

ON SOME NEW IDENTITIES FOR THE FIBONOMIAL COEFFICIENTS

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ABSTRACT. Let F_n be the n th Fibonacci number. The Fibonomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_F$ are defined for $n \geq k > 0$ as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_F = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_F = 0$ for $n < k$. In this paper, we shall provide several identities among Fibonomial coefficients. In particular, we prove that

$$\sum_{j=0}^{4l+1} \operatorname{sgn}(2l-j) \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_{n-4l-1},$$

holds for all non-negative integers n and l .

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1. Introduction

In 1915, Fontené published a one-page note [1] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{a_n\}$ of real or complex numbers. Thus the generalized binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \frac{a_n a_{n-1} \cdots a_{n-k+1}}{a_1 a_2 \cdots a_k}.$$

Setting $a_n = n$ we recover the ordinary binomial coefficients, while $a_n = q^n - 1$ we obtain the q -binomial coefficients studied by Gauss, Euler, Cauchy and which

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were shortly called q -Gaussian coefficients (Gauss q -binomial coefficients). The sequence $\{a_n\}$ is essentially arbitrary but we do require that $a_n \neq 0$ for $n \geq 1$.

The generalized binomial coefficients have many interesting properties. Obviously, for an integer $n \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_a = \begin{bmatrix} n \\ n-k \end{bmatrix}_a, \quad \begin{bmatrix} n \\ 1 \end{bmatrix}_a = a_n \quad \text{and} \quad \begin{bmatrix} n \\ n \end{bmatrix}_a = 1.$$

Specially, take a sequence $\{a_n\}$ of positive integers generated by

$$a_{n+2} = g a_{n+1} - h a_n \quad (1)$$

for integers $n \geq 0$, where $g, h \neq 0$ are real numbers. Let α and β , $\alpha \neq \beta$, be the roots of the characteristic equation $x^2 - gx + h = 0$ and let u_n, v_n be the solutions of (1) defined by $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $v_n = \alpha^n + \beta^n$. Jarden [3] found the following formula

$$\sum_{j=0}^k (-1)^j \begin{bmatrix} n \\ k \end{bmatrix}_a h^{\frac{j-1}{2}} z_{n+k-j} = 0,$$

where z_n is the product of the n th terms of $k-1$ sequences satisfying (1).

Since 1964, there has been an accelerated interest in *Fibonomial coefficients*, which correspond to the choice $a_n = F_n$, where F_n are the Fibonacci numbers defined by (1) for $g = 1, h = -1$.

It is easy to write the key recurrence formula in the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + F_{n-k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F.$$

We refer the reader to recent papers [4] and [5] for results on the spacing and perfect powers among Fibonomial coefficients, respectively.

For the signed Fibonomial coefficients $(-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k \\ j \end{bmatrix}_F$ (named A055870 in Sloane's encyclopedia of sequences), we can find the following identity, see [14], (a little rewritten)

$$\sum_{j=0}^k (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k \\ j \end{bmatrix}_F F_{n-j}^{k-1} = 0,$$

where n, k are any positive integers with $n \geq k+1$.

In [6], we obtain for any non-negative integer $l \equiv 0, 2 \pmod{4}$ the following identity

$$\sum_{j=0}^l (-1)^{\frac{j}{2}(j+(-1)^{l/2})} \begin{bmatrix} l \\ j \end{bmatrix}_F F_{n+l-j} = 0.$$

In this paper, we shall study the similar sums of the Fibonomial coefficients, but for $l \equiv 1, 3 \pmod{4}$. The following theorem is our main result.

THEOREM 1. *Let l, n be any non-negative integers. Then*

$$\sum_{j=0}^{4l+3} \text{sgn}(2l+1-j) \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n-j} = \frac{F_{2l}}{F_{4l+3}} \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{n-4l-3} \quad (2)$$

and

$$\sum_{j=0}^{4l+1} \text{sgn}(2l-j) \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_{n-4l-1}. \quad (3)$$

As usual, in the above statement, $\text{sgn}(x)$ denotes the *sign function* of x , defined by $\text{sgn}(0) = 0$ and $\text{sgn}(x) = x/|x|$, for $x \neq 0$.

