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Mathematica Slovaca

DOI: 10.2478/s12175-014-0249-z
Math. Slovaca **64** (2014), No. 4, 931–940

GLOBAL BEHAVIOR OF A HIGHER ORDER DIFFERENCE EQUATION

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(*Communicated by Michal Fečkan*)

ABSTRACT. The aim of this paper is to investigate the global stability and periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D\prod\limits_{i=l}^{k} x_{n-2i}}, \qquad n = 0, 1, 2, \dots,$$

where A, B are nonnegative real numbers, $C, D > 0$ and l, k are nonnegative integers such that $l \leq k$.

1. Introduction and preliminaries

The mathematical modeling of a physical or economical problem very often leads to difference equations (for partial review of the theory of difference equations and their applications see [1, 4, 6, 7, 10]).

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. It is worthwhile to point out that although several approaches have been developed for finding the global character of difference equations, relatively a large number of difference equations have not been thoroughly understood yet [7, 11–13].

Cinar [5] examined the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}, \qquad n = 0, 1, 2, \dots.$$

2010 Mathematics Subject Classification: Primary 39A11.
Keywords: difference equation, periodic solution, globally asymptotically stable.

Xiaofan yang et all [14] investigated the global attractivity of all solutions of the difference equation

$$x_{n+1} = \frac{ax_n + bx_{n-1}}{c + dx_n x_{n-1}}, \qquad n = 0, 1, 2, \dots,$$

where $a \geq o, b, c, d > 0$.

C. H. Gibbons et all [9] investigated the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}, \qquad n = 0, 1, 2, \dots,$$

where $\beta > 0$ and $\alpha, \gamma \geq 0$.

In [2] we have discussed global asymptotic stability of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n^2}, \qquad n = 0, 1, 2, \dots,$$

where A, B are nonnegative real numbers and $C, D > 0$.

In this paper, we study the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D\prod_{i=l}^{k} x_{n-2i}}, \qquad n = 0, 1, 2, \dots, \tag{1.1}$$

where A, B are nonnegative real numbers and $C, D > 0$.

When $l = k = 0$, equation (1.1) reduces to the difference equation

$$x_{n+1} = \frac{A + Bx_{n-1}}{C + Dx_n}, \qquad n = 0, 1, 2, \dots,$$

which has been investigated in [11].

When $A = 0$ equation (1.1) reduces to the difference equation

$$x_{n+1} = \frac{Bx_{n-2k-1}}{C + D\prod_{i=l}^{k} x_{n-2i}}, \qquad n = 0, 1, 2, \dots,$$

which has been investigated as a special case in [3].

We give some preliminaries which will be needed in this paper.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \qquad n = 0, 1, \dots, \tag{1.2}$$

where $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$.

Definition 1.1. ([10]) An *equilibrium point* for equation (1.2) is a point $\bar{x} \in \mathbb{R}$ such that $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$.

Definition 1.2. ([10])

(1) An equilibrium point $\bar{x}$ for equation (1.2) is called *locally stable* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x}-\delta, \bar{x}+\delta[$ is such that $x_n \in]\bar{x}-\varepsilon, \bar{x}+\varepsilon[$ for all $n \in \mathbb{N}$. Otherwise $\bar{x}$ is said to be *unstable*.

(2) The equilibrium point $\bar{x}$ of equation (1.2) is called *locally asymptotically stable* if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in]\bar{x}-\gamma, \bar{x}+\gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

(3) An equilibrium point $\bar{x}$ for equation (1.2) is called a *global attractor* if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.

(4) The equilibrium point $\bar{x}$ for equation (1.2) is called *globally asymptotically stable* if it is locally asymptotically stable and global attractor.

