VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0246-2
Math. Slovaca **64** (2014), No. 4, 879–892

# ON REFINED HARDY-TYPE INEQUALITIES WITH FRACTIONAL INTEGRALS AND FRACTIONAL DERIVATIVES

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(*Communicated by Ján Borsík*)

ABSTRACT. The aim of this paper is to present new more general Hardy-type inequalities for different kinds of fractional integrals and fractional derivatives.



## 1. Introduction

Let us start with the well known definitions of the Riemann-Liouville fractional integrals, see [5] and [14].

Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^\alpha_{a^+}f$ and $I^\alpha_{b^-}f$ of order $\alpha > 0$ are defined by

$$I^\alpha_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}\,\mathrm{d}y, \qquad (x > a)$$

and

$$I^\alpha_{b^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y-x)^{\alpha-1}\,\mathrm{d}y, \qquad (x < b)$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function, i.e. $\Gamma(\alpha) = \int_0^\infty \mathrm{e}^t t^{\alpha-1}\,\mathrm{d}t$. These integrals are called the left-sided and right-sided fractional integrals respectively.

2010 Mathematics Subject Classification: Primary 26D15, 26D10.
Keywords: Hardy-type inequalities, convex function, kernel, fractional derivative, fractional integrals.
This research is partially funded by Higher Education Commission of Pakistan.
The research of the second and third authors has been fully supported by Croatian Science Foundation under the project 5435.

The first result yields that the fractional integral operators are bounded in $L_p(a,b)$, $-\infty < a < b < \infty$, $1 \le p \le \infty$, that is

$$\|I^{\alpha}_{a^+} f\|_p \le K\|f\|_p, \qquad \|I^{\alpha}_{b^-} f\|_p \le K\|f\|_p \tag{1.1}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

G. H. Hardy proved the inequality (1.1) involving the left-sided fractional integral in one of his early paper, see [8]. The calculation for the constant $K$ is hidden inside the proof. Inequality (1.1) refers to as inequality of G. H. Hardy.

Many mathematicians gave generalizations and improvements of Hardy-type inequalities by giving applications for fractional integrals and fractional derivatives. They discover important and useful Hardy-type inequalities for convex functions as well as for superquadratic functions, (see [4], [6], [9], [11], [12], [13], [16]). In this paper, we obtain some more general Hardy-type inequalities for different kinds of fractional integrals and fractional derivatives like Riemann-Liouville fractional integrals, Caputo fractional derivative, fractional integral of a function with respect to an increasing function, Erdelyi-Kóber fractional integrals and Hadamard-type fractional integrals.

Let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [1], [7].

Let $0 < a < b \le \infty$. By $C^m([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order $m$, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of $\alpha$ (the integer $k$ satisfying $k \le \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of $\alpha$ ($\min\{n \in \mathbb{N} : \ n \ge \alpha\}$). By $L_1(a,b)$ we denote the space of all functions integrable on the interval $(a,b)$, and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on $(a,b)$. Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma$-finite measures and $A_k$ be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y)\, \mathrm{d}\mu_2(y), \tag{1.2}$$

where $k\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f$ is measurable function on $\Omega_2$, and

$$K(x) := \int_{\Omega_2} k(x,y)\, \mathrm{d}\mu_2(y), \qquad x \in \Omega_1. \tag{1.3}$$

The following theorem is given in [15].

