VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0250-6
Math. Slovaca **64** (2014), No. 4, 941–960

COMMON FIXED POINT OF GENERALIZED WEAK CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED B-METRIC SPACES

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(*Communicated by Gregor Dolinar*)

ABSTRACT. We prove some common fixed point results for four mappings satisfying generalized weak contractive condition in partially ordered complete b-metric spaces. Our results extend and improve several comparable results in the existing literature

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1. Introduction and preliminaries

The concept of b-metric space was introduced by Czerwik in [9]. Since then, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in b-metric spaces (see also [5, 6, 23]). Pacurar [19] proved some results on sequences of almost contractions and fixed points in b-metric spaces. Hussain and Shah [12] obtained some results on KKM mappings in cone b-metric spaces. Recently, Khamsi [15] and Khamsi and Hussain [16] have dealt with spaces of this kind, although under different names (in the spaces called "metric-type") and obtained (common) fixed point results. In particular, they showed that most of the new fixed point existence results of contractive mappings defined on such metric spaces are merely copies of the classical ones.

Consistent with [9] and [23], the following definitions and results will be needed in the sequel.

2010 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Keywords: common fixed point, compatible maps, b-metric space, partially ordered set.

Definition 1.1. ([9]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric iff, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$.

Here we present an example to show that in general a b-metric need not necessarily be a metric, (see also [23: p. 264]):

***Example* 1.** Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. we show that ρ is a b-metric with $s = 2^{p-1}$.

Obviously conditions (b_1) and (b_2) of Definition 1.1 are satisfied. If $1<p<\infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right),$$

and hence, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus, for each $x, y, z \in X$ we obtain

$$\begin{aligned}\rho(x, y) = (d(x, y))^p &\leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) = 2^{p-1}(\rho(x, z) + \rho(z, y)).\end{aligned}$$

So condition (b_3) of Definition 1.1 is satisfied and ρ is a b-metric. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$.

Also the following example of a b-metric space is given in [16].

***Example* 2.** ([16]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 \, \mathrm{d}x < \infty.$$

Define $D\colon X \times X \to [0, \infty)$ by

$$D(f, g) = \int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x.$$

Then D satisfies the following properties

(b_4) $D(f, g) = 0$ if and only if $f = g$,

(b_5) $D(f, g) = D(g, f)$, for any $f, g \in X$,

(b_6) $D(f, g) \le 2(D(f, h) + D(h, g))$, for any points $f, g, h \in X$.

M. A. Khamsi [15] also showed that each cone metric space has a b-metric structure. In fact he proved the following interesting result.

Theorem 1.1. ([15]) *Let (X, d) be a metric cone over the Banach space E with the cone P which is normal with the normal constant K. The mapping $D\colon X \times X \to [0, \infty)$ defined by $D(x, y) = |d(x, y)|$ satisfies the following properties*

(b_7) $D(x, y) = 0$ *if and only if* $x = y$,

(b_8) $D(x, y) = D(y, x)$, *for any* $x, y \in X$.

(b_9) $D(x, y) \le K(D(x, z_1) + D(z_1, z_2) + \ldots + D(z_n, y))$, *for any points* $x, y, z_i \in X$, $i = 1, 2, \ldots, n$.

Now let us review some results on weak contraction mappings, related to the existing literature. Alber and Guerre-Delabriere [3] defined weakly contractive mappings on a Hilbert spaces and established a fixed point theorem for such mappings.

Let (X, d) be a metric space. A self-mapping f on X is said to be weakly contractive if

$$d(fx, fy) \le d(x, y) - \varphi(d(x, y)),$$

for all $x, y \in X$, where $\varphi\colon [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function so that $\varphi(t) = 0$ if and only if $t = 0$.

Rhoades [22] obtained the following result.

Theorem 1.2. ([22]) *Let (X, d) be a complete metric space. If $f\colon X \to X$ is a weakly contractive mapping, then f has a unique fixed point.*

Dutta and Choudhury [11] generalized the weak contractive condition and proved the following fixed point theorem which in turn extends Theorem 1.3 and the corresponding result in [3].

Theorem 1.3. ([11]) *Let (X,d) be a complete metric space. If $f\colon X\to X$ satisfies*

$$\psi(d(fx,fy))\leq\psi(d(x,y))-\varphi(d(x,y)),$$

for all $x,y\in X$, where $\psi,\varphi\colon[0,\infty)\to[0,\infty)$ are both continuous and monotone non-decreasing functions with $\psi(t)=\varphi(t)=0$ if and only if $t=0$. Then f has a unique fixed point.

Recently, Zhang and Song [24] introduced generalized φ-weak contractive mappings and obtained the following common fixed point result.

