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REAL HYPERSURFACES IN A COMPLEX SPACE FORM WITH A CONDITION ON THE STRUCTURE JACOBI OPERATOR

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ABSTRACT. In this paper we classify the real hypersurfaces in a non-flat complex space form with its structure Jacobi operator R_ξ satisfying $(\nabla_X R_\xi)\xi = 0$, for all vector fields X in the maximal holomorphic distribution D. With this result, we prove the non-existence of real hypersurfaces with D-parallel as well as D-recurrent structure Jacobi operator in complex projective and hyperbolic spaces. We can also prove the non-existence of real hypersurfaces with recurrent structure Jacobi operator in a non-flat complex space form as a corollary.

1. Introduction

Let $M_n(c)$ be an n-dimensional non-flat complex space form with constant holomorphic sectional curvature $4c$. It is known that a complete and simply connected non-flat complex space form is either a complex projective space ($c > 0$), denoted by $\mathbb{C}P^n$, or a complex hyperbolic space ($c < 0$), denoted by $\mathbb{C}H^n$. Without loss of generality, we always assume $c = 1$ for $\mathbb{C}P^n$ and $c = -1$ for $\mathbb{C}H^n$.

Let M be a real hypersurface in $M_n(c)$. The Jacobi operator R_X, with respect to a tangent vector field X on an open subset of M, is defined by $R_X(Y) = R(Y, X)X$, for vector fields Y tangent to M, where R is the curvature tensor

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on M. In particular, for the structure vector field $\xi = -JN$, where N is locally a unit normal vector field of M, R_ξ is called the structure Jacobi operator of M. Let D denote the distribution determined by tangent vectors perpendicular to ξ at each point of M. D is called the maximal holomorphic distribution on M. If the structure Jacobi operator satisfies $(\nabla_X R_\xi)Y = 0$, for any vector field X in D and Y tangent to M, then it is said to be D-parallel; and if it satisfies $(\nabla_X R_\xi)Y = \omega(X)R_\xi(Y)$ for any vector field X in D and Y tangent to M, where ω is a 1-form on M, then the structure Jacobi operator is said to be D-recurrent.

In [5], J. D. Pérez, F. G. Santos and Y. J. Suh studied the non-existence of real hypersurfaces with D-parallel structure Jacobi operator in a complex projective space, leaving the complex hyperbolic case open. The concept of D-recurrent structure Jacobi operator was first considered in [6], which gives the original idea to our paper. However, a part of the proof given for the classification of real hypersurfaces in $\mathbb{C}P^n$ with its structure Jacobi operator D-recurrent ([6: §5, p. 221]) cannot be justified. Actually, there does not exist any real hypersurface in $M_n(c)$, for $n > 2$, with D-recurrent structure Jacobi operator as will be shown.

Recently T. Theofanidis and P. J. Xenos proved in [8] that there does not exist any real hypersurface in $M_n(c)$, $n > 2$, with recurrent structure Jacobi operator. They also studied real hypersurfaces in $M_2(c)$ with D-recurrent structure Jacobi operator in [9].

In this article, we consider a condition weaker than D-parallelism and D-recurrence on the structure Jacobi operator and prove the following theorem.

Theorem 1.1. *Let M be a real hypersurface in $M_n(c)$, $n > 2$, satisfying $(\nabla_X R_\xi)\xi = 0$, for all vector fields X in D. Then M is locally congruent to a ruled real hypersurface.*

The above theorem will lead to the proof of non-existence of real hypersurfaces with D-parallel or D-recurrent structure Jacobi operator in $M_n(c)$.

Theorem 1.2. *There does not exist any real hypersurface M in $M_n(c)$, $n > 2$, with its structure Jacobi operator D-recurrent: $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$, for all vector fields X in D and Y tangent to M. Here ω denotes a 1-form on M.*

From Theorem 1.2, we have the following results.