**Theorem 1.1.** *Let $0 < p \leq q < \infty$, or $-\infty < q \leq p < 0$, $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with $\sigma$-finite measures, $u$ be a weight function on $\Omega_1$, $k$ be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and $K$ be defined on $\Omega_1$ by (1.3), and that the function $x \mapsto u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on $\Omega_1$ for each fixed $y \in \Omega_2$, and that $v$ is defined on $\Omega_2$ by*

$$v(y) := \left( \int\limits_{\Omega_1} u(x) \left( \frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} \mathrm{d}\mu_1(x) \right)^{\frac{p}{q}} < \infty.$$

*If $\Phi$ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, and $\varphi \colon I \to \mathbb{R}$ is any function, such that $\varphi \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\int\limits_{\Omega_1} u(x) [\Phi(A_k f(x))]^{\frac{q}{p}} \mathrm{d}\mu_1(x) + \frac{q}{p} \int\limits_{\Omega_1} \frac{u(x)}{K(x)} \Phi^{\frac{q}{p}-1}(A_k f(x)) \int\limits_{\Omega_2} k(x,y) r(x,y) \,\mathrm{d}\mu_2(y) \,\mathrm{d}\mu_1(x) \leq \left( \int\limits_{\Omega_2} v(y) \Phi(f(y)) \,\mathrm{d}\mu_2(y) \right)^{\frac{q}{p}}, \tag{1.4}$$

*holds for all measurable functions $f \colon \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$ for all $y \in \Omega_2$, where $A_k$ is defined by (1.2) and $r \colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ is non-negative function defined by*

$$r(x,y) = \big| |\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))| \, |f(y) - A_k f(x)| \big|. \tag{1.5}$$

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a nonnegative measurable function on the actual interval or more general set.

The paper is organized in the following way. After introduction, in Section 2 and Section 3, we give more generalized results related to inequalities of G. H. Hardy for different kind of fractional integrals and fractional derivatives. Also we obtain some particular results for inequalities of G. H. Hardy which are discussed in [11] (see also [10], [12]).

## 2. Hardy-type inequalities for fractional integrals

Let us continue by taking the non-negative difference of the left-hand side and the right-hand side of the inequality given in (1.4) by taking $\Phi \colon \mathbb{R}^+ \to \mathbb{R}^+$,

$\Phi(x) = x^s$, $s \geq 1$ as:

$$\rho(s) = \left( \int_{\Omega_2} v(y) f^s(y) \, \mathrm{d}\mu_2(y) \right)^{\frac{q}{p}} - \int_{\Omega_1} u(x) (A_k f(x))^{\frac{sq}{p}} \, \mathrm{d}\mu_1(x) - \frac{q}{p} \int_{\Omega_1} \frac{u(x)}{K(x)} (A_k f(x))^{s(\frac{q}{p}-1)} \int_{\Omega_2} k(x,y) r(x,y) \, \mathrm{d}\mu_2(y) \, \mathrm{d}\mu_1(x), \tag{2.1}$$

where $r(x,y)$ is defined by (1.5).

Using Theorem 1.1, we will construct new Hardy type inequalities for different fractional integrals and fractional derivatives.

We continue with definitions and some properties of the *fractional integrals of a function $f$ with respect to given function $g$*. For details see e.g. [14: p. 99]:

Let $(a,b)$, $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $g$ be an increasing function on $(a,b)$ and $g'$ be a continuous function on $(a,b)$. The left- and right-sided fractional integrals of a function $f$ with respect to another function $g$ in $(a,b)$ are given by

$$\left(I^{\alpha}_{a+;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) \, \mathrm{d}t}{[g(x) - g(t)]^{1-\alpha}}, \qquad (x > a)$$

and

$$\left(I^{\alpha}_{b-;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) \, \mathrm{d}t}{[g(t) - g(x)]^{1-\alpha}}, \qquad (x < b),$$

respectively.

In the following theorem, our first result involving fractional integral of $f$ with respect to another increasing function $g$ is given. We give results for the Riemann-Liouville fractional integrals and Hadamard-type fractional integrals as an applications of this theorem.

