarithm (LIL) and the law of the logarithm have also been investigated by many authors under the dependent assumptions. For example, Li, Rao, and Tomkins [8] noted that the law of the iterated logarithm does not hold under the assumption that the array is row-wise i.i.d instead of being an i.i.d array. Hu and Weber [6] and Qi [10] studied the logarithmic law for the row-wise i.i.d array. Su, Hu, and Liang [15] studied the logarithm law of row-wise NA arrays.

Using the Hoffmann-Jørgensen inequality given in the paper, the complete convergence result for NA random variables is derived consequently. As an application, the logarithm law given by Su, Hu and Liang [15] is extended and the nonclassical logarithm law of NA arrays is established. We refer to Li, Rao, Jiang and Wang [8] for complete convergence, Hu and Weber [6], Qi [10] and Su, Hu and Liang [15] for the logarithm law, Klesov [7] for the nonclassical LIL.

Throughout the paper, we assume that $\{\overline{X}, \overline{X}_n :\ n \geq 1\}$ is an independent copy of $\{X, X_n :\ n \geq 1\}$, and we will consider the symmetrized random variables $X^s = X - \overline{X}$, $X_n^s = X_n - \overline{X_n}$, $S_n^s = X_1^s + \cdots + X_n^s$, $n \geq 1$. And let $\{X^*, X_n^* :\ n \geq 1\}$ denote a sequence of independent random variables, $X \stackrel{d}{=} X^*$, $X_n \stackrel{d}{=} X_n^*$, $n \geq 1$. In the sequel, $\log^+ x = \log(e \vee x)$, $x \geq 0$, $\sim$ between expressions will mean that the limit of their ratio is 1, and $[x]$ will denote the greatest integer less or equal than x.

At the end of this section, we state the following Hoffmann-Jørgensen type inequality for NA random variables which is used as our main tool in the application part.

Theorem 1.1. *Let $\{X_k :\ k \in Z\}$ be a NA-sequence of symmetric random variables, and $\{X_k^* :\ k \in Z\}$ be a sequence of independent real random variables such that X_k^* has the same distribution as X_k for all $k \in Z$ and the series $\sum_{k\in Z} X_k^*$ converges a.s. Let $\varepsilon > 0$ and $j \in N$ be given and let $Y_k^* = \left(-\frac{\varepsilon}{10j}\right) \vee \left(X_k^* \wedge \frac{\varepsilon}{10j}\right)$ denote the standard truncation of Y_k at level $\frac{\varepsilon}{10j}$. Then the series $\sum_{k\in Z} Y_k^*$ and $\sum_{k\in Z} X_k$ converges a.s. and if we define $S_k = \sum_{i\leq k} X_i$, then we have*

$$\begin{aligned} &\mathsf{P}\Big(\sup_{k\in Z} |S_k| > 2\varepsilon\Big) \\ &\leq \mathsf{P}\Big(\sup_{k\in Z} |X_k| > \frac{\varepsilon}{10j}\Big) + \frac{4^j}{\varepsilon}\int_{\varepsilon}^{\infty}\bigg(\mathsf{P}\Big(\Big|\sum_{k\in Z} Y_k^*\Big| > \frac{x}{2j}\Big)\bigg)^j \mathrm{d}x \qquad (1.1) \\ &\leq \mathsf{P}\Big(\sup_{k\in Z} |X_k| > \frac{\varepsilon}{10j}\Big) + \frac{24^j}{\varepsilon}\int_{\varepsilon}^{\infty}\bigg(\mathsf{P}\Big(\Big|\sum_{k\in Z} X_k^*\Big| > \frac{x}{4j}\Big)\bigg)^j \mathrm{d}x, \qquad (1.2) \end{aligned}$$

The rest of the paper is organized as follows. Section 2 is the applications of the Hoffmann-Jørgensen inequality for NA random variables. Some extensive results will be shown in the part. In Section 3, some preliminary lemmas are stated first, then we use them to prove the main theorems.

2. Applications to the logarithm law of NA arrays

Now we will apply the NA type Hoffmann-Jørgensen inequality (Theorem 1.1) to the logarithm law of NA arrays. Some general assumptions are given in the first. The results are the extensive version of [15].