Suppose that f is continuously differentiable in some open neighborhood of $\bar{x}$. Let

$$a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x}), \qquad \text{for} \quad i = 0, 1, \ldots, k$$

denote the partial derivatives of $f(x_n, x_{n-1}, \ldots, x_{n-k})$ with respect to x_{n-i} evaluated at the equilibrium point $\bar{x}$ of equation (1.2). Then the equation

$$z_{n+1} = \sum_{i=0}^{k} a_i z_{n-i}, \qquad n = 0, 1, \ldots, \tag{1.3}$$

is called the linearized equation associated with equation (1.2) about the equilibrium point $\bar{x}$, and the equation

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0 \tag{1.4}$$

is called the characteristic equation associated with equation (1.3) about the equilibrium point $\bar{x}$.

Theorem 1.3. ([10]) *Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation* (1.2)*. Then the following statements are true:*

(1) *If all roots of equation* (1.4) *lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.*

(2) *If at least one root of equation* (1.4) *has absolute value greater than one, then $\bar{x}$ is unstable.*

Theorem 1.4. ([8]) *Assume that $\sum_{i=0}^{k} |a_i| < 1$. Then every root of equation* (1.4) *has absolute value less than one.*

Definition 1.5. ([11]) A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -k, \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} < \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} < \bar{x}.$$

Definition 1.6. ([11]) A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than or equal to the equilibrium $\bar{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either} \quad l = -k, \quad \text{or} \quad l > -k \quad \text{and} \quad x_{l-1} \geq \bar{x}$$

and

$$\text{either} \quad m = \infty, \quad \text{or} \quad m < \infty \quad \text{and} \quad x_{m+1} \geq \bar{x}.$$

The change of variables $x_n = \sqrt[k-l+1]{\frac{C}{D}}\, y_n$ reduces equation (1.1) to the difference equation

$$y_{n+1} = \frac{r + s y_{n-2k-1}}{1 + \prod_{i=l}^{k} y_{n-2i}}, \qquad n = 0, 1, 2, \ldots, \tag{1.5}$$

where $r = \frac{A}{C} \sqrt[k-l+1]{\frac{D}{C}}$, $s = \frac{B}{C}$.

2. Linearized stability analysis

Now we determine the equilibrium points of equation (1.5) and discuss their local asymptotic behavior. It is clear that the values of the equilibrium points depends on r and s.

The equilibrium points of equation (1.5) are the zeros of the function $f(y) = \bar{y}^{k-l+2} + (1-s)\bar{y} - r$ and clearly depends on whether $k - l$ even or odd.

- *case $k - l$ even*: When $s > 1$, equation (1.5) has a unique positive equilibrium point $\bar{y} > \sqrt[k-l+1]{s-1}$. When $s < 1$, equation (1.5) has a unique positive equilibrium point $\bar{y}$ such that

$$\bar{y} > \sqrt[k-l+1]{\frac{1-s}{k-l}} \qquad \text{if} \quad r > (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}$$

and

$$0 < \bar{y} < \sqrt[k-l+1]{\frac{1-s}{k-l}} \qquad \text{if} \quad r < (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}.$$

- *case $k-l$ odd:* When $s > 1$, equation (1.5) has a unique positive equilibrium point $\bar{y} > \sqrt[k-l+1]{s-1}$. When $s < 1$, equation (1.5) has a unique positive equilibrium point $\bar{y}$ such that

$$\bar{y} > \sqrt[k-l+1]{\frac{1-s}{k-l}} \qquad \text{if} \quad r > (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}$$

and

$$0 < \bar{y} < \sqrt[k-l+1]{\frac{1-s}{k-l}} \qquad \text{if} \quad r < (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}.$$

The linearized equation associated with equation (1.5) about $\bar{y}$ is

$$z_{n+1} + \frac{\bar{y}^{k-l+1}}{1+\bar{y}^{k-l+1}} \sum_{i=l}^{k} z_{n-2i} - \frac{s}{1+\bar{y}^{k-l+1}} z_{n-2k-1} = 0, \qquad n = 0, 1, 2, \ldots \tag{2.1}$$

the characteristic equation associated with this equation is

$$\lambda^{2k+2} + \frac{\bar{y}^{k-l+1}}{1+\bar{y}^{k-l+1}} \sum_{i=l}^{k} \lambda^{2k-2i+1} - \frac{s}{1+\bar{y}^{k-l+1}} = 0. \tag{2.2}$$

We summarize the results of this section in the following theorem.