Corollary 1.3. *There does not exist any real hypersurface M in $M_n(c)$, $n > 2$, with its structure Jacobi operator D-parallel: $(\nabla_X R_\xi)Y = 0$, for all vector fields X in D and Y tangent to M.*

Corollary 1.4. *There does not exist any real hypersurface M in $M_n(c)$, $n > 2$, with recurrent structure Jacobi operator, i.e., $\nabla R_\xi = \omega \otimes R_\xi$.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $M_n(c)$, $n > 2$, without boundary. Let N be a locally defined unit normal vector field on M. Denote by ∇ the Levi-Civita connection on M induced from $M_n(c)$. Let $\langle\cdot,\cdot\rangle$ denote the Riemannian metric of M induced from the Riemannian metric of $M_n(c)$ and A be the shape operator of M in $M_n(c)$. Now, we define a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then it is seen that $\langle \xi, X\rangle = \eta(X)$. Furthermore, the set of tensors $(\phi, \xi, \eta, \langle\cdot,\cdot\rangle)$ is an almost contact metric structure on M, i.e., they satisfy the following

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1.$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi$$

and

$$\nabla_X \xi = \phi AX.$$

Let R be the curvature tensor of M. Then the Gauss and Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = c\big\{&\langle Y,Z\rangle X - \langle X,Z\rangle\, Y + \langle \phi Y, Z\rangle \phi X - \langle \phi X, Z\rangle \phi Y \\ &-2\langle \phi X, Y\rangle \phi Z\big\} + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\big\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle \xi\big\}.$$

From the Gauss equation, we have

$$R_\xi Y = c\{Y - \eta(Y)\xi\} + \alpha AY - \eta(AY)A\xi, \tag{2.1}$$

$$\begin{aligned} (\nabla_X R_\xi)Y = &-c\langle Y, \phi AX\rangle\xi - c\eta(Y)\phi AX + (X\alpha)AY + \alpha(\nabla_X A)Y \\ &-\langle (\nabla_X A)Y, \xi\rangle A\xi - \langle Y, A\phi AX\rangle A\xi \\ &-\eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX, \end{aligned} \tag{2.2}$$

for all vector fields X, Y tangent to M.

For a unit vector field U in D, where D is the maximal holomorphic distribution, let D_U denote the collection of tangent vectors orthogonal to ξ, U and ϕU at each point of M. Then according to our assumption $n > 2$, D_U is a subdistribution of D with a positive dimension. Denote $\alpha = \langle A\xi, \xi\rangle$ and $\beta = \|\phi A\xi\|$.

A real hypersurface is said to be Hopf if $A\xi = \alpha\xi$. A ruled real hypersurface is a non-Hopf real hypersurface satisfying the condition $\langle AX, Y\rangle = 0$, for all vector fields X, Y in D. A real hypersurface is ruled if and only if its shape operator could be expressed as $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $AX = 0$ for $X \perp \operatorname{span}\{\xi, U\}$, $\langle U, U\rangle = 1$ and $\beta \neq 0$ on an open dense subset.

We recall the following results for later use.

Theorem 2.1. ([3]) *Let M be a Hopf hypersurface in $M_n(c)$, $n > 1$, and let X be in D. If $AX = \lambda X$, and $A\phi X = \lambda\phi X$, then $\lambda^2 = \alpha\lambda + c$.*

Lemma 2.2. ([1]) *Let M be a real hypersurface in $M_n(c)$, $n > 2$. Suppose $\langle(\phi A - A\phi)X, Y\rangle = 0$ for all vector fields X, Y in D. Let $G_1 = \{x \in M : \|\phi A\phi\|_x \neq 0\}$. Then on G_1, we have $\operatorname{grad}\alpha = \alpha V - 2AV$, where $V = \phi A\xi$. Furthermore, if we suppose $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in D, and β is a nonvanishing function on G_1, and $AV = 0$, then we have $\operatorname{grad}\beta = (c + \beta^2)\phi U$.*

Lemma 2.3. ([2], [7]) *If M is a ruled real hypersurface in $M_n(c)$, $n > 2$, then we have*

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle\}\xi$$

for all tangent vectors X, Y in D, where $V = \phi A\xi$.

3. Some lemmas

We begin with some lemmas in preparation for the proof of Theorem 1.1.