Let $\xi_0, \xi_1, \xi_2, \ldots$ be a strictly stationary sequence which is NA. Let $\{X_{nk} : k = 1, \ldots, n,\ n \in N\}$ be a row-wise NA array with

$$(X_{n1}, X_{n2}, \ldots, X_{nn}) \stackrel{d}{=} (\xi_1, \xi_2, \ldots, \xi_n), \qquad n \in N.$$

Assumption 2.1. *For some $t \geq 1$, define $p(x) = g^{-1}(x)/x^t$, for some $c_0 > 0$*

$$p(x_2) \geq c_0 p(x_1), \qquad \text{for all} \quad x_2 \geq x_1 > 0.$$

Assumption 2.2. $\mathsf{E}\big[(g^{-1}(|\xi_0|))^2\big] < \infty$ *and* $\mathsf{E}\,\xi_0 = 0$, *where* $g(n) = \sqrt{n}\phi(n)$, *$\phi(n)$ is a slowly varying function defined later.*

Assumption 2.3. *$\beta(n)$ is decreasing and $\beta(n) = O(n^{-\theta})$ for some $\theta > 5$, where $\beta(n) = -\mathsf{E}(\xi_0\xi_n) = -\mathsf{E}(\xi_j\xi_{j+n})$ for all $j \in N$;*

Assumption 2.4. $0 < \sigma^2 := \mathsf{E}\,\xi_0^2 - 2\sum_{n=1}^{\infty}\beta(n) < \infty.$

The following theorem gives a complete convergence results of NA sequence, it will be used partially in the proof of the next theorem, but is also of independent interest.

Theorem 2.1. *Let $\{X, X_n : n \geq 1\}$ be a zero mean strictly stationary NA sequence, and $S_n = \sum_{k=1}^{n} X_k$. Suppose $g(x)$ satisfies* Assumption 2.1, *for $r \geq 1$, $rt > 1$. We assume the following conditions hold.*

(1) *There exist $\varepsilon_0 \geq 0$ and $m \in N$ such that for every $\varepsilon > \varepsilon_0$,*

$$\sum_{n=1}^{\infty} \frac{n^{r-2}\sqrt{[n/m]\,\mathsf{E}\,\xi_1^2}}{g(n)} \exp\left(-\frac{\varepsilon^2 g^2(n)}{2[n/m]\,\mathsf{E}\,\xi_1^2}\right) < \infty; \tag{2.1}$$

(2) $\mathsf{E}(g^{-1}(|X|))^r < \infty$, $M^*_{[n/m]}/g(n) \xrightarrow{\mathsf{P}} 0$ *for m in* (1) *and*

$$\sup_{n\geq 1} \mathsf{P}\big(M^*_{[n/m]} \geq xg(n)/(12j)\big) = O(x^{-\delta}) \quad \text{as } x \to \infty \qquad \text{for some} \quad \delta > 0;$$

(3) $\sum\limits_{n=1}^{k} n^{r-1}/g(n) \le Ck^r/g(k)$.

Then, we have

$$\sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\Big(\max_{1\le k\le n}|S_k| \ge \varepsilon g(n)\Big) < \infty \qquad \text{for every} \quad \varepsilon > \varepsilon_0. \tag{2.2}$$

Corollary 2.1. *Let $\{X, X_n : n \ge 1\}$ be a strictly stationary NA sequence , and $S_n = \sum\limits_{k=1}^{n} X_k$. Assume $r > 1$ and $\mathsf{E}\big(|X|^{2r}/\log^+|X|\big)^r < \infty$. Then*

$$\sum_{n\ge 3} n^{r-2} P\Big(\max_{1\le k\le n}|S_k| > \varepsilon\sqrt{n\log n}\,\Big) < \infty \qquad \text{for every} \quad \varepsilon > \sigma\sqrt{2(r-1)},$$

where $\sigma^2 = \mathsf{E}\,X^2 + 2\sum\limits_{k=1}^{\infty}\mathsf{E}\,X_1X_{k+1}$.

Corollary 2.2. *Let $\{X, X_n : n \ge 1\}$ be a strictly stationary sequence which is NA and $S_n = \sum\limits_{k=1}^{n} X_k$. Assume $\mathsf{E}\,X^2 < \infty$. Then*

$$\sum_{n\ge 3}\frac{1}{n} P\Big(\max_{1\le k\le n}|S_k| > \varepsilon\sqrt{n\log\log n}\,\Big) < \infty \qquad \text{for every} \quad \varepsilon > \sqrt{2}\sigma,$$

where $\sigma^2 = \mathsf{E}\,X^2 + 2\sum\limits_{k=1}^{\infty}\mathsf{E}\,X_1X_{k+1}$.

Proof of Corollaries 2.1 and 2.2. By noting that $\mathsf{E}\,\xi_1^2/m \to \sigma^2$, and checking the conditions in Theorem 2.1, we get the corollaries immediately. □

By employing the NA type Hoffmann-Jørgensen inequality (Theorem 1.1), the next two theorems extent the results of Su, Hu and Liang [15].