Theorem 2.1. *Assume that $\bar{y}$ is the positive positive equilibrium point of equation* (1.5). *Then the following statements are true:*

(1) *If $s > 1$, then $\bar{y}$ is is a saddle point.*

(2) *If $s < 1$, then*

  (a) *$\bar{y}$ is locally asymptotically stable if $r < (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}$.*

  (b) *$\bar{y}$ is a saddle point if $r > (k-l+1)\Big(\frac{1-s}{k-l}\Big)^{\frac{k-l+2}{k-l+1}}$.*

Proof.

(1) Assume that $s > 1$ and consider the function

$$g(\lambda) = \lambda^{2k+2} + \frac{\bar{y}^{k-l+1}}{1+\bar{y}^{k-l+1}} \sum_{i=l}^{k} \lambda^{2k-2i+1} - \frac{s}{1+\bar{y}^{k-l+1}}.$$

It is clear that the function $g(\lambda)$ has a real root in the interval $(0,1)$ and other real root in the interval $(-\infty, -1)$. So $\bar{y}$ is a saddle point.

(2) Assume that $s < 1$.

(1) If $r < (k-l+1)(1-s)^{\frac{k-l+2}{k-l+1}}$, that is $\bar{y} < \sqrt[k-l+1]{\frac{1-s}{k-l}}$, then using Theorem 1.4 we get the result.
(2) The proof is similar to that of (1) and will be omitted.

$\square$

3. Global behavior of equation (1.5)

This section is devoted to study the oscillation and global stability of the positive equilibrium point $\bar{y}$. Also we investigate the existence of unbounded solutions under certain conditions.

Theorem 3.1. *Assume that $\bar{y}$ denote the unique positive equilibrium of equation* (1.5) *and Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a nontrivial solution of equation* (1.5).

(1) *If either one of the conditions*
 (C_1) $y_{-2k-1}, y_{-2k+1}, \dots, y_{-1} < \bar{y} \le y_{-2k}, y_{-2k+2}, \dots, y_0$
 or
 (C_2) $y_{-2k}, y_{-2k+2}, \dots, y_0 < \bar{y} \le y_{-2k-1}, y_{-2k+1}, \dots, y_{-1}$
 is satisfied.
 Then $\{y_n\}_{n=-2k-1}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.
(2) *Every oscillatory solution of equation* (1.5) *has semicycles of length at most* $2k+1$.

Theorem 3.2. *Assume that $s < 1$ and $r < (k-l+1)(\frac{1-s}{k-l})^{\frac{k-l+2}{k-l+1}}$. Then the positive equilibrium point $0 < \bar{y} < \sqrt[k-l+1]{\frac{1-s}{k-l}}$ is globally asymptotically stable.*

Proof. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of equation (1.5). Then

$$y_{n+1} = \frac{r + s y_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}} < r + s y_{n-1}, \qquad n = 0, 1, 2, \dots.$$

Then there exists a real number $\beta > 0$ such that $y_n < \beta$, $n = 1, 2, \dots$

This implies that

$$y_{n+1} = \frac{r + s y_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}} > \frac{r}{1 + \beta^{k-l+1}}.$$

Let $\lambda = \liminf y_n$ and $\Lambda = \limsup y_n$. Hence we have

$$\frac{r + s\lambda}{1 + \Lambda^{k-l+1}} \le \lambda \le \Lambda \le \frac{r + s\Lambda}{1 + \lambda^{k-l+1}}.$$