**Theorem 2.1.** *Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, $f \geq 0$, $g$ be increasing function on $(a,b)$ such that $g'$ be continuous on $(a,b)$. Then the following inequality holds true:*

$$0 \leq \rho_1(s) \leq \overline{H}_1(s) - M_1(s) \leq \overline{H}_1(s),$$

*where*

$$\rho_1(s) = \frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left( (I^{\alpha+\frac{p}{q}}_{a+;g} f^s)(b) \right)^{\frac{q}{p}} - \alpha^{\frac{sq}{p}} (\Gamma(\alpha))^{\frac{sq}{p}+1} \left( I^{\alpha\frac{q}{p}(1-s)+1}_{b-;g} [(I^{\alpha}_{a+;g} f)(x)]^{\frac{sq}{p}} \right)(a) - M_1(s),$$

$$M_1(s) = \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p} \int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha(q-p)(1-s)}{p}} \left(I^{\alpha}_{a+;g} f(x)\right)^{s(\frac{q}{p}-1)} \times \left(I^{\alpha}_{a+;g} r_1\right)(x,\cdot)\,\mathrm{d}x,$$

$$r_1(x,y) = \left|\left| f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right)^s \right| - s\left|\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)\right|\right|,$$

*and*

$$\overline{H}_1(s) = (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}}(g(b)-g(a))^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b g'(y) f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}}\,\mathrm{d}x\right].$$

P r o o f. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(a,b)$, $\mathrm{d}\mu_1(x)=\mathrm{d}x$, $\mathrm{d}\mu_2(y)=\mathrm{d}y$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a<y\le x; \\ 0, & x<y\le b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$. For the particular weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x\in(a,b)$, we get $v(y) = \left(\alpha g'(y)\big(g(b)-g(y)\big)^{\alpha-1+\frac{p}{q}}\right)/\left(\left((\alpha-1)\frac{q}{p}+1\right)^{\frac{p}{q}}\right)$, then (2.1) takes the form

$$\rho_1(s) = \frac{\alpha^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)\big(g(x)-g(a)\big)^{\frac{\alpha q(1-s)}{p}} \left(I^{\alpha}_{a+;g} f(x)\right)^{\frac{sq}{p}}\,\mathrm{d}x - M_1(s).$$

Since $\frac{\alpha q}{p}(1-s)\le 0$, $g$ is increasing and $M_1(s)\ge 0$, we obtain that

$$0 \le \rho_1(s) \le \frac{\alpha^{\frac{q}{p}}(g(b)-g(a))^{(\alpha-1)\frac{q}{p}+1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b g'(y) f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}} - (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)}(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b g'(x)(I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}}\,\mathrm{d}x - M_1(s)$$

$$= \overline{H}_1(s) - M_1(s) \le \overline{H}_1(s).$$

This completes the proof. □

**Remark 1.** Similar result can be obtained for the right sided fractional integral of $f$ with respect to another increasing function $g$, but here we omit the details.

Here, we give a first special case for the Riemman-Liouville fractional integral. If $g(x) = x$, then $I^{\alpha}_{a+;x} f(x)$ reduces to $I^{\alpha}_{a+} f(x)$ left-sided Riemann-Liouville fractional integral and the following result follows.

**Corollary 2.1.1.** *Let $0 < p \le q < \infty$, $\alpha > 1 - \frac{p}{q}$, $s \ge 1$, $f \ge 0$. Then the following inequality holds true:*

$$0 \le \rho_2(s) \le \overline{H}_2(s) - M_2(s) \le \overline{H}_2(s),$$

*where*

$$\begin{aligned}\rho_2(s) = {} & \frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left( (I^{\alpha+\frac{p}{q}}_{a^+} f^s)(b) \right)^{\frac{q}{p}} \\ & - \alpha^{\frac{sq}{p}} (\Gamma(\alpha))^{\frac{sq}{p}+1} \left( I^{\alpha\frac{q}{p}(1-s)+1}_{b^-} [(I^{\alpha}_{a^+} f)(x)]^{\frac{sq}{p}} \right)(a) - M_2(s),\end{aligned}$$

$$M_2(s) = \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p} \int_a^b (x-a)^{\frac{\alpha(q-p)(1-s)}{p}} (I^{\alpha}_{a^+} f(x))^{s(\frac{q}{p}-1)} (I^{\alpha}_{a^+} r_2)(x,\cdot)\,\mathrm{d}x,$$

$$\begin{aligned}r_2(x,y) = {} & \left| \left| f^s(y) - \left( \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x) \right)^s \right| \right. \\ & \left. - s \left| \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a_+} f(x) \right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a^+} f(x) \right| \right|,\end{aligned}$$

