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CHARACTERIZATIONS OF STRONG MORITA EQUIVALENCE FOR ORDERED SEMIGROUPS WITH LOCAL UNITS

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ABSTRACT. We prove that partially ordered semigroups S and T with local units are strongly Morita equivalent if and only if there exists a surjective strict local isomorphism to T from a factorizable Rees matrix posemigroup over S. We also provide two similar descriptions which use Cauchy completions and Morita posemigroups instead.

1. Introduction

The Morita theory of semigroups without identity is relatively new, going back to the paper by Talwar ([9]). It has seen quite a bit of recent development by Lawson ([7]) and Laan and Márki ([6]), who present a number of alternative characterizations of Morita equivalent semigroups. Our work considers how those results can be applied to the Morita theory of posemigroups and is a sequel to [12]. We use the main result of [12] to examine how strong Morita equivalence is connected to the existence of a strict local isomorphism between a posemigroup T and several posemigroups constructed from a given posemigroup S: the Cauchy completion $C(S)$ viewed as a posemigroup, a Rees matrix posemigroup over S and a Morita posemigroup over S. Along the way, we describe the Rees matrix posemigrops corresponding to unipotent pomonoids and show that

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Morita posemigroups over S are always strongly Morita equivalent to S. Most of the results are refinements of the Morita theory of semigroups with (various kinds of) local units as found in [6], [7] and [10].

We use the symmetric monoidal closed category Pos of partial orders and monotone maps (using cartesian product for monoidal tensor product) and consider various categories enriched over Pos. Our reference for details on Pos-categories, Pos-functors and Pos-equivalences is [3]. We just remark that a full and faithful Pos-functor must provide a poset isomorphism instead of a bijection between the corresponding posets of morphisms. Categorial compositions will be written from right to left.

A partially ordered semigroup S (a *posemigroup* for short) is a (nonempty) semigroup that is endowed with a partial order so that its operation is monotone. For a fixed posemigroup S, (one-sided) *S-posets* are partially ordered S-acts where the S-action is monotone in both arguments. A right S-poset X is said to be *unitary* if $XS = X$. The notion for left S-posets is dual. A poset is called an *(S,T)-biposet* if it is a left S- and a right T-poset and its S- and T-actions commute with each other. (S,T)-biposets are called *unitary* if they are unitary as both left S- and right T-posets. *Posemigroup homomorphisms* are monotone semigroup homomorphisms. A number of basic facts about S-posets over pomonoids can be found in [1].

A (po)semigroup S is said to have *local units* if for any $s \in S$ there exist idempotents e and f such that

$$es = s = sf.$$

A (po)semigroup S is said to have *weak local units* if for any $s \in S$ there exist not necessarily idempotent $e \in S$ and $f \in S$ such that $es = s = sf$. A (po)semigroup S is said to have *common (weak) local units* (cf. [5]) if for any $s, s' \in S$ there exist $e \in E(S)$ and $f \in E(S)$ ($e \in S$ and $f \in S$) such that $es = s = sf$ and $es' = s' = s'f$. We say that a posemigroup S has *ordered (weak) local units* if for all $s, s' \in S$, $s \leq s'$, there exist $e, e', f, f' \in E(S)$ ($e, e', f, f' \in S$) such that

$$es = s = sf, \quad e's' = s' = s'f', \quad e \leq e', \quad f \leq f'.$$

A (po)semigroup is called *factorizable* if $S^2 = S$. Having local units implies having weak local units, which in turn implies factorizability. Also, a (po)semigroup with (weak) common local units has (weak) local units. A (po)semigroup is *unipotent* if it contains exactly one idempotent.

The *tensor product* $A \otimes_S B$ of a right S-poset A and a left S-poset B is the quotient poset $(A \times B)/\sim$, where $(a,b) \sim (a',b')$ iff $(a,b) \preceq (a',b')$ and $(a',b') \preceq (a,b)$, and $(a,b) \preceq (a',b')$ iff there exist $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$,

$a_1, \ldots, a_n \in A$ and $b_2, \ldots, b_n \in B$ such that

$$\begin{aligned} a &\le a_1 s_1 & & \\ a_1 t_1 &\le a_2 s_2 & s_1 b &\le t_1 b_2 \\ &\vdots & &\vdots \\ a_n t_n &\le a' & s_n b_n &\le t_n b', \end{aligned} \tag{1}$$

where we take $xu = x$ for every $x \in \{a_1, \ldots, a_n\}$ and $uy = y$ for every $y \in \{b, b'\} \cup \{b_2, \ldots, b_n\}$ if $u \in S^1$ is the externally adjoined identity. For a pair $(a, b) \in A \times B$, the equivalence class $[(a, b)]_\sim$ is denoted by $a \otimes b$. The order relation on $A \otimes_S B$ is defined by setting, for $a \otimes b$, $a' \otimes b' \in A \otimes_S B$,

$$a \otimes b \le a' \otimes b' \iff (a, b) \preceq (a', b').$$

If A is a (T, S)-biposet, then $A \otimes_S B$ is a left T-poset, where the action is defined by $t(a \otimes b) = (ta) \otimes b$. Similarly, if B is an (S, T)-biposet, then $A \otimes_S B$ is a right T-poset.