2. The preliminary results

First, we extend the definition of the Fibonomial coefficients, for any integers n, k , by the following way

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} 0, & \text{if } k < 0; \\ 1, & \text{if } k = 0; \\ \prod_{i=0}^{k-1} \frac{F_{n-i}}{F_{k-i}}, & \text{in other cases.} \end{cases}$$

Throughout what follows, $\{L_n\}$ denotes the sequence of *Lucas numbers* which follows the same recursive pattern as Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

LEMMA 2. *Let k, l, m be any integers. Then*

$$F_{2l} - (-1)^l = L_{l-1} F_{l+1}, \quad (4)$$

$$F_{2l+1} - (-1)^l = L_{l+1} F_l, \quad (5)$$

$$F_{l+2} F_{2l} + (-1)^l F_{l-1} = F_{l+1} F_{2l+1}, \quad (6)$$

$$F_{k+1} F_{l-k} - F_k F_{l-k+1} = (-1)^k F_{l-2k}, \quad (7)$$

$$F_{l-k} F_{l-k+1} - F_{l+1} F_{l-2k} = (-1)^l F_k F_{k+1}. \quad (8)$$

$$F_{l+1} F_{2l+1} + (-1)^{l+1} F_{l-1} = F_{l+2} F_{2l}, \quad (9)$$

$$F_{2l+1} L_{l+1} + (-1)^l L_{l+2} = F_{l+2} L_l^2, \quad (10)$$

$$F_{k+1} F_{l-k+2} - F_{k+2} F_{l-k+1} = (-1)^k F_{l-2k}, \quad (11)$$

$$F_{m+k+l}^2 + (-1)^{l+1} F_{m-k}^2 = F_{2m+l} F_{2k+l}. \quad (12)$$

Proof. Identities (4) and (5) follows from identity

$$F_{n+m} - (-1)^m F_{n-m} = F_m L_n,$$

see (15b) in [11], setting $m = l + 1$, $n = l - 1$ and $m = l$, $n = l + 1$ respectively.

Identity (6) follows from identity

$$F_{a+b} F_{a+c} - F_a F_{a+b+c} = (-1)^a F_b F_c, \quad (13)$$

see (20a) in [11], setting $c = 1$, $a = l$, $b = l - 1$ and using basic recurrence.

Identity (7) follows from identity (13) setting $c = 1$, $a = k$, $b = l - 2k$.

Identity (8) follows from identity (13) setting $a = l + 1$, $b = k + 1$, $c = -k - 1$ and using identity $F_{-a} = (-1)^{a+1} F_a$ (see [11], identity (2)).

Identity (9) follows from identity (13) setting $a = l + 3$, $b = l - 1$, $c = 1$ and using Fibonacci recurrence relation.

Identity (10) can be rewritten by identities $L_{2n+2} \cdot (-1)^n = L_n^2$ and $F_n + L_n = 2F_{n+1}$ (see identities (17c) and (7b) in [11]), to the form

$$F_{l+2} L_{2l} - F_{2l+1} L_{l+1} = (-1)^l F_{l-1},$$

which follows from identity (13) setting $a = l + 1$, $b = l - 1$, $c = 1$.

Identity (11) follows from (7) using the basic recurrence.

Identity (12) follows identity (13) setting $c = m - p$, $b = m - p$ and $a = n + 2p$. $\square$

LEMMA 3. *Let $l, k \neq -1$ be any integers. Then the following holds*

$$(-1)^l \left[\begin{matrix} l \\ k+1 \end{matrix} \right]_F + \left[\begin{matrix} l+1 \\ k \end{matrix} \right]_F \frac{F_{2k-l}}{F_{k+1}} = \left[\begin{matrix} l \\ k-1 \end{matrix} \right]_F.$$

Proof. After overwriting the Fibonomial coefficients using their definition, we obtain the identity

$$(-1)^l F_{l-k+1} F_{l-k} + F_{2k-l} F_{l+1} = F_k F_{k+1},$$

which follows from identity (8) using again identity $F_{-a} = (-1)^{a+1} F_a$. $\square$

LEMMA 4. *Let l, k be any integers. Then the following holds*

$$\begin{aligned} \left[\begin{matrix} 4l+2 \\ 2k \end{matrix} \right]_F \frac{F_{4l+2} F_{2l+3}}{F_{2l+1}} + \left[\begin{matrix} 4l+3 \\ 2k+1 \end{matrix} \right]_F (F_{4l-2k+3} - F_{2k+2}) \\ + \left[\begin{matrix} 4l+3 \\ 2k+2 \end{matrix} \right]_F (F_{4l-2k+2} - F_{2k+3}) = \left[\begin{matrix} 4l+2 \\ 2k+2 \end{matrix} \right]_F \frac{F_{4l+2} F_{2l+3}}{F_{2l+1}}. \end{aligned}$$