*and*

$$\begin{aligned}\overline{H}_2(s) = (b-a)^{\frac{\alpha q}{p}(1-s)} & \left[ \frac{\alpha^{\frac{q}{p}} (b-a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left( \int_a^b f^s(y)\,\mathrm{d}y \right)^{\frac{q}{p}} \right. \\ & \left. -(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b (I^{\alpha}_{a^+} f(x))^{\frac{sq}{p}}\,\mathrm{d}x \right].\end{aligned}$$

Now we continue with the definition of the Hadamard-type fractional integrals. Let $(a,b)$ be finite or infinite interval of $\mathbb{R}^+$ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ is given by

$$(J^{\alpha}_{a_+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x}{y} \right)^{\alpha-1} \frac{f(y)\,\mathrm{d}y}{y}, \qquad x > a$$

and

$$(J_{b_-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{y}{x}\right)^{\alpha-1} \frac{f(y)\,\mathrm{d}y}{y}, \qquad x < b$$

respectively.

Notice that the Hadamard fractional integrals of order $\alpha$ are special case of the left- and right-sided fractional integrals of a function $f$ with respect to another function $g(x) = \log x$ in $(a,b)$ where $0 \le a < b \le \infty$, the following result follows.

**Corollary 2.1.2.** *Let $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1 - \frac{p}{q}$, $f \ge 0$. Then the following inequality holds*

$$0 \le \rho_3(s) \le \overline{H}_3(s) - M_3(s) \le \overline{H}_3(s),$$

*where*

$$\rho_3(s) = \frac{(\Gamma(\alpha+1))^{\frac{q}{p}}}{(\alpha-1)\frac{q}{p}+1} \left((J_{a_+}^{\alpha+\frac{p}{q}} f^s)(b)\right)^{\frac{q}{p}} - \alpha^{\frac{sq}{p}} (\Gamma(\alpha))^{\frac{sq}{p}+1} \left(J_{b_-}^{\alpha\frac{q}{p}(1-s)+1} [(J_{a_+}^{\alpha} f)(x)]^{\frac{sq}{p}}\right)(a) - M_3(s),$$

$$M_3(s) = \frac{q(\Gamma(\alpha+1))^{s(\frac{q}{p}-1)+1}}{p} \int_a^b (\log x - \log a)^{\frac{\alpha(q-p)(1-s)}{p}} \left(J_{a_+}^{\alpha} f(x)\right)^{s(\frac{q}{p}-1)} \times (J_{a_+}^{\alpha} r_3)(x,\cdot)\,\frac{\mathrm{d}x}{x},$$

$$r_3(x,y) = \left|\left|f^s(y) - \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right)^s\right| - s\left|\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right|^{s-1} \cdot \left|f(y) - \frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J_{a_+}^{\alpha} f(x)\right|\right|,$$

*and*

$$\overline{H}_3(s) = (\log b - \log a)^{\frac{\alpha q}{p}(1-s)} \left[\frac{\alpha^{\frac{q}{p}} (\log b - \log a)^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1} \left(\int_a^b f^s(y) \frac{\mathrm{d}y}{y}\right)^{\frac{q}{p}} - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b (J_{a_+}^{\alpha} f(x))^{\frac{sq}{p}} \frac{\mathrm{d}x}{x}\right].$$

Now we present definitions and some properties of the *Erdélyi-Kober type fractional integrals.* Some of these definitions and results were presented in Samko et al. in [17].