When S is a sub(po)semigroup (with inherited order) of R, then R is called an *enlargement* of S if

$$S = SRS \quad \text{and} \quad R = RSR.$$

For (po)semigroups S, T and R, R is said to be a *joint enlargement* (cf. [7]) of S and T if it is an enlargement of its sub(po)semigroups $S' \cong S$ and $T' \cong T$.

For a Pos-category $\mathcal{C}$, we will denote by $\mathcal{C}_0$ its set of objects and by $\mathcal{C}(A, B)$ its poset of morphisms from object A to object B.

If S is a posemigroup, then the *Cauchy completion* of S (cf. [7]) is the small Pos-category $C(S)$ that has $C(S)_0 = E(S)$, morphism posets

$$C(S)(f, e) = \big\{(e, s, f) \mid s \in S,\ \ esf = s\big\},$$

with the order $(e, s, f) \le (e, s', f)$ iff $s \le s'$ in S, and the composition rule $(e, s, f) \circ (f, s', g) = (e, ss', g)$.

A category $\mathcal{C}$ is called *strongly connected* if for all $A, B \in \mathcal{C}_0$ there always exists a morphism $f\colon A \to B$. If $\mathcal{C}$ is a strongly connected category, then a *consolidation* on $\mathcal{C}$ is a map

$$p\colon \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}$$

(denoted by $p_{B,A} := p(B, A)\colon A \to B$) such that $p_{A,A} = 1_A$. A small Pos-category $\mathcal{C}$ (actually its set of morphisms, which we will again denote by $\mathcal{C}$) with a consolidation p can be made into a posemigroup by defining

$$g \diamond f = g \circ p_{\operatorname{dom} g, \operatorname{cod} f} \circ f$$

and taking the order from the morphism-poset order of $\mathcal{C}$. This posemigroup will be denoted by $\mathcal{C}^p$. It is easy to see that if the composite $g \circ f$ exists for morphisms f and g in $\mathcal{C}$, then $g \diamond f = g \circ f$.

If S and T are posemigroups, then we say that a 6-tuple

$$\big(S, T, P, Q, \langle -, - \rangle, [-, -]\big)$$

is a *Morita context* if the following conditions hold:

(M1) P is an (S,T)-biposet and Q is a (T,S)-biposet;

(M2) $\langle -, - \rangle \colon P \otimes_T Q \to S$ is an (S,S)-biposet morphism and $[-,-] \colon Q \otimes_S P \to T$ is a (T,T)-biposet morphism;

(M3) the following two conditions hold for all $p, p' \in P$ and $q, q' \in Q$:
  (i) $\langle p, q \rangle p' = p[q, p']$,
  (ii) $q \langle p, q' \rangle = [q, p] q'$.

A Morita context is called *unitary* if the biposets P and Q are unitary. We say that two posemigroups S and T are *strongly Morita equivalent* (as introduced by Talwar in [10]) if there exists a unitary Morita context $(S, T, P, Q, \langle -, - \rangle, [-, -])$ such that the mappings $\langle -, - \rangle$ and $[-, -]$ are surjective.

We recall from [11: Lemma 3.2] that:

Lemma 1.1. *Strongly Morita equivalent posemigroups are factorizable.*

If S and T are arbitrary posemigroups, then we call a posemigroup homomorphism $\tau \colon S \to T$ a *strict local isomorphism* (cf. [7] and [6]) if the following conditions hold:

(LI1) τ restricted to aSb is a posemigroup isomorphism with $\tau(a)T\tau(b)$ for any $a \in Sa$, $b \in bS$,

(LI2) *idempotents lift along* τ, i.e. if $e' = \tau(s) \in E(T)$ for some $s \in S$, then there exists $e \in E(S)$ such that $e' = \tau(e)$,

(LI3) for any $e \in E(T)$ there exists $f \in E(T)$ such that $f = \tau(s)$ for some $s \in S$ and $e\mathcal{D}f$.

When S and T have trivial order, then a strict local isomorphism $S \to T$ is a local isomorphism in the sense of [7] and every surjective strict local isomorphism that lifts idempotents (as in [6]) is a strict local isomorphism.