Lemma 3.1. *Let M be a non-Hopf real hypersurface in $M_n(c)$, $n > 2$. Suppose M satisfies $A\xi = \alpha\xi + \beta U$, where β is non-vanishing and U is a unit vector field in D, and there exists a unit vector field Z in $D_U = \{X \in TM : X \perp \xi, U, \phi U\}$ such that $AZ = \lambda Z$ and $A\phi Z = \lambda\phi Z$.*

(a) *If M satisfies*

$$A\phi U = \delta\phi U \tag{3.1}$$

then

$$(\lambda - \delta)(\lambda^2 - \alpha\lambda - c) = \beta\phi U\lambda. \tag{3.2}$$

(b) *If M satisfies*

$$AU = \beta\xi + \gamma U \tag{3.3}$$

then

$$(\lambda - \gamma)(\lambda^2 - \alpha\lambda - c) - \beta^2\lambda = 0. \tag{3.4}$$

(c) *If M satisfies both* (3.1) *and* (3.3) *then*

$$\beta\lambda(\lambda-\delta)-(\lambda-\gamma)\phi U\lambda=0. \tag{3.5}$$

P r o o f. Suppose M satisfies (3.1). Taking inner product in the Codazzi equation

$$(\nabla_Z A)\xi-(\nabla_\xi A)Z=-c\phi Z$$

with ϕZ, we obtain

$$\beta\langle\nabla_Z U,\phi Z\rangle=\lambda^2-\alpha\lambda-c. \tag{3.6}$$

Taking inner product in the Codazzi equation

$$(\nabla_Z A)\phi U-(\nabla_{\phi U}A)Z=0$$

with Z, we obtain

$$(\delta-\lambda)\langle\nabla_Z\phi U,Z\rangle=\phi U\lambda.$$

By using

$$\nabla_Z\phi U=\phi\nabla_Z U,$$

we have (3.2).

Next, suppose M satisfies (3.3). Taking inner product in the Codazzi equation $(\nabla_Z A)\phi Z-(\nabla_{\phi Z}A)Z=-2c\xi$ with ξ and U respectively, we obtain

$$\langle\nabla_{\phi Z}Z-\nabla_Z\phi Z,U\rangle=\frac{2(\lambda^2-\alpha\lambda-c)}{\beta},$$

$$(\lambda-\gamma)(\langle\nabla_{\phi Z}Z,U\rangle-\langle\nabla_Z\phi Z,U\rangle)=2\beta\lambda.$$

Combining these two equations, we obtain (3.4).

Finally, if M satisfies both (3.1) and (3.3) then by using (3.2) and (3.4), we get (3.5). □

It is stated in [4] that there exist no real hypersurfaces M in $\mathbb{C}P^n$, $n>2$, with shape operator given by $A\xi=\xi+\beta U$, $AU=\beta\xi+(\beta^2-1)U$, $AX=-X$ for all $X\perp\xi,U$, where U is a unit vector field in D and β is a nonvanishing function. We shall generalize this statement to $M_n(c)$ and give an alternative proof.

Lemma 3.2. *Suppose M is a real hypersurface in $M_n(c)$, $n>1$, such that the shape operator satisfies $A\xi=c\xi+\beta U$, $AU=\beta\xi+(\beta^2-c)U$, $A\phi U=-c\phi U$, where U is a unit vector field in D and β is a nonvanishing function defined on M. Then $c>0$. Furthermore, if $n>2$, then there exists a vector field X in D_U such that $AX\neq-X$.*

P r o o f. Taking inner product in the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2c\xi$ with U and ξ respectively, we obtain

$$-\beta\langle\nabla_U \phi U, U\rangle + \beta^2 - 3c - 2\phi U\beta = 0, \tag{3.7}$$
$$-\beta\langle\nabla_U \phi U, U\rangle + 3c\beta^2 - 4c^2 + 2c - \phi U\beta = 0.$$

From these two equations we obtain

$$\beta^2 - 3c\beta^2 + 4c^2 - 5c - \phi U\beta = 0. \tag{3.8}$$

Taking inner product in the Codazzi equation $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = cU$ with U and ξ respectively, we obtain

$$\beta^2\langle\nabla_\xi \phi U, U\rangle + 2c^2 - c\beta^2 - \beta^2 - c + \phi U\beta = 0,$$
$$\langle\nabla_\xi \phi U, U\rangle - 4c = 0. \tag{3.9}$$