P r o o f. After overwriting the Fibonomial coefficients using their definition we get

$$\begin{aligned} & F_{4l+2}F_{2l+3}F_{2k+1}F_{2k+2} + F_{4l+3}F_{2l+1}F_{2k+2}(F_{4l-2k+3} - F_{2k+2}) \\ & + F_{4l+3}F_{2l+1}F_{4l-2k+2}(F_{4l-2k+2} - F_{2k+3}) \\ & = F_{4l-2k+2}F_{4l-2k+1}F_{4l+2}F_{2l+3} \end{aligned}$$

which can be simplified by basic recurrence for the Fibonacci numbers and using identities (4), (7), where we replace k by $2k$ and l by $4l$, and (8), where we replace k by $2k + 1$ and l by $4l + 2$, to the form

$$F_{4l-2k+2}^2 - F_{2k+2}^2 = F_{4(l+1)}F_{4(l-k)}$$

which is the special form of identity (12) for $m = 0$, $n = 4(l - k)$ and $p = 2k + 2$. $\square$

LEMMA 5. *Let $l \neq 0$, k be any integers. Then the following holds*

$$\begin{aligned} \left[\begin{matrix} 4l \\ 2k-1 \end{matrix} \right]_F \frac{F_{4l}F_{2l+2}}{F_{2l}} + \left[\begin{matrix} 4l+1 \\ 2k \end{matrix} \right]_F (F_{4l-2k+2} - F_{2k+1}) \\ + \left[\begin{matrix} 4l+1 \\ 2k+1 \end{matrix} \right]_F (F_{4l-2k+1} - F_{2k+2}) = \left[\begin{matrix} 4l \\ 2k+1 \end{matrix} \right]_F \frac{F_{4l}F_{2l+2}}{F_{2l}}. \end{aligned}$$

P r o o f. The proof is very similar as in Lemma 4. After overwriting the Fibonomial coefficients using their definition we get

$$\begin{aligned} & F_{4l}F_{2l+2}F_{2k}F_{2k+1} + F_{4l+1}F_{2l}F_{2k+1}(F_{4l-2k+2} - F_{2k+1}) \\ & + F_{4l+1}F_{2l}F_{4l-2k+1}(F_{4l-2k+1} - F_{2k+2}) \\ & = F_{4l-2k+1}F_{4l-2k}F_{4l}F_{2l+2}, \end{aligned}$$

which can be simplified by basic recurrence for the Fibonacci numbers and using identities (4), (11), where we replace k by $2k$ and l by $4l$, and (8), where we replace k by $2k$ and l by $4l$, to the form

$$F_{4l-2k+1}^2 - F_{2k+1}^2 = F_{2(2l+1)}F_{4(l-k)},$$

which is the special form of identity (12) for $m = 0$, $n = 4(l - k)$ and $p = 2k + 1$. $\square$

LEMMA 6. *Let l , k be any integers. Then*

$$(-1)^l \left[\begin{matrix} l \\ k-1 \end{matrix} \right]_F + \left[\begin{matrix} l+1 \\ k+1 \end{matrix} \right]_F \frac{F_{l-2k}}{F_{l-k+1}} = \left[\begin{matrix} l \\ k+1 \end{matrix} \right]_F$$

for $l \neq k - 1$.

P r o o f. After overwriting the Fibonomial coefficients using their definition we obtain identity (8). $\square$

THEOREM 7. *Let l be any integer and n any non-negative integer. Then*

$$\sum_{j=0}^{2n} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+2 \\ 2n \end{bmatrix}_F \frac{F_{4l+2} F_{2l+3}}{F_{2l+1}}. \quad (14)$$