Lemma 1.2. *Let S and T be posemigroups with local units and let $\tau \colon S \to T$ be a surjective posemigroup homomorphism. Then* (LI1) *is equivalent to*

(LI1') *τ restricted to eSf is a posemigroup isomorphism with $\tau(e)T\tau(f)$ for any $e, f \in E(S)$.*

Proof. It is clear that (LI1) implies (LI1'). For the converse, take $a \in Sa$ and $b \in bS$. As S has local units, $a = ea$ and $b = bf$ for some $e, f \in E(S)$. So $\tau|_{eSf}\colon eSf \to \tau(e)T\tau(f)$ is a posemigroup isomorphism by (LI1'). Since $aSb \subseteq eSf$, $\tau|_{aSb} = (\tau|_{eSf})|_{aSb}\colon aSb \to \tau(a)T\tau(b)$ is a surjective posemigroup embedding, i.e. a posemigroup isomorphism. □

Remark 1.1. Note that by [2: Proposition 2.3.5], the requirement $e\mathcal{D}f$ in condition (LI3) can be rephrased as follows: there exist $x \in S$ and $x' \in V(x)$ such that $xx' = e$ and $x'x = f$. Also, (LI3) is obviously satisfied for a surjective homomorphism τ.

2. Strong Morita equivalence

The principal result of the article is the following theorem. Roughly half of it has been proved in [12] and the subsequent sections of the article are devoted to examining each of the three remaining conditions in detail and establishing their equivalence with the others.

Theorem 2.1. (cf. [6], [7: Theorems 1.1, 1.2]) *Let S and T be posemigroups with local units. Then the following are equivalent:*

(1) *S and T are strongly Morita equivalent;*
(2) *S and T have a joint enlargement;*
(3) *the categories $C(S)$ and $C(T)$ are* Pos*-equivalent;*
(4) *there exist a consolidation q on $C(S)$ and a strict local isomorphism $C(S)^q \to T$;*
(5) *there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ for which $S = S\,\mathrm{Im}(M)S$ and a surjective strict local isomorphism $\mathcal{M} \to T$.*

Moreover, if S and T have ordered local units then the following condition is also equivalent to (1)–(5)*:*

(6) *there exist a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $Q \otimes_S P \to T$.*

Proof. [12: Theorem 2.1, Remark 2.3] established that (1) $\iff$ (2) $\iff$ (3). From below, we use Proposition 3.2 for (2) $\Longrightarrow$ (4), Proposition 3.1 for (4) $\Longrightarrow$ (3), Theorem 4.2 for (1) $\iff$ (5) and Theorem 6.3 for (1) $\iff$ (6). □

The following is an important observation about the structure of strongly Morita equivalent posemigroups, namely that they have the same local structure.

Corollary 2.1. (cf. [4: Corollary 6]) *If S and T are strongly Morita equivalent posemigroups with local units, then each local subpomonoid of S is isomorphic to a local subpomonoid of T.*

Proof. By Theorem 2.1(2), $C(S)$ and $C(T)$ are Pos-equivalent. Any local subpomonoid eSe of S is isomorphic to the pomonoid $C(S)(e,e)$, which is isomorphic to some pomonoid $C(T)(f,f)$, which in turn is isomorphic to the subpomonoid fTf of T for some $f \in E(T)$. □

3. Consolidations and strict local isomorphisms

In this section we transfer more of Lawson's work on semigroups in [7] to posemigroups and prove that condition (4) of Theorem 2.1 is equivalent to conditions (1)–(3).

Lemma 3.1. *Let S and T be two factorizable posemigroups and let T be an enlargement of S. Then each idempotent of T is $\mathcal{D}$-related to an idempotent of S.*

Proof. This follows from [7: Lemma 4.1]. □

Lemma 3.2.

(1) *Let $\tau\colon S \to T$ be a surjective strict local isomorphism and $\tau'\colon T \to U$ a strict local isomorphism between posemigroups S, T and U, all with local units. Then $\tau'\tau\colon S \to U$ is also a strict local isomorphism.*

(2) *Let T be a posemigroup with local units and let S be a subposemigroup of T which also has local units. Then T is an enlargement of S if and only if the posemigroup embedding of S in T is a strict local isomorphism.*

(3) *Let S and T be posemigroups with local units and let $\tau\colon S \to T$ be a strict local isomorphism. Then the posemigroup T is an enlargement of $\tau(S)$.*

Proof. The proof is almost exactly the same as in [7: Lemma 4.2]. □

Proposition 3.1. *Let S and T be posemigroups with local units. If there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau\colon C(S)^q \to T$, then $C(S)$ and $C(T)$ are Pos-equivalent Pos-categories.*