Theorem 1.4. ([24]) *Let (X,d) be a complete metric space. If $f,g\colon X\to X$ satisfies*

$$d(fx,gy)\leq M(x,y)-\varphi(M(x,y)),$$

for all $x,y\in X$, where $\varphi\colon[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)>0$ for $t\in(0,\infty)$, $\varphi(0)=0$, and

$$M(x,y)=\max\bigl\{d(x,y),d(x,fx),d(y,gy),\tfrac{1}{2}[d(x,gy)+d(y,fx)]\bigr\}.$$

Then there exists a unique point $u\in X$ so that $u=fu=gu$.

Dorić [10] extended Theorem 1.2 and obtained the following theorem.

Theorem 1.5. ([10]) *Let (X,d) be a complete metric space and let $f,g\colon X\to X$ be two self-mappings so that*

$$\psi(d(fx,gy))\leq\psi(M(x,y))-\varphi(M(x,y)),$$

for all $x,y\in X$ where

- (c_1) *$\psi\colon[0,\infty)\to[0,\infty)$ is a continuous monotone non-decreasing function with $\psi(t)=0$ if and only if $t=0$,*
- (c_2) *$\varphi\colon[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$,*
- (c_3) *M is the same as taken in* Theorem 1.4.

Then there exists the unique point $u\in X$ such that $u=fu=gu$.

The functions $\psi,\varphi\colon[0,\infty)\to[0,\infty)$ given in the above theorem are called control functions.

The existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [21], and then by Nieto and Lopez [18]. Further results in this direction were proved in [2, 4, 8, 17].

Recently, Radenović and Kadelburg [20] proved the following result.

Theorem 1.6. ([20]) *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $f, g\colon X \to X$ be two weakly increasing maps. Suppose that there exists a pair of control functions ψ and φ so that for every two comparable elements $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)).$$

Then f and g have a common fixed point provided that one of the following conditions is satisfied:

(d_1) *f or g is continuous, or*

(d_2) *if a non-decreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \preceq x$ for all n.*

The purpose of this paper is to obtain a common fixed point theorem for four self-mappings satisfying a generalized weak contractive condition in ordered b-metric spaces.

Our results extend, unify and generalize the comparable results in [10, 17, 20, 24].

We also need the following definitions:

Definition 1.2. ([14]) Let f and g be two self-maps on a nonempty set X. If $w = fx = gx$, for some x in X, then x is called the coincidence point of f and g, where w is called the point of coincidence of f and g.

Definition 1.3. ([14]) Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Definition 1.4. Let X be a nonempty set. Then $(X, d, \preceq)$ is called partially ordered b-metric space if and only if d is a b-metric on a partially ordered set $(X, \preceq)$.

A subset $\mathcal{K}$ of a partially ordered set X is said to be well ordered if every two elements of $\mathcal{K}$ are comparable.

Definition 1.5. ([2]) Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each x in X.

***Example* 3.** ([2]) Let $X = [0, 1]$ be endowed with usual ordering and $f\colon X \to X$ be defined by $fx = \sqrt[3]{x}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in X$. Therefore f is a dominating map.

Definition 1.6. Let $(X, \preceq)$ be a partially ordered set. A mapping f is called dominated if $fx \preceq x$ for each x in X.

***Example* 4.** Let $X = [0, 1]$ be endowed with the usual ordering and $f\colon X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$. Therefore f is a dominated map.

Before stating and proving our results, we need the following definitions and propositions in b-metric space. We recall first the notions of convergence, closedness and completeness in a b-metric space.

Definition 1.7. ([7]) Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to +\infty$.

Proposition 1.1. ([7: Remark 2.1]) *In a b-metric space (X, d) the following assertions hold:*

(p_1) *a convergent sequence has a unique limit,*

(p_2) *each convergent sequence is Cauchy,*

(p_3) *in general, a b-metric is not continuous.*

Definition 1.8. ([7]) Let (X, d) be a b-metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y , i.e.,

$$\overline{Y} = \left\{ x \in X : \text{ there exists a sequence } \{x_n\} \text{ in Y so that } \lim_{n\to\infty} x_n = x \right\}.$$

Taking into account of the above definition, we have the following concepts.

Definition 1.9. ([7]) Let (X, d) be a b-metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$ (i.e. $\overline{Y} = Y$).

Definition 1.10. ([7]) The b-metric space (X, d) is complete if every Cauchy sequence in X converges.

Definition 1.11. Let (X, d) be a b-metric space. Then the pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

2. Common fixed point results

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 2.1. *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y respectively, then we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y),$$

In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z),$$

P r o o f. Using the triangle inequality in a b-metric space it is easy to see that

$$d(x,y) \leq sd(x,x_n) + s^2 d(x_n,y_n) + s^2 d(y_n,y),$$

and

$$d(x_n,y_n) \leq sd(x_n,x) + s^2 d(x,y) + s^2 d(y,y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result. Similarly, using again the triangle inequality the last assertion follows. □

Now we state the main result of this section.