Theorem 2.2. *Let $\{\xi_0, \xi_n : n \in N\}$ and $\{X_{nk} : k = 1,\dots,n,\ n \in N\}$ be defined as above. Suppose $\sum\limits_{n=1}^{k} n^{r-1}/g(n) \le Ck^r/g(k)$ and $g(x)$ satisfy* Assumption 2.1. *Then*

(1) *under* Assumptions 2.2 *and* 2.4, *we have*

$$\sum_{n=1}^{\infty}\frac{1}{\phi(n)}\exp\Big(-\varepsilon^2\frac{\phi^2(n)}{2}\Big) < \infty \qquad \text{for every} \quad \varepsilon > \varepsilon_0$$

implies

$$\limsup_{n\to\infty}\frac{|S_n|}{\sigma\sqrt{n}\phi(n)} \le \varepsilon_0\,; \tag{2.3}$$

(2) *under the* Assumptions 2.2–2.4, *we have*

$$\sum_{n=1}^{\infty} \frac{n}{\phi(n^2)} \exp\Big(-\frac{\phi^2(n^2)}{2}\Big) = \infty$$

implies

$$\limsup_{n\to\infty} \frac{|S_n|}{\sigma\sqrt{n}\phi(n)} \ge 1. \tag{2.4}$$

Remark 2.1. Let $\phi(n) = \sqrt{2\log n}$ in (1) and $(1-\varepsilon)\sqrt{2\log n}$ in (2), we get

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2\sigma^2 n\log n}} = 1$$

under the Assumptions 2.2–2.4. This is the case which Su, Hu and Liang [15] have studied. Let $\phi(n) = \sqrt{n}$ in (i), we have $\lim\limits_{n\to\infty} S_n/n = 0$ under $\mathsf{E}\,\xi_0^4 < \infty$. In fact, $\mathsf{E}\,\xi_0^4 < \infty$ is also necessary for $\lim\limits_{n\to\infty} S_n/n = 0$.

By applying Theorem 2.2, we can get a nonclassical logarithm law of NA arrays. For more about the nonclassical LIL, we refer to Klesov's work [7].

Let $M > 1$ and define

$$N_0 = 1, \qquad N_k = \min\{n:\ \log n \ge M\log N_{k-1}\}, \quad k \ge 1,$$
$$b(n) = M\log N_k, \qquad N_k \le n < N_{k+1}, \quad k \ge 0.$$

Obviously, we have

$$\log n \le b(n) \le M\log n, \quad n \ge 1. \qquad \limsup_{n\to\infty} \frac{b(n)}{\log n} = M, \qquad \liminf_{n\to\infty} \frac{b(n)}{\log n} = 1.$$

Theorem 2.3. *Under the conditions of* Theorem 2.2, *and let $b(n)$ be defined as above, we have*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sigma\sqrt{2nb(n)}} = 1.$$

3. Proofs

3.1. Preliminary lemmas

In this section, we state some lemmas which will be used in establishing the general results. The first one is a useful version of the maximal inequality of Hoffmann-Jørgensen (c.f., see Li, Rao, Jiang and Wang [8]).

Lemma 3.1. *Let $\{V_k:\ 1 \le k \le n\}$ be a finite sequence of independent symmetric real random variables and $S_n = \sum_{k=1}^{n} V_k$. Then for each integer $j \ge 1$, there exist positive numbers C_j and D_j depending only on j such that for all $t > 0$,*

$$\mathsf{P}\big(|S_n| \ge 2jt\big) \le C_j\,\mathsf{P}\Big(\max_{1\le k\le n} |V_k| \ge t\Big) + D_j\big(\mathsf{P}\left(|S_n| \ge t\right)\big)^j.$$

Using [4: Theorem 2.12, p. 71], we obtain the following contraction lemma (Lemma 3.2).

Lemma 3.2 (Contraction Lemma). *Let $X = (X_1, X_2, \dots, X_n)$ be a symmetric random vector and let $\phi_1, \dots, \phi_n \colon \mathbf{R} \to \mathbf{R}$ be odd Borel functions satisfying $|\phi_k(x)| \le x$ for all $x \in \mathbf{R}$ and all $k = 1, \dots, n$. Then we have*

$$\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \phi_k(X_k)\Big| > 5c\bigg) \le 6\,\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} X_k\Big| > c\bigg), \qquad \textit{for all} \quad c > 0.$$

P r o o f. Let $c > 0$ be given and let $(\varepsilon_1, \dots, \varepsilon_n)$ be a Bernoulli vector which is independent of $(X_1, \dots, X_n)$. By [4: Theorem 2.15, p. 71], We have

$$\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k\phi_k(y_k)\Big| > 5c\bigg) \le 6\,\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k y_k\Big| > c\bigg), \quad \text{for all} \quad (y_1, \dots, y_n) \in \mathbf{R}^n.$$