Proof. The proof carries over from [7: Theorem 1.2], with very minor additional verifications. □

Proposition 3.2. *Let S and T be posemigroups with local units and a joint enlargement R. Then there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau\colon C(S)^q \to T$.*

P r o o f. Without loss of generality, assume that S and T are subposemigroups of R. We will construct a subposemigroup T' of T such that T is an enlargement of T', a consolidation q on $C(S)$ and a surjective strict local isomorphism $\tau\colon C(S)^q \to T'$. Then we have a strict local isomorphism from $C(S)^q$ to T by Lemma 3.2(1) and (2). Since we use the construction from [7: Theorem 1.2], we need not check the purely algebraic conditions, such as τ being a homomorphism, lifting idempotents, T being an enlargement of T', etc.

For each $e \in E(S) \subseteq E(R)$ there exists $i \in E(T)$ such that $e\mathcal{D}i$ in R due to Lemma 3.1. Therefore Remark 1.1 provides $x_e \in R$ and $x'_e \in V(x_e)$ such that

$$x'_e x_e = e \in E(S) \quad \text{and} \quad x_e x'_e = i \in E(T). \tag{2}$$

Define the consolidation q on $C(S)$ by $q_{e,f} = (e, x'_e x_f, f)$. Now define a mapping $\tau\colon C(S)^q \to T$ by

$$\tau(e,s,f) = x_e s x'_f.$$

If $(e_1,s_1,f_1) \leq (e_2,s_2,f_2)$ in $C(S)$, then $e_1 = e_2$, $f_1 = f_2$ and $s_1 \leq s_2$ in S. Thus

$$\tau(e_1,s_1,f_1) = x_{e_1} s_1 x'_{f_1} \leq x_{e_2} s_2 x'_{f_2} = \tau(e_2,s_2,f_2).$$

So τ is a posemigroup homomorphism. Take $T' = \operatorname{Im}\tau$.

We show that $\tau\colon C(S)^q \to T'$ is a strict local isomorphism. By Lemma 1.2, we can replace condition (LI1) with (LI1'). So we only need to examine the restriction $\overline{\tau}\colon (e_0,e,e_1)C(S)^q(f_0,f,f_1) \to \tau(e_0,e,e_1)T'\tau(f_0,f,f_1)$ for arbitrary $(e_0,e,e_1), (f_0,f,f_1) \in E(C(S)^q)$. We already know that $\overline{\tau}$ is a surjective posemigroup homomorphism. Thus we only need to verify that it reflects order. Now take (s_0,s,s_1), $(z_0,z,z_1) \in C(S)$ and assume that

$$\tau(e_0,e,e_1)\tau(s_0,s,s_1)\tau(f_0,f,f_1) \leq \tau(e_0,e,e_1)\tau(z_0,z,z_1)\tau(f_0,f,f_1)$$

in T, i.e.

$$\tau(e_0, ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}f, f_1) \leq \tau(e_0, ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}f, f_1).$$

Then by (2)

$$\begin{aligned} ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}f &= e_0ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}ff_1 = x'_{e_0}(x_{e_0}ex'_{e_1}x_{s_0}sx'_{s_1}x_{f_0}fx'_{f_1})x_{f_1} \\ &\leq x'_{e_0}(x_{e_0}ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}fx'_{f_1})x_{f_1} = e_0ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}ff_1 \\ &= ex'_{e_1}x_{z_0}zx'_{z_1}x_{f_0}f \end{aligned}$$

in S. So

$$(e_0,e,e_1) \diamond (s_0,s,s_1) \diamond (f_0,f,f_1) \leq (e_0,e,e_1) \diamond (z_0,z,z_1) \diamond (f_0,f,f_1)$$

in $C(S)^q$, as required. $\square$

Theorem 3.1. (cf. [7: Theorem 1.2]) *Let S and T be posemigroups with local units. Then S and T satisfy conditions* (1)–(3) *of* Theorem 2.1 *if and only if there exist a consolidation q on $C(S)$ and a strict local isomorphism $\tau\colon C(S)^q \to T$.*

P r o o f. Sufficiency is proved in Proposition 3.1 and necessity in Proposition 3.2. □

4. Rees matrix covers

We will now investigate how the existence of a Rees matrix cover connection relates to strong Morita equivalence.

For a Rees matrix semigroup $\mathcal{M}(S,U,V,M)$ over a posemigroup S, let $\operatorname{Im}(M)$ denote the image of the mapping $M\colon V \times U \to S$. Being a *classical Rees matrix posemigroup* over a pomonoid S means that for every $u \in U$ there exists $v \in V$ such that $M(v,u)$ is invertible and for every $v \in V$ there also exists a $u \in U$ such that $M(v,u)$ is invertible (i.e. every row and every column of M contains an invertible element).