Proof. We use the induction with respect to n . For $n = 0$ the assertion follows from (4) replacing l by $2l + 2$ and using well-known equality $F_l L_l = F_{2l}$. Let us suppose that the identity holds for $n = k$ and prove it for $n = k + 1$. The left-hand side can be rewritten as

$$\begin{aligned} & \sum_{j=0}^{2k+2} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) \\ &= \sum_{j=0}^{2k} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) + \begin{bmatrix} 4l+3 \\ 2k+1 \end{bmatrix}_F (F_{4l-2k+3} - F_{2k+2}) \\ & \quad + \begin{bmatrix} 4l+3 \\ 2k+2 \end{bmatrix}_F (F_{4l-2k+2} - F_{2k+3}) \end{aligned}$$

and the proof follows from Lemma 4. $\square$

THEOREM 8. *Let l be any integer and n any positive integer. Then*

$$\sum_{j=1}^n \begin{bmatrix} 4l-1 \\ 2j-1 \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2j}} = \begin{bmatrix} 4l-2 \\ 2n \end{bmatrix}_F - 1. \quad (15)$$

Proof. We use the induction with respect to n . For $n = 1$ the assertion is implied by (8) putting $k = 1$ and replacing l by $4l - 2$. Let us suppose that proved identity holds for $n = k$ and prove it for $n = k + 1$. The left side of (15) has the form

$$\sum_{j=1}^{k+1} \begin{bmatrix} 4l-1 \\ 2j-1 \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2j}} = \sum_{j=1}^k \begin{bmatrix} 4l-1 \\ 2j-1 \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2j}} + \begin{bmatrix} 4l-1 \\ 2k+1 \end{bmatrix}_F \frac{F_{4(l-k-1)}}{F_{2(k+1)}}$$

and the proved identity follows from Lemma 3 replacing k by $2k + 1$ and l by $4l - 2$. $\square$

COROLLARY 9. *Let n be any positive integer. Then*

$$\begin{aligned} & \sum_{j=1}^n (-1)^j L_{2j} = (-1)^n F_{2n+1} - 1, \\ & \sum_{j=1}^n (-1)^j F_{2j+1} F_{2j+2} F_{2j+3} F_{4(j+1)} = \frac{(-1)^n}{5} F_{2n+1} F_{2n+2} F_{2n+3} F_{2n+4} F_{2n+5} - 6. \end{aligned}$$

Proof. We get the first relation setting $l = 0$ in (15) and using identities $\begin{bmatrix} -2 \\ 2n \end{bmatrix}_F = (-1)^n F_{2n+1}$ and $\begin{bmatrix} -1 \\ 2j-1 \end{bmatrix}_F = (-1)^{j+1}$, which follow from our extension of definition of the Fibonomial coefficient. The second relation we obtain similarly setting $l = -1$ in (15). $\square$

THEOREM 10. *Let l and n be any integers. Then*

$$\sum_{j=1}^n \sigma(j, l) = \begin{bmatrix} 4l-2 \\ 2n \end{bmatrix}_F - 1,$$

where

$$\sigma(j, l) = \begin{cases} -1, & j = 2l; \\ \begin{bmatrix} 4l-1 \\ 2j \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2(2l-j)}}, & j \neq 2l, \end{cases}$$

and

$$\sum_{j=0}^n \begin{bmatrix} 4l \\ 2j \end{bmatrix}_F F_{4(l-j)} = F_{4l} \begin{bmatrix} 4l-2 \\ 2n \end{bmatrix}_F.$$

Proof. By overwriting the Fibonomial coefficients we obtain

$$\begin{bmatrix} 4l-1 \\ 2j-1 \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2j}} = \begin{cases} -1, & j = 2l; \\ \begin{bmatrix} 4l-1 \\ 2j \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2(2l-j)}}, & j \neq 2l \end{cases}$$

and the first identity follows from (15). The second identity clearly holds for $l = 0$ and for $l \neq 0$ we use the relation

$$\begin{bmatrix} 4l-1 \\ 2j-1 \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{2j}} = \begin{bmatrix} 4l \\ 2j \end{bmatrix}_F \frac{F_{4(l-j)}}{F_{4l}},$$

which can be derived by overwriting the Fibonomial coefficients again, and the assertion is implied by (15). $\square$

THEOREM 11. *Let $l \neq 0$ be any integer and n any positive integer. Then*

$$\sum_{j=0}^{2n-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l \\ 2n-1 \end{bmatrix}_F \frac{F_{2l+2} F_{4l}}{F_{2l}}. \quad (16)$$

Proof. We use the induction with respect to n . For $n = 1$ we obtain after overwriting the Fibonomial coefficients the following identity

$$F_{4l+2} - 1 + F_{4l+1}^2 - F_{4l+1} = \frac{F_{4l}^2 F_{2l+2}}{F_{2l}}.$$

Using relations $F_{2l} = F_l L_l$ and (5) we can rewrite it to the identity $L_{2l+2} + F_{4l+1} L_{2l+1} = L_{2l}^2 F_{2l+2}$, which follows from (10) replacing l by $2l$.