Theorem 2.1. *Let $(X,d,\preceq)$ be an ordered complete b-metric space. Let f, g, S and T be self-maps on X, $\{f,g\}$ and $\{S,T\}$ be dominated and dominating maps, respectively with $fX \subseteq TX$ and $gX \subseteq SX$.*

Suppose that there exist control functions ψ and φ so that $\psi\colon [0,\infty) \to [0,\infty)$ is a continuous monotone non-decreasing function with $\psi(t) = 0$ iff $t = 0$, and $\varphi\colon [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ iff $t = 0$, and for every two comparable elements $x, y \in X$,

$$\psi(s^4 d(fx,gy)) \leq \psi(M_s(x,y)) - \varphi(M_s(x,y)), \tag{2.1}$$

is satisfied where

$$M_s(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy) + d(fx,Ty)}{2s}\right\}. \tag{2.2}$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ we have $u \preceq x_n$ and either

(a_1) *$\{f,S\}$ are compatible, f or S is continuous and $\{g,T\}$ is weakly compatible or*

(a_2) *$\{g,T\}$ are compatible, g or T is continuous and $\{f,S\}$ is weakly compatible,* *then f, g, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$ and $gX \subseteq SX$, we can define inductively the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n+1} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+2} = gx_{2n+1} = Sx_{2n+2}, \qquad n = 0, 1, 2, \dots .$$

By given assumptions, $x_{2n+1} \preceq Tx_{2n+1} = fx_{2n} \preceq x_{2n}$ and $x_{2n} \preceq Sx_{2n} = gx_{2n-1} \preceq x_{2n-1}$. Thus, we have $x_{n+1} \preceq x_n$ for all $n \geq 0$. We suppose that $d(y_{2n}, y_{2n+1}) > 0$ for every n. If not, then for some k, $y_{2k+1} = y_{2k}$, and from (2.1), we obtain

$$\begin{aligned}\psi(d(y_{2k+1}, y_{2k+2})) &\leq \psi(s^4 d(y_{2k+1}, y_{2k+2})) \\ &= \psi(s^4 d(fx_{2k}, gx_{2k+1})) \\ &\leq \psi(M_s(x_{2k}, x_{2k+1})) - \varphi(M_s(x_{2k}, x_{2k+1})),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}M_s(x_{2k}, x_{2k+1}) =\ &\max\Big\{d(Sx_{2k}, Tx_{2k+1}), d(fx_{2k}, Sx_{2k}), d(gx_{2k+1}, Tx_{2k+1}), \\ &\qquad \frac{d(Sx_{2k}, gx_{2k+1}) + d(fx_{2k}, Tx_{2k+1})}{2s}\Big\} \\ =\ &\max\Big\{d(y_{2k}, y_{2k+1}), d(y_{2k+1}, y_{2k}), \\ &\qquad d(y_{2k+2}, y_{2k+1}), \frac{d(y_{2k}, y_{2k+2}) + d(y_{2k+1}, y_{2k+1})}{2s}\Big\} \\ =\ &\max\Big\{0, 0, d(y_{2k+2}, y_{2k+1}), \frac{d(y_{2k+1}, y_{2k+2})}{2s}\Big\} \\ =\ &d(y_{2k+1}, y_{2k+2}).\end{aligned} \tag{2.4}$$

So from (2.3) and (2.4), we obtain

$$\psi(d(y_{2k+1}, y_{2k+2})) \leq \psi(d(y_{2k+1}, y_{2k+2})) - \varphi(d(y_{2k+1}, y_{2k+2})),$$

which gives $\varphi(d(y_{2k+1}, y_{2k+2})) \leq 0$ and so $y_{2k+1} = y_{2k+2}$ which further implies that $y_{2k+2} = y_{2k+3}$. Thus $\{y_n\}$ becomes a constant sequence and y_{2k} is a common fixed point of f, g, S and T.

Now take $d(y_{2n}, y_{2n+1}) > 0$ for each n. Since x_{2n} and x_{2n+1} are comparable, from (2.1) we have

$$\begin{aligned}\psi(d(y_{2n+1}, y_{2n+2})) &\leq \psi(s^4 d(y_{2n+1}, y_{2n+2})) \\ &= \psi(s^4 d(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})).\end{aligned} \tag{2.5}$$

Hence

$$d(y_{2n+1}, y_{2n+2}) \leq M_s(x_{2n}, x_{2n+1}), \tag{2.6}$$

where

$$\begin{aligned}M_s(x_{2n}, x_{2n+1}) =\ & \max\Big\{d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), \\ & \qquad \frac{d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})}{2s}\Big\} \\ =\ & \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ & \qquad \frac{d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})}{2s}\Big\} \\ \leq\ & \max\Big\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ & \qquad \frac{sd(y_{2n}, y_{2n+1}) + sd(y_{2n+1}, y_{2n+2})}{2s}\Big\} \\ =\ & \max\Big\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ & \qquad \frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\Big\} \\ =\ & \max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}.\end{aligned}$$

If for some n, $d(y_{2n+2}, y_{2n+1}) \geq d(y_{2n+1}, y_{2n}) > 0$, then (2.6) gives that $M_s(x_{2n}, x_{2n+1}) = d(y_{2n+2}, y_{2n+1})$ and from (2.1) we have

$$\begin{aligned}\psi(d(y_{2n+2}, y_{2n+1})) &\leq \psi(s^4 d(y_{2n+2}, y_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})) \\ &= \psi(d(y_{2n+2}, y_{2n+1})) - \varphi(d(y_{2n+2}, y_{2n+1})),\end{aligned}$$

and $\varphi(d(y_{2n+2}, y_{2n+1})) \leq 0$ or equivalently $d(y_{2n+2}, y_{2n+1}) = 0$, a contradiction.