When we speak about Rees matrix posemigroups, we assume that S is a posemigroup, but in general we do not impose additional order-related conditions on U, V and M. Such a Rees matrix semigroup $\mathcal{M}(S,U,V,M)$ can be equipped with a compatible order as follows:

$$(u,s,v) \le (u',s',v') \iff u = u' \ \&\ v = v' \ \&\ s \le s'.$$

Lemma 4.1. *Let T, P and Q be subposemigroups of some posemigroup R. Furthermore, let P be a right T-poset and Q a left T-poset with respect to actions defined by multiplication in R. If $p \otimes q \le p' \otimes q'$ in $P \otimes_T Q$, then $pq \le p'q'$ in R.*

P r o o f. Assume that $p \otimes q \le p' \otimes q'$ in $P \otimes_T Q$. Then by (1) we have $u_1, \dots, u_n$, $v_1, \dots, v_n \in T^1$, $p_1, \dots, p_n \in P$ and $q_2, \dots, q_n \in Q$ such that

$$\begin{array}{rr}
p \le p_1 u_1 & \\
p_1 v_1 \le p_2 u_2 & u_1 q \le v_1 q_2 \\
\vdots & \vdots \\
p_n v_n \le p' & u_n q_n \le v_n q'
\end{array}$$

Therefore in R we get that

$$\begin{aligned}
pq &\le (p_1u_1)q = p_1(u_1q) \le p_1(v_1q_2) = (p_1v_1)q_2 \le (p_2u_2)q_2 \\
&= p_2(u_2q_2) \le \dots \le p_n(u_nq_n) \le p_n(v_nq') = (p_nv_n)q' \le p'q'.
\end{aligned}$$

□

Theorem 4.1. (cf. [6: Theorem 3]) *Let S and T be strongly Morita equivalent posemigroups. Then there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S\operatorname{Im}(M)S = S$ and a surjective strict local isomorphism $\tau\colon \mathcal{M} \to T$.*

Proof. Let $(S, T, P, Q, \langle -, - \rangle, [-, -])$ be a Morita context. As T is factorizable due to Lemma 1.1, we can pick subsets $U, V \subseteq T$ such that $T = UT = TV$. For all $x \in U \cup V$, fix $p_x \in P$ and $q_x \in Q$ such that $x = [q_x, p_x]$. Now, define $M\colon V \times U \to S$ by

$$M(v, u) = \langle p_v, q_u \rangle.$$

This gives us a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$. Since S and T are also strongly Morita equivalent as semigroups and we follow the construction in [6: Theorem 3], we deduce that $S\operatorname{Im}(M)S = S$.

We define our strict local isomorphism $\tau\colon \mathcal{M} \to T$ by

$$\tau(u, s, v) = [q_u, sp_v].$$

Since P is a left S-poset and $[-, -]$ is monotone, τ is also monotone. Again, as we are following the construction of [6: Theorem 3], τ is a surjective posemigroup homomorphism along which idempotents and regular elements lift, so (LI2) and (LI3) are satisfied. To prove that (LI1) holds as well, we need to show that τ reflects order when restricted to certain subposemigroups.

Take $(u_1, s_1, v_1), (u_2, s_2, v_2) \in \mathcal{M}$ such that

$$(u_1, s_1', v_1')(u_1, s_1, v_1) = (u_1, s_1, v_1) \quad \text{and} \quad (u_2, s_2, v_2)(u_2', s_2', v_2) = (u_2, s_2, v_2)$$

for some $(u_1, s_1', v_1'), (u_2', s_2', v_2) \in \mathcal{M}$. Suppose that $\tau(u_1, s, v_2) \leq \tau(u_1, z, v_2)$ for some $(u_1, s, v_2), (u_1, z, v_2) \in (u_1, s_1, v_1)\mathcal{M}(u_2, s_2, v_2)$, i.e.

$$[q_{u_1}, sp_{v_2}] \leq [q_{u_1}, zp_{v_2}].$$

As $(u_1, s, v_2), (u_1, z, v_2) \in (u_1, s_1, v_1)\mathcal{M}(u_2, s_2, v_2)$, we get that

$$\begin{aligned} &(u_1, s_1', v_1')(u_1, s, v_2) = (u_1, s, v_2), && (u_1, s, v_2)(u_2', s_2', v_2) = (u_1, s, v_2), \\ &(u_1, s_1', v_1')(u_1, z, v_2) = (u_1, z, v_2) \quad \text{and} && (u_1, z, v_2)(u_2', s_2', v_2) = (u_1, z, v_2). \end{aligned}$$