Since $(X_1, X_2, \dots, X_n)$ is symmetric and independent of $(\varepsilon_1, \dots, \varepsilon_n)$, we have that $(X_1, X_2, \dots, X_n)$ and $(\varepsilon_1 X_1, \dots, \varepsilon_n X_n)$ are equidistributied and notice that ϕ_k is odd, we have $\phi_k(\varepsilon_k X_k) = \varepsilon_k \phi_k(X_k)$. Hence, if P_X denote the distribution of X, we have

$$\begin{aligned}
&\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \phi_k(X_k)\Big| > 5c\bigg) \\
&= \mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \phi_k(\varepsilon_k X_k)\Big| > 5c\bigg) = \mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k\phi_k(X_k)\Big| > 5c\bigg) \\
&= \int_{\mathbf{R}^n} \mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k\phi_k(y_k)\Big| > 5c\bigg)\,\mathsf{P}_X(\mathrm{d}y) \\
&\le 6\int_{\mathbf{R}^n} \mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k y_k\Big| > c\bigg)\,\mathsf{P}_X(\mathrm{d}y) \\
&= 6\,\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} \varepsilon_k X_k\Big| > c\bigg) = 6\,\mathsf{P}\bigg(\Big|\sum_{k=1}^{n} X_k\Big| > c\bigg).
\end{aligned}$$

□

Lemma 3.3. *For each $n \geq 1$, let $\{X_{nk} : k \in Z\}$ be a sequence of NA real random variables, and $\{X^*_{nk} : k \in Z\}$ be a sequence of independent real random variables such that X^*_{nk} has the same distribution as X_{nk} for each $n \geq 1$, $k \in Z$, and the series $\sum_{k \in Z} X^*_k$ converges a.s. Suppose that*

$$S^*_n = \sum_{k \in Z} X^*_{nk} \xrightarrow{\mathsf{P}} 0,$$

$$\sup_{k \in Z} |\operatorname{\mathsf{med}}(X_{nk})| \to 0.$$

Then, we have

$$S_n = \sum_{k \in Z} X_{nk} \xrightarrow{\mathsf{P}} 0.$$

Proof. By the proof of [1: Theorem 1, p. 326] of Chow and Teicher with some necessary modifications, we can get that $S^*_n = \sum_{k \in Z} X^*_{nk} \xrightarrow{\mathsf{P}} 0$ implies

$$\sum_{k \in Z} \mathsf{P}(|X_{nk}| \geq \varepsilon) \to 0,$$

$$\sum_{k \in Z} \mathsf{Var}(X_{nk} I\{|X_{nk}| < \varepsilon\}) \to 0,$$

$$\sum_{k \in Z} \mathsf{E}\, X_{nk} I\{|X_{nk}| < \varepsilon\} \to 0.$$

And by the classical truncated method, we can get $S_n = \sum_{k \in Z} X_{nk} \xrightarrow{\mathsf{P}} 0$. $\square$

The next lemma comes from Li and Spătaru [9].

Lemma 3.4. *Let $\{U_n : n \geq 1\}$ be a sequence of random variables such that $U_n \xrightarrow{P} 0$, $\{U'_n : n \geq 1\}$ be an independent copy of $\{U_n : n \geq 1\}$, and let $\{a_n : n \geq 1\}$ be a sequence of nonnegative numbers, suppose $q > 0$ and $\delta \geq 0$. The following are equivalent:*

(i)
$$\int_{\varepsilon}^{\infty} \Big(\sum_{n \geq 1} a_n P(|U_n| > x^q) \Big) \mathrm{d}x < \infty, \qquad \varepsilon > \delta^{1/q};$$

(ii) *there exists $a > \delta^{1/q}$ such that*

$$\int_{a}^{\infty} \Big(\sum_{n \geq 1} a_n P(|U_n - U'_n| > x^q) \Big) \mathrm{d}x < \infty,$$

and

$$\sum_{n \geq 1} a_n P(|U_n| > \varepsilon) < \infty, \qquad \varepsilon > \delta.$$

The following result on the approximation of the sums of the independent random variables is due to Sakhanenko ([11], [12] and [13]).

Lemma 3.5. *For any sequence of independent random variables $\{\xi_n:\ n\ge 1\}$ with zero mean and finite variance, there exist a sequence of independent normal variables $\{\eta_n:\ n\ge 1\}$ with $\mathsf{E}\,\eta_n=0$ and $\mathsf{E}\,\eta_n^2=\mathsf{E}\,\xi_n^2$ such that, for all $Q>2$ and $y>0$,*

$$\mathsf{P}\Big(\max_{k\le n}\Big|\sum_{i=1}^{k}\xi_i-\sum_{i=1}^{k}\eta_i\Big|\ge y\Big)\le (AQ)^Q y^{-Q}\sum_{i=1}^{n}\mathsf{E}\,|\xi_i|^Q,$$

whenever $\mathsf{E}\,|\xi_i|^Q<\infty$, $i=1,\dots,n$. Here, A is a universal constant.