Hence, $M_s(x_{2n}, x_{2n+1}) \le d(y_{2n+1}, y_{2n})$. Since, $M_s(x_{2n}, x_{2n+1}) \ge d(y_{2n+1}, y_{2n})$ therefore, $d(y_{2n+2}, y_{2n+1}) \le M_s(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$. By similar arguments, we have

$$d(y_{2n+3}, y_{2n+2}) \le M_s(x_{2n+2}, x_{2n+1}) = d(y_{2n+2}, y_{2n+1}). \tag{2.7}$$

Therefore $\{d(y_n, y_{n+1})\}$ is a non-increasing sequence and so there exists $r \ge 0$ so that

$$\lim_{n\to\infty} d(y_{n-1}, y_n) = \lim_{n\to\infty} M_s(x_n, x_{n+1}) = r.$$

Suppose that $r > 0$. Since we have

$$\begin{aligned}\psi(d(y_{2n+2}, y_{2n+1})) &\le \psi(s^4 d(y_{2n+2}, y_{2n+1})) \\ &\le \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})).\end{aligned}$$

then taking the upper limit as $n \to \infty$ it implies that

$$\begin{aligned}\psi(r) &\le \psi(r) - \liminf_{n\to\infty} \varphi(M_s(x_{2n}, x_{2n+1})) \\ &= \psi(r) - \varphi(\liminf_{n\to\infty} M_s(x_{2n}, x_{2n+1})) \\ &= \psi(r) - \varphi(r),\end{aligned}$$

a contradiction. Hence

$$\lim_{n\to\infty} d(y_{n-1}, y_n) = 0. \tag{2.8}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. To do this, it is sufficient to show that the subsequence $\{y_{2n}\}$ is Cauchy in X. Assume on the contrary that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ so that n_k is the smallest index for which $2n_k > 2m_k > k$,

$$d(y_{2m_k}, y_{2n_k}) \ge \varepsilon \tag{2.9}$$

and

$$d(y_{2m_k-2}, y_{2n_k}) < \varepsilon. \tag{2.10}$$

Using the triangle inequality in b-metric space and (2.10) we obtain that

$$\begin{aligned}\varepsilon &\le d(y_{2m_k}, y_{2n_k}) \\ &\le sd(y_{2m_k-2}, y_{2n_k}) + s^2 d(y_{2m_k-1}, y_{2m_k-2}) + s^2 d(y_{2m_k}, y_{2m_k-1}), \\ &< \varepsilon s + s^2 d(y_{2m_k-1}, y_{2m_k-2}) + s^2 d(y_{2m_k}, y_{2m_k-1}).\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.8) we obtain

$$\varepsilon \le \limsup_{k\to\infty} d(y_{2m_k},\ y_{2n_k}) \le \varepsilon s. \tag{2.11}$$

Also

$$\varepsilon \le d(y_{2m_k}, y_{2n_k}) \le sd(y_{2m_k}, y_{2m_k-1}) + sd(y_{2m_k-1}, y_{2n_k}).$$

Hence

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}).$$

On the other hand, we have

$$d(y_{2m_k-1}, y_{2n_k}) \leq sd(y_{2m_k-1}, y_{2m_k}) + sd(y_{2m_k},\ y_{2n_k}).$$

So from (2.8) and (2.11) we have

$$\limsup_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) \leq s \limsup_{k\to\infty} d(y_{2m_k},\ y_{2n_k}) \leq \varepsilon s^2.$$

Consequently,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m_k-1}, y_{2n_k}) \leq \varepsilon s^2. \tag{2.12}$$

Similarly

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(y_{2n_k+1}, y_{2m_k-1}) \leq \varepsilon s^3, \tag{2.13}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(y_{2n_k+1}, y_{2m_k}). \tag{2.14}$$

Using (2.11), (2.12) and (2.13) we get

$$\begin{aligned} \frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} &= \min\left\{\frac{\varepsilon}{s}, \frac{\varepsilon + \frac{\varepsilon}{s^2}}{2s}\right\} \\ &\leq \max\left\{\limsup_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}),\right. \qquad (2.15) \\ &\qquad \left.\frac{\limsup_{k\to\infty} d(y_{2m_k},\ y_{2n_k}) + \limsup_{k\to\infty} d(y_{2n_k+1},\ y_{2m_k-1})}{2s}\right\} \\ &\leq \max\left\{\varepsilon s^2, \frac{\varepsilon s + \epsilon s^3}{2s}\right\} = \varepsilon s^2. \end{aligned}$$