From these two equations we obtain

$$3c\beta^2 + 2c^2 - \beta^2 - c + \phi U\beta = 0. \tag{3.10}$$

By summing up (3.8) and (3.10), we obtain

$$c(c-1) = 0,$$

which cannot happen when $c = -1$. Hence $c = 1$ and (3.8) becomes

$$\phi U\beta = -2\beta^2 - 1. \tag{3.11}$$

From (3.7) and (3.11), we have

$$\beta\langle\nabla_U \phi U, U\rangle = 5\beta^2 - 1. \tag{3.12}$$

From (3.9) we have

$$\langle\nabla_\xi \phi U, U\rangle = 4. \tag{3.13}$$

Now suppose to the contrary that $AX = -X$ for any vector field X in D_U. For any unit vector field Z in D_U, taking inner product in the Codazzi equation $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$ with $U, \phi U, \xi$ respectively, we have

$$Z\beta + \beta^2\langle\nabla_\xi Z, U\rangle = 0, \tag{3.14}$$
$$\langle\nabla_Z U, \phi U\rangle = 0, \tag{3.15}$$
$$\langle\nabla_\xi Z, U\rangle = 0. \tag{3.16}$$

From (3.14) and (3.16), we obtain

$$Z\beta = 0. \tag{3.17}$$

Take inner product in the Codazzi equation $(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$ with ξ,

$$\langle\nabla_Z \phi U, U\rangle = \langle\nabla_{\phi U} Z, U\rangle;$$

then apply (3.15),

$$\langle \nabla_{\phi U} U, Z \rangle = 0. \tag{3.18}$$

Take inner product in the Codazzi equation $(\nabla_Z A)U - (\nabla_U A)Z = 0$ with U,

$$2Z\beta + \beta \langle \nabla_U Z, U \rangle = 0;$$

then apply (3.17),

$$\langle \nabla_U U, Z \rangle = 0. \tag{3.19}$$

Taking inner product in the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$ with ϕU, we have

$$\langle \nabla_{\phi U} U, \phi U \rangle = 0. \tag{3.20}$$

From (3.18), (3.20) and $\langle \nabla_{\phi U} U, \xi \rangle = -1$, we obtain

$$\nabla_{\phi U} U = -\xi; \tag{3.21}$$

hence

$$\nabla_{\phi U} \phi U = 0. \tag{3.22}$$

From (3.12), (3.19) and $\langle \nabla_U U, \xi \rangle = 0$, we obtain

$$\nabla_U U = \frac{1 - 5\beta^2}{\beta} \phi U; \tag{3.23}$$

hence

$$\nabla_U \phi U = (1 - \beta^2)\xi + \frac{5\beta^2 - 1}{\beta} U. \tag{3.24}$$

From (3.13), (3.16) and $\langle \nabla_\xi U, \xi \rangle = 0$, we obtain

$$\nabla_\xi U = -4\phi U. \tag{3.25}$$

We also have

$$\nabla_{\phi U} \xi = U. \tag{3.26}$$

Let $X = U$, $Y = \phi U$, $Z = U$ in the Gauss equation, we have

$$R(U, \phi U)U = (\beta^2 - 5)\phi U. \tag{3.27}$$

On the other hand, applying (3.11), (3.21), (3.22), (3.23), (3.24), (3.25), (3.26) to

$$R(U, \phi U)U = \nabla_U \nabla_{\phi U} U - \nabla_{\phi U} \nabla_U U - \nabla_{[U, \phi U]} U,$$

we have

$$R(U, \phi U)U = (10\beta^2 - 8)\phi U. \tag{3.28}$$

From (3.27) and (3.28), we see that β is constant. This contradicts (3.11). Hence the proof is completed. □

4. Proof of Theorem 1.1

In this section we give a classification of real hypersurfaces in $M_n(c)$ satisfying the condition

$$(\nabla_X R_\xi)\xi = 0 \tag{4.1}$$

for all vector fields X in D. Note that the condition (4.1) is equivalent to

$$c\phi AX + \alpha A\phi AX - \langle \phi AX, A\xi\rangle A\xi = 0, \tag{4.2}$$

for any tangent vector field X in D. Hence by (4.2), for any vector fields X, Y in D,

$$c\langle (\phi A - A\phi)X, Y\rangle = \langle \phi AX, A\xi\rangle\langle A\xi, Y\rangle + \langle X, A\xi\rangle\langle \phi AY, A\xi\rangle. \tag{4.3}$$

Proposition 4.1. *There does not exist any Hopf hypersurface in $M_n(c)$, $n > 2$, satisfying the condition $(\nabla_X R_\xi)\xi = 0$ for any vector field X in D.*