Let $(a, b)$ $(0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0$, $\sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$(I^{\alpha}_{a_+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)\,\mathrm{d}t}{(x^\sigma - t^\sigma)^{1-\alpha}}, \qquad x > a, \tag{2.2}$$

and

$$(I^{\alpha}_{b_-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\,\mathrm{d}t}{(t^\sigma - x^\sigma)^{1-\alpha}}, \qquad x < b, \tag{2.3}$$

respectively. Integrals (2.2) and (2.3) are called the Erdélyi-Kober type fractional integrals.

Now, we give the following result.

**Theorem 2.2.** *Let $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1 - \frac{p}{q}$, $f \ge 0$ and ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function. Then the following inequality holds true:*

$$0 \le \rho_4(s) \le \overline{H}_4(s) - M_4(s) \le \overline{H}_4(s),$$

*where*

$$\begin{aligned}\rho_4(s) = {} & \frac{\alpha^{\frac{q}{p}} \sigma^{\frac{q}{p}-1}}{(\alpha-1)\frac{q}{p}+1} \left( \int_a^b y^{\sigma-1}\, {}_2F_1(y)\, (b^\sigma - y^\sigma)^{\alpha-1+\frac{p}{q}} f^s(y)\,\mathrm{d}y \right)^{\frac{q}{p}} \\ & - (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1} \left((x^\sigma - a^\sigma)^\alpha\, {}_2F_1(x)\right)^{\frac{q(1-s)}{p}} \\ & \times \left(I^{\alpha}_{a_+;\sigma;\eta} f(x)\right)^{\frac{sq}{p}} \,\mathrm{d}x - M_4(s),\end{aligned}$$

$$\begin{aligned}M_4(s) = {} & \frac{\alpha q \sigma (\Gamma(\alpha+1))^{\frac{sq}{p}}}{p} \int_a^b x^{\sigma\alpha s(\frac{q}{p}-1)+\sigma-\sigma\eta-1} \left((x^\sigma - a^\sigma)^\alpha\, {}_2F_1(x)\right)^{\frac{(q-p)(1-s)}{p}} \\ & \times \left(I^{\alpha}_{a_+;\sigma;\eta} f(x)\right)^{s(\frac{q}{p}-1)} \int_a^x \frac{r_4(x,y) y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} \,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

$$r_4(x,y) = \left|\left| f^s(y) - \left( \frac{\Gamma(\alpha+1)}{\left(1 - \left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a_+;\sigma;\eta} f(x) \right)^s \right|\right.$$

$$- s\left|\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right|^{s-1} \cdot \left| f(y) - \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)\right|\Bigg|,$$

$$\overline{H}_4(s) =$$
$$= (b^{\sigma}-a^{\sigma})^{\alpha\frac{q}{p}(1-s)}\left[\frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1} b^{(\sigma-1)\frac{q}{p}} (b^{\sigma}-a^{\sigma})^{\frac{q(\alpha s-1)+p}{p}}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b {}_2F_1(y) f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}}\right.$$
$$\left. - a^{\sigma\alpha s\frac{q}{p}+\sigma-1}(\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b ({}_2F_1(x))^{\frac{q}{p}(1-s)} \left(I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^{\frac{sq}{p}} \mathrm{d}x\right],$$

$${}_2F_1(x) = {}_2F_1\left(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma}\right)$$

*and*

$${}_2F_1(y) = {}_2F_1\left(\eta,\alpha;\alpha+1;1-\left(\frac{b}{y}\right)^{\sigma}\right).$$

P r o o f. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(a,b)$, $\mathrm{d}\mu_1(x)=\mathrm{d}x$, $\mathrm{d}\mu_2(y)=\mathrm{d}y$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)}\frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}} y^{\sigma\eta+\sigma-1}, & a<y\le x\,; \\ 0, & x<y\le b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^{\sigma})$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^{\sigma}\right)^{\alpha}{}_2F_1(x)} I^{\alpha}_{a+;\sigma;\eta} f(x)$.