So

$$\begin{aligned} s &= s_1' M(v_1', u_1) s M(v_2, u_2') s_2' = s_1' \langle p_{v_1'}, q_{u_1} \rangle s \langle p_{v_2}, q_{u_2'} \rangle s_2' \\ &= s_1' \langle p_{v_1'}, q_{u_1} \langle sp_{v_2}, q_{u_2'} \rangle \rangle s_2' = s_1' \langle p_{v_1'}, [q_{u_1}, sp_{v_2}] q_{u_2'} \rangle s_2' \\ &\leq s_1' \langle p_{v_1'}, [q_{u_1}, zp_{v_2}] q_{u_2'} \rangle s_2' = s_1' \langle p_{v_1'}, q_{u_1} \langle zp_{v_2}, q_{u_2'} \rangle \rangle s_2' \\ &= s_1' \langle p_{v_1'}, q_{u_1} \rangle z \langle p_{v_2}, q_{u_2'} \rangle s_2' = s_1' M(v_1', u_1) z M(v_2, u_2') s_2' = z. \end{aligned}$$

□

Theorem 4.1 admits the following converse:

Lemma 4.2. (cf. [6: Lemma 7]) *If S and T are factorizable posemigroups, $\mathcal{M} = \mathcal{M}(S, U, V, M)$ is a Rees matrix posemigroup over S with $S\,\mathrm{Im}(M)S = S$ and there exists a surjective strict local isomorphism $\tau\colon \mathcal{M} \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Proof. Again, we use the fact that all surjective strict local isomorphisms of posemigroups are idempotent-lifting strict local isomorphisms of their underlying semigroups and make use of the construction in [6: Lemma 7] to get a full essentially surjective functor $F\colon C(S) \to C(T)$. The construction itself is as follows. Take $e \in E(S)$ and pick $a_e, b_e \in S$, $u_e \in U$ and $v_e \in V$ such that

$$e = a_e M(v_e, u_e) b_e.$$

Then $\overline{e} = (u_e, b_e e a_e, v_e) \in E(\mathcal{M})$ and $\tau(\overline{e}) \in E(T)$. The assignment

$$\begin{tikzcd} e \arrow[r, mapsto] \arrow[d, "{(f,s,e)}"'] & \tau(\overline{e}) \arrow[d, "{(\tau(\overline{f}),\tau(u_f,b_f s a_e,v_e),\tau(\overline{e}))}"] \\ f \arrow[r, mapsto] & \tau(\overline{f}) \end{tikzcd}$$

defines a functor $F\colon C(S) \to C(T)$. As τ is monotone, F is a Pos-functor. We now only need to verify that it reflects the order of morphisms. Suppose that $\tau(u_f, b_f s a_e, v_e) \le \tau(u_f, b_f s' a_e, v_e)$ for some $(f, s, e), (f, s', e)\colon e \to f$ in $C(S)$. First,

$$\begin{aligned} \overline{f}(u_f, b_f s a_e, v_e)\overline{e} &= (u_f, b_f f a_f M(v_f, u_f) b_f s a_e M(v_e, u_e) b_e e a_e, v_e) \\ &= (u_f, b_f f s e a_e, v_e) = (u_f, b_f s a_e, v_e) \end{aligned}$$

and likewise

$$\overline{f}(u_f, b_f s' a_e, v_e)\overline{e} = (u_f, b_f s' a_e, v_e).$$

Since $\overline{f}, \overline{e} \in E(\mathcal{M})$ and τ is a strict local isomorphism, due to (LI1) we get that $(u_f, b_f s a_e, v_e) \le (u_f, b_f s' a_e, v_e)$ in $\mathcal{M}$ and thus $b_f s a_e \le b_f s' a_e$ in S. But then

$$\begin{aligned} s = fse &= a_f M(v_f, u_f) b_f s a_e M(v_e, u_e) b_e \\ &\le a_f M(v_f, u_f) b_f s' a_e M(v_e, u_e) b_e = f s' e = s'. \end{aligned}$$

□

In conclusion, we have the following description of strong Morita equivalence.