3.2. The proofs of the theorems

Proof of Theorem 1.1. Let $Y_k=(-\varepsilon/(10j))\vee X_k\wedge(\varepsilon/(10j))$. We assume the right of the inequality is finite. We have

$$\mathsf{P}\Big(\sup_{l\in Z}\Big|\sum_{k\le l}X_k\Big|>2\varepsilon\Big)\le \mathsf{P}\Big(\sup_{k\in Z}|X_k|>\frac{\varepsilon}{10j}\Big)+\mathsf{P}\Big(\sup_{l\in Z}\Big|\sum_{k\le l}Y_k\Big|>2\varepsilon\Big),$$

and by the comparison theorem of Shao [14],

$$\begin{aligned}
&\mathsf{P}\Big(\sup_{l\in Z}\Big|\sum_{k\le l}Y_k\Big|>2\varepsilon\Big)=\lim_{m,t\to\infty}\mathsf{P}\Big(\sup_{-t\le l\le m}\Big|\sum_{-t\le k\le l}Y_k\Big|>2\varepsilon\Big)\\
&\le \lim_{m,t\to\infty}\Big\{\mathsf{P}\Big(\sup_{-t\le l\le m}\sum_{-t\le k\le l}Y_k>\varepsilon\Big)+\mathsf{P}\Big(\sup_{-t\le l\le m}\Big(-\sum_{-t\le k\le l}Y_k\Big)>\varepsilon\Big)\Big\}\\
&\le \lim_{m,t\to\infty}\frac{1}{\epsilon}\Big\{\mathsf{E}\Big\{\max_{-t\le l\le m}\sum_{-t\le k\le l}Y_k-\varepsilon\Big\}_+ +\mathsf{E}\Big\{\max_{-t\le l\le m}\Big(-\sum_{-t\le k\le l}Y_k\Big)-\varepsilon\Big\}_+\Big\}\\
&\le \lim_{m,t\to\infty}\frac{1}{\epsilon}\Big\{\mathsf{E}\Big\{\max_{-t\le l\le m}\sum_{-t\le k\le l}Y_k^*-\varepsilon\Big\}_+ +\mathsf{E}\Big\{\max_{-t\le l\le m}\Big(-\sum_{-t\le k\le l}Y_k^*\Big)-\varepsilon\Big\}_+\Big\}\\
&\le \lim_{m,t\to\infty}\frac{2}{\varepsilon}\mathsf{E}\Big\{\max_{-t\le j\le m}\Big|\sum_{-t\le k\le l}Y_k^*\Big|-\varepsilon\Big\}_+\\
&= \lim_{m,t\to\infty}\frac{2}{\varepsilon}\int_{\varepsilon}^{\infty}\mathsf{P}\Big(\max_{-t\le l\le m}\Big|\sum_{-t\le k\le l}Y_k^*\Big|>x\Big)\,\mathrm{d}x\le\frac{4}{\varepsilon}\int_{\varepsilon}^{\infty}\mathsf{P}\Big(\Big|\sum_{k\in Z}Y_k^*\Big|>x\Big)\,\mathrm{d}x,
\end{aligned}$$

where the last inequality follows from Lévy inequality. Since $|Y_k^*|\le\varepsilon/(10j)$, we have $\mathsf{P}(|Y_k^*|\ge x/(2j))=0$ for $x\ge\varepsilon$. Applying Lemma 3.1 with $C_j:=\frac{4^j-1}{3}$ and $D_j:=4^{j-1}$ we get

$$\mathsf{P}\Big(\sup_{l\in Z}\Big|\sum_{k\le l}Y_k\Big|>2\varepsilon\Big)\le\frac{4^j}{\varepsilon}\int_{\varepsilon}^{\infty}\Big(\mathsf{P}\Big(\Big|\sum_{k\in Z}Y_k^*\Big|>\frac{x}{2j}\Big)\Big)^j\,\mathrm{d}x.$$

This proves (1.1). For (1.2), notice that Lemma 3.2 implies

$$\mathsf{P}\Big(\Big|\sum_{k\in Z} Y_k^*\Big| > \frac{x}{2j}\Big) \le 6\,\mathsf{P}\Big(\Big|\sum_{k\in Z} X_k^*\Big| > \frac{x}{10j}\Big),$$

therefore, combining with (1.1) and the Lévy inequality, we fulfill the proof of Theorem 1.1. □

Proof of Theorem 2.1. By Assumption 2.1, we can get

$$\frac{\mathsf{E}\,|X|I\{|X|\ge g(n)\}}{g(n)} \le \frac{\mathsf{E}\,g^{-1}(|X|)I\{|X|\ge g(n)\}}{n}.$$

Hence,

$$\frac{n\,\mathsf{E}(-g(n))\vee X_k\wedge g(n)}{g(n)} \to 0 \qquad \text{as} \quad n\to\infty.$$

Therefore, we have

$$\begin{aligned} I &:= \sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\Big(\max_{1\le k\le n}|S_k| \ge \varepsilon g(n)\Big) \\ &\le \sum_{n=1}^{\infty} n^{r-1}\,\mathsf{P}(|X|\ge g(n)) + C\sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\Big(\max_{1\le k\le n}|S'_k| \ge (1-a)\varepsilon g(n)\Big) \\ &=: I_1 + I_2. \end{aligned}$$