For the particular weight function $u(x)=x^{\sigma-1}\left((x^{\sigma}-a^{\sigma})^{\alpha}{}_2F_1(x)\right)^{\frac{q}{p}}$, $x\in(a,b)$ we get $v(y) = \left(\alpha\sigma^{1-\frac{p}{q}} y^{\sigma-1}{}_2F_1(y)(b^{\sigma}-y^{\sigma})^{\alpha-1+\frac{p}{q}}\right)/\left(\left((\alpha-1)\frac{q}{p}+1\right)^{\frac{p}{q}}\right)$, then (2.1) becomes

$$\rho_4(s) = \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b y^{\sigma-1}\,{}_2F_1(y)\,(b^{\sigma}-y^{\sigma})^{\alpha-1+\frac{p}{q}} f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}}$$
$$- (\Gamma(\alpha+1))^{\frac{sq}{p}} \int_a^b x^{\frac{\sigma\alpha sq}{p}+\sigma-1}\left((x^{\sigma}-a^{\sigma})^{\alpha}\,{}_2F_1(x)\right)^{\frac{q(1-s)}{p}}$$
$$\cdot \left(I^{\alpha}_{a+;\sigma;\eta} f(x)\right)^{\frac{sq}{p}} \mathrm{d}x - M_4(s),$$

Since $\frac{\alpha q}{p}(1-s)\le 0$ and $M_4(s)\ge 0$, we get that

$$\rho_4(s) \le \frac{\alpha^{\frac{q}{p}}\sigma^{\frac{q}{p}-1} b^{(\sigma-1)\frac{q}{p}}(b^{\sigma}-a^{\sigma})^{\frac{q}{p}(\alpha-1)+1}}{(\alpha-1)\frac{q}{p}+1}\left(\int_a^b {}_2F_1(y) f^s(y)\,\mathrm{d}y\right)^{\frac{q}{p}}$$

$$
\begin{aligned}
&- a^{\frac{\sigma\alpha s q}{p}+\sigma-1}(\Gamma(\alpha+1))^{\frac{sq}{p}}(b^\sigma-a^\sigma)^{\frac{\alpha q}{p}(1-s)}\\
&\cdot \int_a^b ({}_2F_1(x))^{\frac{q}{p}(1-s)}\left(I^{\alpha}_{a_+;\sigma;\eta}f(x)\right)^{\frac{sq}{p}}\,\mathrm{d}x - M_4(s)\\
= {} & \overline{H}_4(s) - M_4(s) \le \overline{H}_4(s) \qquad \square
\end{aligned}
$$

**Remark 2.** Similar result can be obtained for the right sided Erdélyi-Kober type fractional integrals, but here we omit the details.

## 3. Hardy-type inequalities for fractional derivatives

Now we define the Canavati-type fractional derivative of $f$ over $[a,b]$ ($\nu$-fractional derivative of $f$), for details see [2]. We consider

$$C_a^{\nu}\big([a,b]\big) = \big\{ f \in C^n([a,b]) :\ I^{1-\bar{\nu}} f^{(n)} \in C^1([a,b]) \big\},$$

$\nu > 0$, $n = [\nu]$, $[\cdot]$ is the integral part, and $\bar{\nu} = \nu - n$, $0 \le \bar{\nu} < 1$.

For $f \in C_a^{\nu}([a,b])$, the Canavati $\nu$-fractional derivative of $f$ is defined by

$$D_a^{\nu} f = D I_a^{1-\bar{\nu}} f^{(n)},$$

where $D = \mathrm{d}/\mathrm{d}x$.

The following lemma gives conditions in composition rule for the Canavati fractional derivative.