Theorem 4.2. (cf. [6: Theorem 8]) *Let S and T be posemigroups with local units. Then S and T are strongly Morita equivalent if and only if there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S\,\mathrm{Im}(M)S = S$ and a surjective strict local isomorphism $\tau\colon \mathcal{M} \to T$.*

Proof. Necessity follows from Theorem 4.1. For sufficiency, we use Lemma 4.2 and [12: Theorem 2.1]. □

Corollary 4.1. (cf. [6: Proposition 2]) *Let S be a factorizable posemigroup. A Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S is strongly Morita equivalent to S if and only if $S \operatorname{Im}(M) S = S$.*

Proof. This follows from Lemma 1.1 and Theorem 4.2, using the identity map $1_{\mathcal{M}}$. □

The preceding theorem can be specified to unipotent pomonoids as follows:

Theorem 4.3. *Let T be a posemigroup with local units. Then T is strongly Morita equivalent to a unipotent pomonoid S if and only if there exist a classical Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S and a surjective strict local isomorphism $\tau \colon \mathcal{M} \to T$.*

Proof. For necessity, let $(S, T, P, Q, \langle -, - \rangle, [-, -])$ be a Morita context. We follow the proof of Theorem 4.1 and use the existence of local units in T to put $U = V = E(T)$. We also use [11: Proposition 4.2] to find an idempotent $e = [q_1, p_1]^2 \in E(T)$ so that $T = TeT$, where $\langle p_1, q_1 \rangle = 1 \in S$. By Corollary 2.1, every local subpomonoid of T is isomorphic to S. Hence eTe is a unipotent pomonoid. For every $u \in U$, fix $p_u \in P$ and $q_u \in Q$ such that $u = [q_u, p_u]$.

We only prove the row part of being a classical Rees matrix posemigroup, as the column part is proved similarly. Take $v \in V = E(T)$, then $v^2 = v$ and $v = [q_v, p_v]$. We will show that $M(v, v) = \langle p_v, q_v \rangle$, as constructed in Theorem 4.1, is invertible in S.

If we take $x = e[q_1, p_v]v[q_v, p_1]e \in eTe$, then

$$\begin{aligned} x^2 &= e[q_1, p_v]v[q_v, p_1]e[q_1, p_v]v[q_v, p_1]e = e[q_1, p_v]v[q_v, \langle p_1, eq_1 \rangle p_v]v[q_v, p_1]e \\ &= e[q_1, p_v]v[q_v, \langle p_1, [q_1, p_1]^2 q_1 \rangle p_v]v[q_v, p_1]e \\ &= e[q_1, p_v]v[q_v, \langle p_1, q_1 \rangle^3 p_v]v[q_v, p_1]e = e[q_1, p_v][q_v, p_v][q_v, 1^3 p_v]v[q_v, p_1]e \\ &= e[q_1, p_v][q_v, (\langle p_v, q_v \rangle 1) p_v]v[q_v, p_1]e = e[q_1, p_v][q_v, \langle p_v, q_v \rangle p_v]v[q_v, p_1]e \\ &= e[q_1, p_v][q_v, p_v]^2 v[q_v, p_1]e = e[q_1, p_v]v^3[q_v, p_1]e = x. \end{aligned}$$

So $x \in E(T)$. Since eTe has only one idempotent, we must have $x = e$. Hence

$$\begin{aligned} \langle p_v, q_v \rangle \langle p_v, vq_v \rangle &= \langle p_1, q_1 \rangle^3 \langle p_v, q_v \rangle \langle p_v, vq_v \rangle \langle p_1, q_1 \rangle^3 \\ &= \langle p_1, e[q_1, p_v][q_v, p_v]v[q_v, p_1]eq_1 \rangle \\ &= \langle p_1, e[q_1, p_v]v^2[q_v, p_1]eq_1 \rangle \\ &= \langle p_1, xq_1 \rangle = \langle p_1, eq_1 \rangle \\ &= \langle p_1, [q_1, p_1]^2 q_1 \rangle = \langle p_1, q_1 \rangle^3 = 1. \end{aligned}$$

For sufficiency, we note that Theorem 4.2 applies, since every $s \in S$ can be factorized as $s = szz^{-1}$ for any invertible $z \in \Im(M)$. $\square$

5. Regular Rees matrix covers

Drawing on [8] for inspiration, we convert our results on Rees matrix covers to the special case of regular posemigroups.

Let S be a regular posemigroup. Following the terminology of [8], we call the subposemigroup of $\mathcal{M}(S, U, V, M)$ consisting of all regular elements a *regular Rees matrix posemigroup* over S and denote it by $\mathcal{RM}(S, U, V, M)$. Note that in our case, the order is weaker than that used by McAlister. [8: Lemma 2.1] shows that $\mathcal{RM}(S, U, V, M)$ is indeed a subsemigroup of $\mathcal{M}(S, U, V, M)$.