It is easy to see that $I_1<\infty$ since $\mathsf{E}(g^{-1}(|X|))^r<\infty$. And for every $0<a<1$,

$$\begin{aligned} I_2 &\le C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\,\mathsf{E}\Big\{\max_{1\le k\le n}|S'_k| - \frac{1-a}{1+a}\varepsilon g(n)\Big\}_+ \\ &\le C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\,\mathsf{E}\Big\{\max_{1\le k\le [n/m]}|T_k| - (1-a')\varepsilon g(n)\Big\}_+ \\ &\quad + C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\,\mathsf{E}\Big\{\max_{1\le i\le [n/m]+1}\ \max_{(i-1)m+1\le l\le im}\Big|\sum_{k=(i-1)m+1}^{l} X'_k\Big| - a\varepsilon g(n)\Big\}_+ \\ &=: I_3 + I_4, \end{aligned}$$

where $1-a' = (1-2a-a^2)/(1+a)$. Using the same argument for proving Theorem 1.1, by [14: Theorem 1] we have

$$\begin{aligned} I_3 &\le C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\,\mathsf{E}\Big\{\max_{1\le k\le [n/m]}|T_k^*| - (1-a')\varepsilon g(n)\Big\}_+ \\ &= C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{(1-a')\varepsilon g(n)}^{\infty}\mathsf{P}\Big(M^*_{[n/m]}\ge x\Big)\,\mathrm{d}x \end{aligned}$$

$$\le C\sum_{n=1}^{\infty} n^{r-2}\,\mathsf{P}\Big(M^*_{[n/m]} \ge (1-a')\varepsilon g(n)\Big) + C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M^*_{[n/m]}\ge x\Big)\,\mathrm{d}x$$
$$=: I_5 + I_6,$$

where $\Delta = Q\varepsilon g(n)$, Q will be specified later. By noting the second condition in Theorem 2.1 and using Lemma 3.1, we get

$$\begin{aligned}
I_6 &\le C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M^{*s}_{[n/m]}\ge x/2\Big)\,\mathrm{d}x\\
&\le C\frac{1}{a}\sum_{n=1}^{\infty}\frac{n^{r-2}}{g(n)}\int_{\Delta}^{\infty}\mathsf{P}\Big(M^{*s}_{[n/m]}\ge x/(6j)\Big)^{j}\,\mathrm{d}x\\
&\le C\frac{1}{a}\sum_{n=1}^{\infty}n^{r-2}\,\mathsf{P}\Big(M^{*}_{[n/m]}\ge Q\varepsilon g(n)/(12j)\Big)\int_{Q\varepsilon}^{\infty}\sup_{n\ge1}\mathsf{P}\Big(M^{*}_{[n/m]}\ge xg(n)/(12j)\Big)^{j-1}\,\mathrm{d}x\\
&\le CI_5,
\end{aligned}$$

where we let Q large enough such that $\Delta > 12jmg(n)$. So we only need to estimate I_4, I_5. For I_5, we apply Lemma 3.4. We have

$$\begin{aligned}
I_5 &\le C\sum_{n=1}^{\infty}n^{r-2}\,\mathsf{P}\Big(\max_{1\le k\le[n/m]}\Big|\sum_{i=1}^{k}Y_i\Big|\ge(1-2a')\varepsilon g(n)\Big)\\
&\quad + C\sum_{n=1}^{\infty}n^{r-2}\,\mathsf{P}\Big(\max_{1\le k\le[n/m]}\Big|\sum_{i=1}^{k}(\xi_i^*-Y_i)\Big|\ge a'\varepsilon g(n)\Big)\\
&=: I_7 + I_8,
\end{aligned}$$

where $Y_i\sim N(0,\mathsf{E}\,\xi_i^2)$. Since $\mathsf{P}(N\ge x)\sim\big(\sqrt{2\pi}x\big)^{-1}\exp(-x^2/2)$ as $x\to\infty$, we have

$$I_7 \le C\sum_{n=1}^{\infty}\frac{n^{r-2}\sqrt{[n/m]\,\mathsf{E}\,\xi_1^2}}{g(n)}\exp\Big(-\frac{(1-2a')^2\varepsilon^2g^2(n)}{2[n/m]\,\mathsf{E}\,\xi_1^2}\Big)<\infty$$

by letting a small enough such that $(1-2a')\varepsilon>\varepsilon_0$, and by Assumption 2.1, we get that for $p\ge t$, $x^{1/p}/g(x)\le Cy^{1/p}/g(y)$ for every $x>y$. Hence