P r o o f. Suppose M is such a Hopf hypersurface. Equation (4.3) becomes

$$\langle (\phi A - A\phi)X, Y\rangle = 0 \tag{4.4}$$

for all vector fields X, Y tangent to M. Hence $A\phi = \phi A$. Pointwise, we get that $D_\lambda = \{X \in D : \ AX = \lambda X\}$ is ϕ-invariant. Hence by Theorem 2.1, we obtain

$$\lambda^2 = \alpha\lambda + c. \tag{4.5}$$

Let X be a unit principal vector field in D such that $AX = \lambda X$. Then by (4.2), we obtain

$$\lambda(c + \alpha\lambda) = 0.$$

From the above two equations, we get $\lambda^3 = 0$. Hence $\lambda = 0$ and this contradicts (4.5). □

In the rest of this section, let M be a real hypersurface in $M_n(c)$ that satisfies the condition (4.1). We also suppose that $A\xi = \alpha\xi + \beta U$ with $\beta \neq 0$ everywhere on M and U a unit vector field in D.

From (4.3) we obtain that for any vector fields X, Y in D,

$$c\langle (\phi A - A\phi)X, Y\rangle = -\beta^2\{\langle Y, U\rangle\langle \phi U, AX\rangle + \langle X, U\rangle\langle \phi U, AY\rangle\}. \tag{4.6}$$

Proposition 4.2. *For a real hypersurface M in $M_n(c)$, $n > 2$, satisfying the condition* (4.1)*, we have*

(a) $A\phi U = \delta\phi U$*, where δ is a function on M,*

(b) $AU = \beta\xi + \left(1 - \dfrac{\beta^2}{c}\right)\delta U$,

(c) $(\beta^2 - c)(c + \alpha\delta)\delta = 0$.

P r o o f. Let $X = Y = \phi U$ in (4.6), then we have

$$\langle AU, \phi U\rangle = 0. \tag{4.7}$$

If we let $X = U$ and Y an arbitrary vector field in D in (4.6), then

$$c\langle \phi AU - A\phi U, Y\rangle = -\beta^2\langle A\phi U, Y\rangle. \tag{4.8}$$

Since (4.8) also holds for $Y = \xi$, we obtain

$$cA\phi U - c\phi AU = \beta^2 A\phi U. \tag{4.9}$$

By putting $X = \phi U$ and replacing Y with ϕY in (4.6), we have

$$c\langle A\phi U - \phi AU, Y\rangle = \beta^2\langle A\phi U, \phi U\rangle\langle \phi U, Y\rangle, \tag{4.10}$$

for any vector field Y in D. Since (4.10) also holds for $Y = \xi$, we obtain

$$cA\phi U - c\phi AU = \beta^2\langle A\phi U, \phi U\rangle\phi U. \tag{4.11}$$

Putting $X = \phi U$ in (4.2) and taking inner product with U, we get

$$(c - \beta^2)\langle \phi A\phi U, U\rangle + \alpha\langle A\phi A\phi U, U\rangle = 0. \tag{4.12}$$

From (4.9) and (4.11), we get (a) and (b). From (a), (b) and (4.12), we get (c). □

From this proposition we know that D_U is invariant under A. In particular, for any vector fields X, Y in D_U, (4.6) becomes

$$\langle (\phi A - A\phi)X, Y\rangle = 0. \tag{4.13}$$

Therefore, let $D_\lambda = \{X \in D_U : AX = \lambda X\}$ denote a pointwise subspace of D_U; then D_λ is ϕ-invariant.

Let Y be a unit vector field in D_U satisfying $AY = \lambda Y$. From (4.2) we have

$$\lambda(c + \alpha\lambda) = 0. \tag{4.14}$$

From (4.14), we consider the following two cases when M is non-Hopf.

Case 1. *$A = 0$ on D_U.*

Hence $D_U = D_0$ at each point of M, i.e., $AY = 0$ for any vector field Y in D_U. From (3.2), we have $\delta = 0$. Therefore, by Proposition 4.2, M satisfies $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi$, $AX = 0$, for all vector fields X perpendicular to ξ and U. Hence M is a ruled real hypersurface.