Let us suppose, by the induction hypothesis, that the proved identity holds for $n = k$ and prove it for $n = k + 1$. The left side we can write as

$$\begin{aligned} & \sum_{j=0}^{2k+1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) \\ &= \sum_{j=0}^{2k-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) + \begin{bmatrix} 4l+1 \\ 2k \end{bmatrix}_F (F_{4l-2k+2} - F_{2k+1}) \\ & \quad + \begin{bmatrix} 4l+1 \\ 2k+1 \end{bmatrix}_F (F_{4l-2k+1} - F_{2k+2}) \end{aligned}$$

and the proved identity follows from Lemma 5. $\square$

3. The proof of Theorem 1

Firstly we prove identity (2). This identity can be rewritten as

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n+4l+3-j} - \sum_{j=2l+2}^{4l+3} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_{n+4l+3-j} = \frac{F_{2l}}{F_{4l+3}} \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_n$$

and replacing j by $(4l+3) - j$ in the second sum and simplifying to the form

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{n+4l+3-j} - F_{n+j}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \left(F_{n+2l+1} + \frac{F_n F_{2l}}{F_{4l+3}} \right).$$

We prove it by induction with respect to n . Let $n = 0$, thus we have to show that

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+3-j} - F_j) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1}.$$

But this identity follows from the relation $\begin{bmatrix} m \\ j \end{bmatrix}_F F_{m-j} = \begin{bmatrix} m \\ j+1 \end{bmatrix}_F F_{j+1}$, where m, j are any non-negative integers, which can be easily obtained overwriting the Fibonomial coefficients, by the following way

$$\begin{aligned} \sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+3-j} - F_j) &= \sum_{j=0}^{2l} \left(\begin{bmatrix} 4l+3 \\ j+1 \end{bmatrix}_F F_{j+1} - \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F F_j \right) \\ &= \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1} - \begin{bmatrix} 4l+3 \\ 0 \end{bmatrix}_F F_0 \\ &= \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F F_{2l+1}. \end{aligned}$$

Let $n = 1$. Then we have to prove the identity

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \left(F_{2l+2} + \frac{F_{2l}}{F_{4l+3}} \right).$$

It can be simplified with respect to (6), where we replace l by $2l+1$, to

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_F \frac{F_{2l+3}F_{4l+2}}{F_{4l+3}}$$

and

$$\sum_{j=0}^{2l} \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_F (F_{4l+4-j} - F_{j+1}) = \begin{bmatrix} 4l+2 \\ 2l \end{bmatrix}_F \frac{F_{4l+2}F_{2l+3}}{F_{2l+1}}$$

but this is a special case of identity (14) for $n = l$ only.

If we consider that (2) holds for n and $n+1$, then adding of appurtant identities and using the basic recurrence $F_{n+2} = F_n + F_{n+1}$ the proof of (2) is over.

Now we prove identity (3). It obviously holds for $l = 0$ therefore we consider $l > 0$ in the next part of proof.

The identity can be rewritten as

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n+4l+1-j} - \sum_{j=2l+1}^{4l+1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F F_{n+4l+1-j} = -\frac{F_{2l-1}}{F_{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F F_n.$$

and replacing j by $(4l+1) - j$ in the second sum and simplifying to the form

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{n+4l+1-j} - F_{n+j}) = \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F \left(F_{n+2l} - \frac{F_n F_{2l-1}}{F_{4l+1}} \right).$$

We again prove it by induction with respect to n . For $n = 0$ the proof can be done analogously as proof of (2) for $n = 1$.

Let $n = 1$. Then we have to show that

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_F \left(F_{2l+1} - \frac{F_{2l-1}}{F_{4l+1}} \right).$$

This relation can be simplified using (9), where we replace l by $2l$, to

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F \frac{F_{2l+2}F_{4l}}{F_{4l+1}}.$$

and

$$\sum_{j=0}^{2l-1} \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_F (F_{4l+2-j} - F_{j+1}) = \begin{bmatrix} 4l \\ 2l-1 \end{bmatrix}_F \frac{F_{2l+2}F_{4l}}{F_{2l}},$$

which is a special case of identity (16). The induction leap is the same as in the proof of identity (2).

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