By the definition of $M_s(x, y)$ and from (2.8) and (2.15) we obtain

$$\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} \leq \limsup_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1}) \leq \varepsilon s^2. \tag{2.16}$$

Indeed we have,

$$\begin{aligned} &M_s(x_{2n_k}, x_{2m_k-1}) \\ &= \max\left\{d(Sx_{2n_k}, Tx_{2m_k-1}), d(fx_{2n_k}, Sx_{2n_k}), d(gx_{2m_k-1}, Tx_{2m_k-1}),\right. \\ &\qquad \left.\frac{d(Sx_{2n_k}, gx_{2m_k-1}) + d(fx_{2n_k}, Tx_{2m_k-1})}{2s}\right\} \end{aligned}$$

$$= \max\bigg\{d(y_{2n_k}, y_{2m_k-1}), d(y_{2n_k+1}, y_{2n_k}), d(y_{2m_k}, y_{2m_k-1}), \frac{d(y_{2n_k}, y_{2m_k}) + d(y_{2n_k+1}, y_{2m_k-1})}{2s}\bigg\}.$$

Taking the upper limit as $n \to \infty$, and using (2.8) and (2.15) we get

$$\begin{aligned}\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} &\le \limsup_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1}) \\ &= \max\bigg\{\limsup_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}), 0, 0, \\ &\qquad \frac{\limsup_{k\to\infty} d(y_{2m_k},\ y_{2n_k}) + \limsup_{k\to\infty} d(y_{2n_k+1},\ y_{2m_k-1})}{2s}\bigg\} \le \ \varepsilon s^2.\end{aligned}$$

Similarly, we obtain

$$\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} \le \liminf_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1}) \le \varepsilon s^2. \tag{2.17}$$

As,

$$\begin{aligned}\psi(s^4 d(y_{2n_k+1}, y_{2m_k})) &= \psi(s^4 d(fx_{2n_k}, gx_{2m_k-1})) \\ &\le \psi(M_s(x_{2n_k}, x_{2m_k-1})) - \varphi(M_s(x_{2n_k}, x_{2m_k-1})),\end{aligned}$$

taking the upper limit as $k \to \infty$, and from (2.14) and (2.16) we obtain

$$\begin{aligned}\psi(\varepsilon s^3) &\le \psi\Big(s^4 \limsup_{k\to\infty} d(y_{2n_k+1}, y_{2m_k})\Big) \\ &\le \psi\Big(\limsup_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1})\Big) - \liminf_{k\to\infty} \varphi(M_s(x_{2n_k}, x_{2m_k-1})) \\ &\le \psi(\varepsilon s^2) - \varphi\Big(\liminf_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1})\Big) \\ &\le \psi(\varepsilon s^3) - \varphi\Big(\liminf_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1})\Big)\end{aligned}$$

which implies that

$$\varphi\Big(\liminf_{k\to\infty} M_s(x_{2n_k}, x_{2m_k-1})\Big) = 0,$$

so $\liminf M_s(x_{2n_k}, x_{2m_k-1}) = 0$, a contradiction to (2.17) and it follows that $\{y_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists $y \in X$ so that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y.$$

Now, we can show that y is a common fixed point of f, g, S and T.

Since S is continuous it follows that

$$\lim_{n\to\infty} S^2x_{2n+2} = Sy, \qquad \lim_{n\to\infty} Sfx_{2n} = Sy.$$

Using the triangle inequality in b-metric space, we have

$$d(fSx_{2n}, Sy) \le s(d(fSx_n, Sfx_n) + d(Sfx_{2n}, Sy)).$$

Since the pair $\{f, S\}$ is compatible, $\lim_{n\to\infty} d(fSx_n, Sfx_n) = 0$. So taking the upper limit when $n \to \infty$ from the above inequality we have

$$\limsup_{n\to\infty} d(fSx_{2n}, Sy) \le s\Big(\limsup_{n\to\infty} d(fSx_n, Sfx_n) + \lim\sup_{n\to\infty} d(Sfx_{2n}, Sy)\Big) = 0.$$

Hence $\lim_{n\to\infty} fSx_{2n} = Sy$.