$$\begin{aligned}
I_8 &\le C\sum_{n=1}^{\infty}n^{r-1}\frac{\mathsf{E}\,|X|^qI\{|X|\le g(n)\}}{g^q(n)}\\
&\le C\sum_{n=1}^{\infty}\frac{n^{r-1}}{g^q(n)}\sum_{j=1}^{n}\mathsf{E}\,|X|^qI\{j-1\le g^{-1}(|X|)\le j\}\\
&\le C\sum_{j=1}^{\infty}\mathsf{E}\,|X|^qI\{j-1\le g^{-1}(|X|)\le j\}\sum_{n=j}^{\infty}n^{r-1-q/p}n^{q/p}/g^q(n)
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}\mathsf{E}\,|X|^q I\{j-1\le g^{-1}(|X|)\le j\}\sum_{n=j}^{\infty}n^{r-1-q/p}j^{q/p}/g^q(j)\\
&\le C\sum_{j=1}^{\infty}g^q(j)\,\mathsf{P}\big(j-1\le g^{-1}(|X|)<j\big)\frac{j^r}{g^q(j)}\\
&\le C\,\mathsf{E}(g^{-1}(|X|))^r<\infty.
\end{aligned}
$$

For I_4, by (3) we have

$$
\begin{aligned}
I_4 &\le C\sum_{n=1}^{\infty}\frac{n^{r-1}}{g(n)}\,\mathsf{E}\,|X|I\big\{|X|\ge a\varepsilon g(n)\big\}\\
&\le C\sum_{n=1}^{\infty}\frac{n^{r-1}}{g(n)}\sum_{n=k}^{\infty}g(k)\,\mathsf{P}\big(a\varepsilon g(k)\le |X|\le a\varepsilon g(k+1)\big)\\
&=\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{n^{r-1}}{g(n)}g(k)\,\mathsf{P}\big(a\varepsilon g(k)\le |X|\le a\varepsilon g(k+1)\big)\\
&\le C\sum_{k=1}^{\infty}k^r\,\mathsf{P}\big(a\varepsilon g(k)\le |X|\le a\varepsilon g(k+1)\big)\\
&\le C\,\mathsf{E}\big(g^{-1}(|X|)\big)^r<\infty.
\end{aligned}
$$

This completes the proof. □

Proof of Theorem 2.2.

(1) Let $r=2$ in Theorem 2.1, and use the Borel-Cantelli lemma, we get (2.3).

(2) By the generalized Borel-Cantelli lemma, we only need to show that

$$\sum_{n=1}^{\infty}\mathsf{P}\big(S_n>(1-5\varepsilon)\sigma g(n)\big)=\infty.$$

Following the proof of Su, Hu and Liang [15] and their notations, we only need to show that

$$\sum_{l=1}^{\infty}\mathsf{P}\Big(S_l^{(2)}>\varepsilon\sigma\sqrt{l}\phi(l)\Big)<\infty,\qquad \varepsilon>0, \tag{3.1}$$

where $S_l^{(2)}$ is the sum of $m_l=O(l^{1-\delta/2})$ identically distributed random variables which are NA, and

$$\sum_{n=1}^{\infty}n\,\mathsf{P}\Big(T_n>(1-\varepsilon)\sigma_n\sqrt{n}\phi(n^2)\Big)=\infty. \tag{3.2}$$

where $T_n=\sum\limits_{j=1}^{n}Z_{nj}$, and $Z_{n1},\dots,Z_{nn}$ be i.i.d. random variables with $\mathsf{E}\,Z_{n1}=0$, $\sigma_n^2=\mathsf{E}\,Z_{n1}^2\to\sigma^2$. We show (3.1) first. By Theorem 1.1, we only need to show that

$$\sum_{l=1}^{\infty} m_l \,\mathsf{P}\big(|\xi_0| \ge \varepsilon\sigma g(l)\big) < \infty, \tag{3.3}$$

and

$$\mathsf{P}\Big(|S_l^{(2)}| > x\sigma\sqrt{l}\phi(l)\Big) \le C\frac{1}{x^q l^t} \qquad \text{for some} \quad q,t > 0. \tag{3.4}$$

(3.4) follows $\mathsf{E}\,\xi_0^2 < \infty$ and the Markov's inequality and (3.3) from Assumption 2.2.

Now we prove (3.2). Let $[x]$ stand for the integer part of x, and let $\{d_n : n \in \mathbf{N}\}$ be a sequence of positive integers, $d_n = [n^{1-\delta}]$ for some $0 < \delta < 1$. For every $n \in \mathbf{N}$, set

$$\begin{aligned}
&a_0 = b_0 = 0, \qquad a_1 = n, \qquad b_1 = a_1 + d_n, \ \ldots, \\
&a_{j+1} = b_j + n, \quad b_{j+1} = a_{j+1} + d_n, \qquad j \in \mathbf{N}, \\
&X'_{lk} = X_{lk} I\big\{|X_{lk}| \le g(n^2)\big\} \\
&T'_n = \sum_{j=1}^{n}\Big(\sum_{k=b_{j-1}+1}^{b_j+n} X'_{lk}\Big)^* - \sum_{j=1}^{n}\mathsf{E}\Big(\sum_{k=b_{j-1}+1}^{b_j+n} X'_{lk}\Big)^*.
\end{aligned}$$