Case 2. *$A \neq 0$ on D_U.*

In this case, there exists a unit vector field Y in D_U, such that $AY = \lambda_1 Y$, where $\lambda_1 \neq 0$ on an open subset of M. We identify this subset with M. By (4.14), $\alpha \neq 0$ and $\lambda_1 = -c/\alpha$. From (3.4), we get

$$\alpha^2\beta^2 = \alpha(c - \beta^2)\delta + c^2. \tag{4.15}$$

By applying (4.15) to Proposition 4.2 (c), we get

$$\delta(\alpha^2 - c) = 0. \tag{4.16}$$

We shall consider the following two subcases.

SUBCASE 2-A. $\delta \neq 0$ *at some point of* M.
By continuity, there exists an open subset of M, such that on this open subset, $\delta \neq 0$ at each point. From (4.16), $\alpha^2 = c$ on this open subset. If necessary, we replace the normal vector field N by $-N$, so that $\alpha = c = 1$. Then $\lambda_1 = -1$ on this open subset. Putting $\lambda = -1$ in (3.2), we obtain $\delta = -1$. From (4.14), for any principal unit vector field Y in D_U such that $AY = \lambda Y$, $\lambda(\lambda+1) = 0$. Hence by continuity, λ is constantly 0 or -1. By using (3.2), we see that $\lambda = -1$. This subcase cannot happen according to Lemma 3.2 and Proposition 4.2.

SUBCASE 2-B. $\delta = 0$ *at every point of* M.
By using $\delta = 0$ and (4.6), we obtain $\langle(\phi A - A\phi)X, Y\rangle = 0$ for all X, Y in D. We use the same notation G_1 as in Lemma 2.2. By continuity of the norm, G_1 is an open subset of M. On G_1, by using Lemma 2.2, we have

$$\phi U\alpha = \alpha\beta \tag{4.17}$$

and

$$\phi U\beta = \beta^2 + c. \tag{4.18}$$

From (4.15), we have $\alpha^2\beta^2 = c^2$; then take the covariant derivative in the direction of ϕU,

$$\beta(\phi U\alpha) + \alpha(\phi U\beta) = 0. \tag{4.19}$$

Putting (4.17), (4.18) into (4.19), with the help of $\alpha \neq 0$, we get

$$2\beta^2 + c = 0.$$

Hence β is constant and by (4.18), we have $\beta^2 + c = 0$. This is a contradiction if G_1 is non-empty.

From the above argument we have G_1 must be empty and $\phi A\phi = 0$ must hold everywhere on M, hence M is a ruled real hypersurface. But this contradicts $D_{-c/\alpha} \neq 0$, which holds in the whole Case 2. So Subcase 2-b is impossible.

Now we have proved that if M is a real hypersurface in $M_n(c)$ satisfying $(\nabla_X R_\xi)\xi = 0$, for all vector fields X in D, then the only possibility for M is that it is a ruled real hypersurface. Conversely, it is easy to check that ruled real hypersurfaces satisfy (4.2). So we have completed the proof of Theorem 1.1.

5. Proof of Theorem 1.2

We only need to verify that the structure Jacobi operator R_ξ of ruled real hypersurfaces cannot be D-recurrent. Suppose there exists a ruled real hypersurface with its structure Jacobi operator D-recurrent. Then its shape operator satisfies $\langle AX, Y\rangle = 0$, for vector fields X, Y in D. From Lemma 2.3, it also satisfies $\langle(\nabla_X A)Y, Z\rangle = 0$ for all vector fields X, Y and Z in D.

We consider X, Y in D for (2.2). Taking inner product on both sides of (2.2) with a unit tangent vector Z in D_U, and applying (2.1), we obtain

$$-\eta(AY)\langle(\nabla_X A)Z, \xi\rangle = c\omega(X)\langle Y, Z\rangle.$$

It follows from Lemma 2.3 that this equation becomes

$$c\eta(AY)\langle\phi X, Z\rangle = c\omega(X)\langle Y, Z\rangle.$$

By putting $Y = U$ and $X = \phi Z$ in the above equation, we obtain $\beta = 0$, which is a contradiction. Hence such a ruled real hypersurface cannot exist.

Remark 5.1. From the proof of Theorem 1.2, we get the following result:

> *In $M_n(c)$, $n > 2$, there does not exist a ruled real hypersurface with its structure Jacobi operator η-recurrent, i.e., $\langle(\nabla_X R_\xi)Y, Z\rangle = \omega(X)\langle R_\xi Y, Z\rangle$ for all vector fields X, Y, Z in D.*

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