As $Sx_{2n+2} = gx_{2n+1} \preceq x_{2n+1}$ so from (2.1) we obtain

$$\psi(s^4 d(fSx_{2n+2}, gx_{2n+1})) \le \psi(M_s(Sx_{2n+2}, x_{2n+1})) - \varphi(M_s(Sx_{2n+2}, x_{2n+1})), \tag{2.18}$$

where

$$\begin{aligned} &M_s(Sx_{2n+2}, x_{2n+1}) \\ &= \max\Big\{d(S^2x_{2n+2}, Tx_{2n+1}), d(fSx_{2n+2}, S^2x_{2n+2}), d(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{d(S^2x_{2n+2}, gx_{2n+1}) + d(fSx_{2n+2}, Tx_{2n+1})}{2s}\Big\}. \end{aligned}$$

Now, by using Lemma 2.1 we get

$$\begin{aligned} &\limsup_{n\to\infty} M_s(Sx_{2n+2}, x_{2n+1}) \\ &\le \max\Big\{s^2 d(Sy, y), 0, 0, \frac{s^2 d(Sy, y) + s^2 d(Sy, y)}{2s}\Big\} = s^2 d(Sy, y). \end{aligned}$$

Hence by taking the upper limit in (2.18) and using Lemma 2.1 we obtain

$$\psi(s^2 d(Sy, y)) \le \psi(s^2 d(Sy, y)) - \varphi(s^2 d(Sy, y)),$$

which gives $\varphi(s^2 d(Sy, y)) \le 0$ or equivalently $Sy = y$. Now, since $gx_{2n+1} \preceq x_{2n+1}$ and $gx_{2n+1} \to y$ as $n \to \infty$, then $y \preceq x_{2n+1}$ and from (2.1) we have,

$$\psi(s^4 d(fy, gx_{2n+1})) \le \psi(M_s(y, x_{2n+1})) - \varphi(M_s(y, x_{2n+1})), \tag{2.19}$$

where,

$$\begin{aligned} M_s(y, x_{2n+1}) = \max\Big\{&d(Sy, Tx_{2n+1}), d(fy, Sy), d(gx_{2n+1}, Tx_{2n+1}), \\ &\frac{d(Sy, gx_{2n+1}) + d(fy, Tx_{2n+1})}{2s}\Big\}. \end{aligned}$$

Taking the upper limit as $n \to \infty$ in (2.19) and using Lemma 2.1 we have

$$\begin{aligned}\psi(s^3 d(fy,y)) = \psi\Big(s^4 \frac{1}{s} d(fy,y)\Big) &\le \psi(d(fy,y)) - \varphi(d(fy,y))\\ &\le \psi(s^3 d(fy,y)) - \varphi(d(fy,y))\end{aligned}$$

which implies $\varphi(d(fy,y) \le 0$, so $fy = y$. Since $f(X) \subseteq T(X)$, there exists a point $v \in X$ so that $fy = Tv$. Suppose that $gv \ne Tv$. Since $v \preceq Tv = fy \preceq y$, from (2.1), we have

$$\psi(d(Tv,gv)) = \psi(d(fy,gv)) \le \psi(M_s(y,v)) - \varphi(M_s(y,v)), \tag{2.20}$$

where

$$\begin{aligned}M_s(y,v) &= \max\bigg\{d(Sy,Tv), d(fy,Sy), d(gv,Tv), \frac{d(Sy,gv)+d(fy,Tv)}{2s}\bigg\}\\ &= d(gv,Tv).\end{aligned}$$

So from (2.20) we have

$$\psi(d(Tv,gv)) \le \psi(d(gv,Tv)) - \varphi(d(gv,Tv)),$$

a contradiction. Therefore $gv = Tv$. Since the pair $\{g,T\}$ is weakly compatible, $gy = gfy = gTv = Tgv = Tfy = Ty$ and y is the coincidence point of g and T. Since $Sx_{2n} \preceq x_{2n}$ and $Sx_{2n} \to y$ as $n \to \infty$, it implies that $y \preceq x_{2n}$ and from (2.1), we obtain

$$\psi(s^4 d(fx_{2n},gy)) \le \psi(M_s(x_{2n},y)) - \varphi(M_s(x_{2n},y)), \tag{2.21}$$

where,

$$\begin{aligned}M_s(x_{2n},y) = \max\bigg\{&d(Sx_{2n},Ty), d(fx_{2n},Sx_{2n}), d(gy,Ty),\\ &\frac{d(Sx_{2n},gy)+d(fx_{2n},Ty)}{2s}\bigg\}.\end{aligned}$$

Taking the upper limit as $n \to \infty$ in (2.21) and using Lemma 2.1 we have

$$\begin{aligned}\psi(s^3 d(y,gy)) = \psi\Big(s^4 \frac{1}{s} d(y,gy)\Big) &\le \psi(sd(y,gy)) - \varphi(sd(y,gy))\\ &\le \psi(s^3 d(y,gy)) - \varphi(sd(y,gy))\end{aligned}$$

which implies that $y = gy$. Therefore, $fy = gy = Sy = Ty = y$. The proof is similar when f is continuous. Similarly, if (a_2) holds then the result follows.