Since

$$\sum_{j=1}^{n}\mathsf{E}\Big(\sum_{k=b_{j-1}+1}^{b_j+n} X'_{lk}\Big)^* / g(n^2) \to 0 \qquad \text{as} \quad n \to \infty,$$

we can get for n large enough,

$$\mathsf{P}\big(|T_n| > (1-\varepsilon)\sigma_n g(n^2)\big) \ge \mathsf{P}\big(|T'_n| > (1-2\varepsilon)\sigma_n g(n^2)\big) - n^2\,\mathsf{P}\big(|\xi_0| \ge g(n^2)\big).$$

One can easily show that

$$\sum_{n=1}^{\infty} n^3\,\mathsf{P}\big(|\xi_0| \ge g(n^2)\big) \le C\,\mathsf{E}\big(g^{-1}(|\xi_0|)\big)^2 < \infty.$$

Also,

$$\begin{aligned}
&\mathsf{P}\big(|T'_n| > (1-2\varepsilon)\sigma_n g(n^2)\big) \\
&\ge \mathsf{P}\bigg(\Big|\sum_{i=1}^{n} Y_i\Big| > (1-3\varepsilon)\sigma_n g(n^2)\bigg) - \mathsf{P}\bigg(|T'_n - \sum_{i=1}^{n} Y_i| > \varepsilon\sigma_n g(n^2)\bigg).
\end{aligned}$$

where $Y_i \backsim N\Big(0, \mathsf{E}\Big(\sum\limits_{i=1}^{n}(\xi'_i - \mathsf{E}\,\xi'_i)\Big)^2\Big)$. By Lemma 3.5, we have

$$\sum_{n=1}^{\infty} n\,\mathsf{P}\Big(\Big|T'_n - \sum_{i=1}^{n} Y_i\Big| > \varepsilon\sigma_n g(n^2)\Big) \le C\sum_{n=1}^{\infty} n^3\frac{\mathsf{E}\Big|\sum\limits_{i=1}^{n}(\xi'_i - \mathsf{E}\,\xi'_i)\Big|^Q}{g^Q(n^2)}$$

$$\leq C \sum_{n=1}^{\infty} n^3 \frac{\sum_{i=1}^{n} \mathsf{E}\,|\xi_i' - \mathsf{E}\,\xi_i'|^Q}{g^Q(n^2)} + \sum_{n=1}^{\infty} \frac{n^3 n^{Q/2} (\mathsf{E}\,\xi_1'^2)^{Q/2}}{g^Q(n^2)}$$
$$\leq C\,\mathsf{E}(g^{-1}(|\xi_1|))^2 + C < \infty$$

and

$$\mathsf{P}\biggl(\Bigl|\sum_{i=1}^{n} Y_i\Bigr| > (1-3\varepsilon)\sigma_n g(n^2)\biggr) \geq \mathsf{P}\biggl(\Bigl|\sum_{i=1}^{n} Y_i\Bigr| > (1-4\varepsilon) g(n^2)\biggr)$$
$$\sim C \frac{1}{\phi(n^2)} \exp\biggl(-\frac{(1-4\varepsilon)^2 \phi^2(n^2)}{2}\biggr).$$

We get

$$\sum_{n=1}^{\infty} n\,\mathsf{P}\bigl(|T_n| > (1-\varepsilon)\sigma_n g(n^2)\bigr) = \infty.$$

The proof is now completed. □

P r o o f o f T h e o r e m 2.3. Notice $b(n) \geq \log n$. By Theorem 2.2, it suffices to show that

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{b(n^2)}} \exp\biggl(-\frac{(1-\varepsilon) b(n^2)}{2}\biggr) = \infty \qquad \text{for every} \quad \varepsilon > 0.$$

By noticing that $N_k \geq (N_{k-1})^M$, we have now

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{b(n^2)}} \exp\biggl(-\frac{(1-\varepsilon) b(n^2)}{2}\biggr) = \sum_{k=1}^{\infty} \sum_{n=\sqrt{N_{k-1}}}^{\sqrt{N_k}} \frac{n}{\sqrt{b(n^2)}} \exp\biggl(-\frac{(1-\varepsilon) b(n^2)}{2}\biggr)$$
$$\geq C \sum_{k=1}^{\infty} \frac{N_k - N_{k-1}}{\sqrt{M \log N_{k-1}}} (N_{k-1})^{-(1-\varepsilon)M}$$
$$\geq C \sum_{k=1}^{\infty} \frac{(N_{k-1})^{\varepsilon M}}{\sqrt{M \log N_{k-1}}} = \infty.$$

□

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