**Lemma 3.1.** *Let $\nu > \gamma > 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C_a^{\nu}([a,b])$, be such that $f^{(i)}(a) = 0$, $i = m, m+1, \dots, n-1$. Then*

(i) $f \in C^{\gamma}([a,b])$

(ii) $(D^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_a^x (x-t)^{\nu-\gamma-1} (D^{\nu} f)(t)\,\mathrm{d}t,$

*for every $x \in [a,b]$.*

In the following Theorem, we will construct new inequality for the Canavati-type fractional derivative.

**Theorem 3.1.** *Let $0 < p \le q < \infty$, $s \ge 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and assumptions in* Lemma 3.1 *be satisfied. Then for non-negative functions $D_a^{\nu} f$ and $D_a^{\gamma} f$ the following inequality holds true:*

$$0 \le \rho_5(s) \le \overline{H}_5(s) - M_5(s) \le \overline{H}_5(s),$$

*where*

$$\rho_5(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left( \int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}} (D_a^{\nu} f(y))^s \,\mathrm{d}y \right)^{\frac{q}{p}}$$

$$-(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b(x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_a^\gamma f(x)\right)^{\frac{sq}{p}}\,\mathrm{d}x-M_5(s),$$

$$M_5(s)=\frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p}\int_a^b(x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}}\left(D_a^\gamma f(x)\right)^{s(\frac{q}{p}-1)}\times\int_a^x r_5(x,y)(x-y)^{\nu-\gamma-1}\,\mathrm{d}y\,\mathrm{d}x,$$

$$r_5(x,y)=\Bigg|\bigg|(D_a^\nu f(y))^s-\left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^\gamma}D_a^\gamma f(x)\right)^s\bigg| - s\left|\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|^{s-1}\left|D_a^\nu f(y)-\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)\right|\Bigg|,$$

*and*

$$\overline{H}_5(s)=(b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)}\left[\frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b(D_a^\nu f(y))^s\,\mathrm{d}y\right)^{\frac{q}{p}} - (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b\left(D_a^\gamma f(x)\right)^{\frac{sq}{p}}\,\mathrm{d}x\right].$$

P r o o f. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(a,b)$, $\mathrm{d}\mu_1(x)=\mathrm{d}x$, $\mathrm{d}\mu_2(y)=\mathrm{d}y$,

$$k(x,y)=\begin{cases}\frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a<y\le x\,;\\ 0, & x<y\le b,\end{cases}$$

we get that $K(x)=\frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$ and $A_kf(x)=\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}}D_a^\gamma f(x)$. Replace $f$ by $D_a^\nu f$. For the particular weight function $u(x)=(x-a)^{\frac{(\nu-\gamma)q}{p}}$, $x\in(a,b)$ we get $v(y)=\left((\nu-\gamma)(b-y)^{\nu-\gamma-1+\frac{p}{q}}\right)/\left(\left((\nu-\gamma-1)\frac{q}{p}+1\right)^{\frac{p}{q}}\right)$, then (2.1) takes the form

$$\rho_5(s)=\frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b(b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_a^\nu f(y))^s\,\mathrm{d}y\right)^{\frac{q}{p}} - (\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b(x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}}\left(D_a^\gamma f(x)\right)^{\frac{sq}{p}}\,\mathrm{d}x-M_5(s).$$

Since $\frac{(\nu-\gamma)q}{p}(1-s) \leq 0$ and $M_5(s) \geq 0$, we obtain that

$$\begin{aligned}\rho_5(s) \leq{}& \frac{(\nu-\gamma)^{\frac{q}{p}}(b-a)^{(\nu-\gamma-1)\frac{q}{p}+1}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (D_a^\nu f(y))^s \,\mathrm{d}y\right)^{\frac{q}{p}} \\ & - (b-a)^{\frac{(\nu-\gamma)q}{p}(1-s)}(\Gamma(\nu-\gamma+1))^{\frac{sq}{p}}\int_a^b (D_a^\gamma f(x))^{\frac{sq}{p}} \,\mathrm{d}x - M_5(s) \\ ={}& \overline{H}_5(s) - M_5(s) \leq \overline{H}_5(s).\end{aligned}$$

This complete the proof. □

Next, we give the result for the Caputo fractional derivative, for details see [1: p. 449]. The Caputo fractional derivative is defined as:

Let $\alpha \geq 0$, $n = [\alpha]+1$, $g \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}}\,\mathrm{d}y,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

We continue with the following lemma that is given in [3].