Now suppose that the set of common fixed points of f, g, S and T is well ordered. We show that they have a unique common fixed point. Assume on the contrary that, $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$ but $u \neq v$. By assumption, we can apply (2.1) to obtain

$$\begin{aligned}\psi(d(u,v)) = \psi(d(fu,gv)) &\leq \psi(s^4 d(fu,gv))\\ &\leq \psi(M_s(u,v)) - \varphi(M_s(u,v)),\end{aligned}$$

where

$$\begin{aligned}M_s(u,v) &= \max\left\{d(Su,Tv), d(fu,Su), d(gv,Tv), \frac{d(Su,gv)+d(fu,Tv)}{2s}\right\}\\ &= \max\left\{d(u,v), 0, 0, \frac{d(u,v)+d(u,v)}{2s}\right\}\\ &= d(u,v).\end{aligned}$$

Hence

$$\psi(d(u,v)) \leq \psi(d(u,v)) - \varphi(d(u,v)),$$

a contradiction. Therefore $u = v$. The converse is obvious. □

Remark 1. As the referee has pointed out, J. Jachymski [13] showed that the usage of ψ in the contractive conditions is often redundant for several classes of mappings both on metric spaces and ordered metric spaces. Checking the proof of Theorem 4 in [13], it turns out also that in this case (ordered b-metric space) we can replace ψ in (2.1) by the identity map to obtain an equivalent contraction condition, if we add the extra condition $\lim\inf_{t\to\infty} \varphi(t) > 0$ on the control function φ. Indeed, to do this, a similar proof as the proof of Theorem 4 in [13] can be used if we replace there, $D(x,y) = \{(M(x,y), d(Tx,Sy)) : x,y \in X\}$ with $D(x,y) := \{(M_s(x,y), d(fx,gy)) : x,y \in X,\ x \preceq y \text{ or } y \preceq x\}$.

Now we give tow examples to support our result.

***Example* 5.** Let $X = [0,1]$ be endowed with the b-metric $d(x,y) = (|x-y|)^2 = (x-y)^2$, where $s = 2$.

Define self-maps f, g, S and T on X by

$$f(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{4},\\ \frac{1}{16}, & \text{if } x \in \left(\frac{1}{4}, 1\right]. \end{cases}$$

$$gx = 0, \quad \text{for all} \quad x \in X,$$

$$T(x) = \begin{cases} 0, & \text{if } x = 0,\\ x, & \text{if } x \in \left(0, \frac{1}{4}\right],\\ 1, & \text{if } x \in \left(\frac{1}{4}, 1\right]. \end{cases}$$

$$Sx = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{4}, & \text{if } x \in \left(0, \frac{1}{4}\right], \\ 1, & \text{if } x \in \left(\frac{1}{4}, 1\right]. \end{cases}$$

Then f and g are dominated maps and S and T are dominating maps with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, i.e.,

	f is dominated map	*g is dominated map*
for each $x \in X$	$fx \leq x$	$gx \leq x$
$x = 0$	$f(0) = 0$	$g(0) = 0$
$x \in \left(0, \frac{1}{4}\right]$	$fx = 0 < x$	$gx = 0 < x$
$x \in \left(\frac{1}{4}, 1\right]$	$fx = \frac{1}{16} < x$	$gx = 0 < x$

	S is dominating map	*T is dominating map*
for each $x \in X$	$x \leq Sx$	$x \leq Tx$
$x = 0$	$0 = S(0)$	$0 = T(0)$
$x \in \left(0, \frac{1}{4}\right]$	$x \leq \frac{1}{4} = S(x)$	$x = T(x)$
$x \in \left(\frac{1}{4}, 1\right]$	$x \leq 1 = S(x)$	$x \leq 1 = T(x)$

Also, the pair $\{g, T\}$ is compatible, g is continuous and $\{f, S\}$ is weakly compatible.

The control functions $\psi, \varphi \colon [0, \infty) \to [0, \infty)$ are defined as $\psi(t) = \frac{5}{4}t$ and $\varphi(t) = \frac{1}{4}t$ for all $t \in [0, \infty)$.

Now we shall show that f, g, S and T satisfy (2.1). We consider the following cases:

(i) If $x \in \left[0, \frac{1}{4}\right]$ and $y \in [0, 1]$, then $d(fx, gy) = 0$ and (2.1) is satisfied.

(ii) If $x \in \left(\frac{1}{4}, 1\right]$ and $y = 0$, then

$$\begin{aligned} \psi(s^4 d(fx, gy)) &= \psi(16 d(fx, gy)) = \psi\left(\frac{1}{16}\right) = \frac{5}{64} < 1 \\ &= d(Sx, Ty) \leq M_2(x, y) = \psi(M_2(x, y)) - \varphi(M_2(x, y)). \end{aligned}$$

(iii) If $x \in \left(\frac{1}{4}, 1\right]$ and $y \in \left(0, \frac{1}{4}\right]$,

$$\begin{aligned} \psi(s^4 d(fx, gy)) &= \psi(16 d(fx, gy)) = \psi\left(\frac{1}{16}\right) = \frac{5}{64} < \frac{9}{16} \\ &\leq d(Sx, Ty) \leq M_2(x, y) = \psi(M_2(x, y)) - \varphi(M_2(x, y)). \end{aligned}$$

(iv) For $x \in \left(\frac{1}{4}, 1\right]$ and $y \in \left(\frac{1}{4}, 1\right]$,

$$\begin{aligned}\psi(s^4 d(fx, gy)) &= \psi(16d(fx, gy)) = \psi\Big(\frac{1}{16}\Big) = \frac{5}{64} < 1 \\ &= d(gy, Ty) \leq M_2(x, y) = \psi(M_2(x, y)) - \varphi(M_2(x, y)).\end{aligned}$$

Thus (2.1) is satisfied for all $x, y \in X$. Therefore, all conditions of Theorem 2.1 are satisfied. Moreover, 0 is the unique common fixed point of f, g, S and T.