This implies that

$$r + s\lambda \leq \lambda + \lambda\Lambda^{k-l+1}$$

and

$$\Lambda + \Lambda\lambda^{k-l+1} \leq r + s\Lambda.$$

Then

$$r\lambda^{k-l} + s\lambda^{k-l+1} \leq \lambda^{k-l+1} + \lambda^{k-l+1}\Lambda^{k-l+1}$$

and

$$\Lambda^{k-l+1} + \Lambda^{k-l+1}\lambda^{k-l+1} \leq r\Lambda^{k-l} + s\Lambda^{k-l+1}.$$

Then we get that

$$r\lambda^{k-l} + \lambda^{k-l+1}(s-1) \leq r\Lambda^{k-l} + \Lambda^{k-l+1}(s-1).$$

That is

$$\lambda^{k-l+1}(1-s) - r\lambda^{k-l} \geq \Lambda^{k-l+1}(1-s) - r\Lambda^{k-l}. \tag{3.1}$$

Consider the function $h(x) = (1-s)x^{k-l+1} - rx^{k-l}$. As $r < (k-l+1)\left(\frac{1-s}{k-l}\right)^{\frac{k-l+2}{k-l+1}}$, we have $\frac{r(k-l)}{(1-s)(k-l+1)} < \bar{y} < \sqrt[k-l+1]{\frac{1-s}{k-l}}$ and $h(x)$ is increasing on $\left(\frac{r(k-l)}{(1-s)(k-l+1)}, \infty\right)$. In view of equation (3.1), we have a contradiction. Therefore $\lambda = \Lambda = \bar{y}$. This completes the proof. □

Lemma 3.3. *Assume that $s > 2$ and $r > 0$. Then the following statements are true:*

(1) *If $x > \sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}}$, then $\sqrt[k-l+1]{s-1} > \frac{r}{x^{k-l+1}-s+1}$.*

(2) *If $x > \sqrt[k-l+1]{s-1}$ and $y > \frac{r}{x^{k-l+1}-s+1}$, then $y > \frac{r+sy}{1+x^{k-l+1}}$.*

Theorem 3.4. *Assume that $s > 2$. Then equation (1.5) has solutions which are neither bounded nor persist.*

P r o o f. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of equation (1.5) with initial conditions

$$\frac{r}{y_{-2k-1}^{k-l+1} - s + 1} < y_0 < y_{-2} < \cdots < y_{-2k} < \sqrt[k-l+1]{s-1}$$

and

$$\sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}} < y_{-2k-1} < y_{-2k+1} < \cdots < y_{-1}.$$

Then

$$y_1 = \frac{r + sy_{-2k-1}}{1 + \prod_{i=l}^{k} y_{-2i}} > \frac{r + sy_{-1}}{s} = \frac{r}{s} + y_{-2k-1},$$

where $\prod_{i=l}^{k} y_{-2i} < s-1$.

$$y_2 = \frac{r+sy_{-2k}}{1+\prod_{i=l}^{k} y_{1-2i}} < \frac{r+sy_{-2k}}{1+y_{-2k-1}^{k-l+1}} < y_{-2k},$$

where $\prod_{i=l}^{k} y_{1-2i} > y_{-2k-1}^{k-l+1}$, $\quad y_{-2k-1} > \sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}}$,
and $\quad y_{-2k} > \frac{r}{y_{-2k-1}^{k-l+1}-s+1}$.

Now consider the subsequences

$$\{y_{(2k+2)n-2k+2j-1}\}_{n=0}^{\infty} \qquad \text{and} \qquad \{y_{(2k+2)n-2k+2j}\}_{n=0}^{\infty},$$

where $0 \le j \le k$.

We claim that for each integer j with $0 \le j \le k$,

$\{y_{(2k+2)n-2k+2j-1}\}_{n=0}^{\infty}$ is monotonically increasing subsequence of $\{y_n\}_{n=-2k-1}^{\infty}$,
$\{y_{(2k+2)n-2k+2j}\}_{n=0}^{\infty}$ is monotonically decreasing subsequence of $\{y_n\}_{n=-2k-1}^{\infty}$.