**Lemma 3.2.** *Let $\nu > \gamma \geq 0$, $n = [\nu]+1$, $m = [\gamma]+1$ and $f \in AC^n([a,b])$. Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, \dots, n-1$

(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m, \dots, n-2$

(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, \dots, n-1$

(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(a) = 0$ *for* $i = m-1, \dots, n-2$.

*Then*

$$D_{*a}^\gamma f(x) = \frac{1}{\Gamma(\nu-\gamma)}\int_a^x (x-y)^{\nu-\gamma-1} D_{*a}^\nu f(y)\,\mathrm{d}y$$

*for all $a \leq x \leq b$.*

**Theorem 3.2.** *Let $0 < p \leq q < \infty$, $s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$ and assumptions in* Lemma 3.2 *be satisfied. Then for non-negative functions $D_{*a}^\nu f$ and $D_{*a}^\gamma f$ the following inequality holds true:*

$$0 \leq \rho_6(s) \leq \overline{H}_6(s) - M_6(s) \leq \overline{H}_6(s),$$

*where*

$$\rho_6(s) = \frac{(\nu-\gamma)^{\frac{q}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1}\left(\int_a^b (b-y)^{\nu-\gamma-1+\frac{p}{q}}(D_{*a}^\nu f(y))^s\,\mathrm{d}y\right)^{\frac{q}{p}}$$

$$- \left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}} \int_a^b (x-a)^{\frac{(\nu-\gamma)q(1-s)}{p}} \left(D_{*a}^{\gamma} f(x)\right)^{\frac{sq}{p}} \,\mathrm{d}x - M_6(s),$$

$$M_6(s) = \frac{q(\nu-\gamma)(\Gamma(\nu-\gamma+1))^{s(\frac{q}{p}-1)}}{p} \int_a^b (x-a)^{\frac{(\nu-\gamma)(q-p)(1-s)}{p}} \left(D_{*a}^{\gamma} f(x)\right)^{s(\frac{q}{p}-1)} \times \int_a^x r_6(x,y)(x-y)^{\nu-\gamma-1} \,\mathrm{d}y \,\mathrm{d}x,$$

$$r_6(x,y) = \Bigg| \left| (D_{*a}^{\nu} f(y))^s - \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\gamma}} D_{*a}^{\gamma} f(x)\right)^s \right| - s \left| \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x) \right|^{s-1} \cdot \left| D_{*a}^{\nu} f(y) - \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_{*a}^{\gamma} f(x) \right| \Bigg|,$$

*and*

$$\overline{H}_6(s) = (b-a)^{(\nu-\gamma)\frac{q}{p}(1-s)} \left[ \frac{(\nu-\gamma)^{\frac{q}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{p}}}{(\nu-\gamma-1)\frac{q}{p}+1} \left( \int_a^b (D_{*a}^{\nu} f(y))^s \,\mathrm{d}y \right)^{\frac{q}{p}} - \left(\Gamma(\nu-\gamma+1)\right)^{\frac{sq}{p}} \int_a^b \left(D_{*a}^{\gamma} f(x)\right)^{\frac{sq}{p}} \,\mathrm{d}x \right].$$

Proof. The proof is similar to the proof of Theorem 3.2. □

**Acknowledgement.** We thank the referee for some valuable advice, which have improved the final version of this paper.

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Received 25. 10. 2011
Accepted 30. 4. 2012

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