***Example* 6.** Consider the non-negative real numbers $X = [0, \infty)$ equipped with the b-metric $d(x, y) = |x - y|^3$, $x, y \in X$ where $s = 2^{3-1} = 4$ according to Example 1, and suppose that "$\leq$" is the usual ordering on $\mathbb{R}$. It is easy to see that $(X, d, \leq)$ is an ordered complete b-metric space. Let f, g, S and $T\colon X \to X$ be defined by the formulas

$$\begin{aligned}f(x) &= \ln(1 + x), \qquad & g(x) &= \ln\left(1 + \frac{x}{7}\right), \\ T(x) &= \mathrm{e}^x - 1, & S(x) &= \mathrm{e}^{7x} - 1.\end{aligned}$$

For each $x \in X$, we have $1 + x \leq \mathrm{e}^x$ and $1 + \frac{x}{7} \leq \mathrm{e}^x$ so $f(x) = \ln(1 + x) \leq x$, $g(x) = \ln\left(1 + \frac{x}{7}\right) \leq x$, $x \leq \mathrm{e}^x - 1 = T(x)$ and $x \leq \mathrm{e}^{7x} - 1 = S(x)$. Thus f and g are dominated maps and T and S are dominating maps with $f(X) = g(X) = S(X) = T(X) = [0, \infty)$.

Also, the pair $\{g, T\}$ is compatible, g is continuous and $\{f, S\}$ is weakly compatible.

The control functions $\psi, \varphi\colon [0, \infty) \to [0, \infty)$ are defined as $\psi(t) = bt$ and $\varphi(t) = (b - 1)t$, for all $t \in [0, \infty)$ where $1 < b \leq \frac{343}{256}$.

In order to show that f, g, S and T are satisfied in (2.1), using the mean value theorem we have

$$\begin{aligned}\psi\big(s^4 d(f(x), g(y))\big) &= \psi\big(256\,|f(x) - g(y)|^3\big) \\ &= 256b\left|\ln(1 + x) - \ln\left(1 + \frac{y}{7}\right)\right|^3 \leq 343 \cdot \frac{1}{7^3}\,|7x - y|^3 \\ &\leq \left|\mathrm{e}^{7x} - \mathrm{e}^y\right|^3 = |S(x) - T(y)|^3 \\ &\leq M_4(x, y) = \psi\big(M_4(x, y)\big) - \varphi\big(M_4(x, y)\big),\end{aligned}$$

for all $x, y \in X$.

Thus f, g, S and T satisfy all the conditions of Theorem 2.1, moreover, 0 is a unique common fixed point of f, g, S and T.

Corollary 2.1. *Let $(X, d, \preceq)$ be an ordered complete b-metric space. Let f and g be dominated self-maps on X, and suppose that there exist control functions ψ and φ as in* Theorem 2.1 *so that for every two comparable elements $x, y \in X$,*

$$\psi(s^4 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)) \tag{2.22}$$

is satisfied where

$$M_s(x, y) = \max\left\{d(x, y), d(fx, x), d(gy, y), \frac{d(x, gy) + d(fx, y)}{2s}\right\}. \tag{2.23}$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ we have $u \preceq x_n$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Taking S and T as the identity maps on X, the result follows from Theorem 2.1. □

Corollary 2.2. *Let $(X, d, \preceq)$ be an ordered complete b-metric space. Let f and g be dominated self-maps on X, and suppose that $\varphi\colon [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ iff $t = 0$. Also for every two comparable elements $x, y \in X$,*

$$s^4 d(fx, gy) \leq M_s(x, y) - \varphi(M_s(x, y)) \tag{2.24}$$

is satisfied where

$$M_s(x, y) = \max\left\{d(x, y), d(fx, x), d(gy, y), \frac{d(x, gy) + d(fx, y)}{2s}\right\}. \tag{2.25}$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$, it implies that $u \preceq x_n$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. If we take S and T as the identity maps on X and $\psi(t) = t$ for $t \in [0, \infty)$, then from Theorem 2.1 it follows that f and g have a unique common fixed point. □

Remark 2. Since a b-metric is a metric when $s = 1$, so our results can be viewed as the generalization and extension of corresponding results in [10,11,24] and several other comparable results.

Acknowledgement. The authors are grateful to anonymous referee whose constructive and insightful comments have very much helped to improve the quality of the manuscript.

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Received 15. 10. 2011
Accepted 25. 4. 2012

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