Proof of the claim: For $n=1$, we have

$$y_{2j+1} = \frac{r+sy_{-(2k+2)+2j+1}}{1+\prod_{i=l}^{k} y_{2j-2i}} > \frac{r+sy_{-(2k+2)+2j+1}}{s} = \frac{r}{s} + y_{-(2k+2)+2j+1},$$

where $\prod_{i=l}^{k} y_{2j-2i} < y_{2j-2k}^{k-l+1} < s-1$.

$$y_{2j+2} = \frac{r+sy_{-(2k+2)+2j+2}}{1+\prod_{i=l}^{k} y_{2j-2i+1}} < \frac{r+sy_{-(2k+2)+2j+2}}{1+y_{2j-2k+1}^{k-l+1}} < y_{-(2k+2)+2j+2},$$

where $\prod_{i=l}^{k} y_{2j-2i+1} > y_{2j-2k+1}^{k-l+1}$, $\quad y_{2j-2k+1} > \sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}}$,
and $\quad y_{-(2k+2)+2j+2} > \frac{r}{y_{-2k-1}^{k-l+1}-s+1} > \frac{r}{y_{2j-2k+1}^{k-l+1}-s+1}$.

Now suppose that for a certain n we have

$$\sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}} < y_{(2k+2)(n-1)-2k+2j-1} < y_{(2k+2)n-2k+2j-1}$$

and

$$\frac{r}{y_{(2k+2)n-2k+2j+1}^{k-l+1}-s+1} < y_{(2k+2)n-2k+2j} < y_{(2k+2)(n-1)-2k+2j} < \sqrt[k-l+1]{s-1}.$$

Then

$$\begin{aligned} y_{(2k+2)(n+1)-2k+2j-1} &= \frac{r + s y_{(2k+2)n-2k+2j-1}}{1 + \prod_{i=l}^{k} y_{(2k+2)n+2j-2i}} \\ &> \frac{r}{s} + y_{(2k+2)n-2k+2j-1} \end{aligned} \tag{3.2}$$

where

$$\prod_{i=l}^{k} y_{(2k+2)n+2j-2i} < y^{k-l+1}_{(2k+2)n+2j-2k} < s-1.$$

Also we have

$$\begin{aligned} y_{(2k+2)(n+1)-2k+2j} &= \frac{r + s y_{(2k+2)n-2k+2j}}{1 + \prod_{i=l}^{k} y_{(2k+2)n+2j-2i+1}} \\ &< \frac{r + s y_{(2k+2)n-2k+2j}}{1 + y^{k-l+1}_{(2k+2)n+2j-2k+1}} < y_{(2k+2)n-2k+2j}, \end{aligned}$$

where

$$\sqrt[k-l+1]{s-1} + \frac{r}{\sqrt[k-l+1]{s-1}} < y_{(2k+2)n-2k+2j+1}$$

and

$$\frac{r}{y^{k-l+1}_{(2k+2)n-2k+2j+1} - s + 1} < y_{(2k+2)n-2k+2j}.$$

Finally using equation (3.2) we have

$$y_{(2k+2)n-2k+2j-1} > \frac{r}{s} + y_{(2k+2)(n-1)-2k+2j-1} > \cdots > n\frac{r}{s} + y_{-2k+2j-1}.$$

It follows that

$$\lim_{n\to\infty} y_{(2k+2)n-2k+2j-1} = \infty, \qquad 0 \le j \le k,$$

and so

$$\lim_{n\to\infty} y_{(2k+2)n-2k+2j} = 0, \qquad 0 \le j \le k.$$

Therefore

$$\lim_{n\to\infty} y_{2n-1} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} y_{2n} = 0.$$

This completes the proof. $\square$

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Received 15. 12. 2011
Accepted 19. 5. 2012

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