Theorem 5.1. (cf. [8: Theorem 2.9] and Theorem 4.1) *Let S and T be strongly Morita equivalent regular posemigroups. Then there exist a regular Rees matrix posemigroup $\mathcal{R} = \mathcal{RM}(S, U, V, M)$ with $S\Im(M)S = S$ and a surjective strict local isomorphism $\tau\colon \mathcal{R} \to T$.*

Proof. By Theorem 4.1, there is a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ with $S\Im(M)S = S$ and a surjective strict local isomorphism $\tau'\colon \mathcal{M} \to T$ that lifts regular elements. Take $\mathcal{R} = \mathcal{RM}(S, U, V, M)$ and $\tau = \tau'|_{\mathcal{R}}$. Then τ is surjective because τ' lifts regular elements. Being a restriction of an idempotent-lifting map, it still lifts idempotents. Since the idempotents of $\mathcal{R}$ are idempotents of $\mathcal{M}$, condition (LI1) can be verified by an argument similar to the one used for proving Lemma 1.2. $\square$

Lemma 5.1. (cf. Lemma 4.2) *Let S and T be two regular posemigroups. If $\mathcal{R} = \mathcal{RM}(S, U, V, M)$ is a regular Rees matrix posemigroup over S such that $S\Im(M)S = S$ and there exists a surjective strict local isomorphism $\tau\colon \mathcal{R} \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Proof. We use the proof of Lemma 4.2 and check that if we assume S to be regular, we can use $\tau\colon \mathcal{R} \to T$ instead of the more general morphism $\mathcal{M}(S, U, V, M) \to T$. This amounts to checking that if $s = fse$ $(e, f \in E(S))$ is a regular element of S, then $(u_f, b_f s a_e, v_e) \in \mathcal{M}$ is also a regular element; and if $s = sM(v, u)s$ and $e = sM(v, u) \in E(S)$, then (u, ea_e, v_e) and $(u_e, b_e s, v)$ are regular elements of $\mathcal{M}$. Note that the latter requirement is necessary to retain the validity of the proof in [6: Lemma 7] that we obtain an essentially surjective functor. Indeed, it is straightforward to check that if $s = fse$ then $(u_e, b_e s' a_f, v_f) \in V(u_f, b_f s a_e, v_e)$ for all $s' \in V(s)$. Moreover, if $s = sM(v, u)s$ and $e = sM(v, u)$ then $(u_e, b_e s, v) \in V(u, ea_e, v_e)$. $\square$

Theorem 5.2. (cf. Theorem 4.2) *Let S and T be regular posemigroups. Then S and T are strongly Morita equivalent if and only if there exist a regular Rees matrix posemigroup $\mathcal{R} = \mathcal{RM}(S, U, V, M)$ with $S\Im(M)S = S$ and a surjective strict local isomorphism $\tau\colon \mathcal{R} \to T$.*

P r o o f. Necessity follows from Theorem 5.1. Sufficiency is immediate due to Lemma 5.1 and Theorem 2.1. □

Corollary 5.1. (cf. Cor 4.1) *If S is a regular posemigroup, then any regular Rees matrix posemigroup $\mathcal{R} = \mathcal{RM}(S, U, V, M)$ is strongly Morita equivalent to S if and only if $S\Im(M)S = S$.*

P r o o f. Like in Corollary 4.1, the identity morphism $1_{\mathcal{R}}$ satisfies the requirements of Theorem 5.2. □

We conclude this section with two results about extended orders on (regular) Rees matrix posemigroups. Because the order in Cauchy completions only features comparisons of the type $(f, s, e) \leq (f, s', e)$, we can generalize Lemmas 4.2 and 5.1 to the following corollaries, the latter of which can be considered a converse to [8: Theorem 2.9].

Corollary 5.2. *Let S and T be two factorizable posemigroups. Suppose that $\mathcal{M} = \mathcal{M}(S, U, V, M)$ is a Rees matrix semigroup over S with a partial order that coincides with that of the corresponding Rees matrix posemigroup on local subpomonoids, such that $S\Im(M)S = S$, and that there exists a surjective strict local isomorphism $\tau\colon \mathcal{M} \to T$. Then the* Pos*-categories $C(S)$ and $C(T)$ are* Pos*-equivalent.*

Corollary 5.3. *If S and T are two regular posemigroups, $\mathcal{R} = \mathcal{R}(S, U, V, M)$ is a regular Rees matrix semigroup over S with a partial order that coincides with that of the corresponding Rees matrix posemigroup on local subpomonoids, $S\Im(M)S = S$ and there is a surjective strict local isomorphism $\tau\colon \mathcal{R} \to T$, then $C(S)$ and $C(T)$ are* Pos*-equivalent.*

6. Morita posemigroups

We extend a construction from [10] to the ordered situation. Let S be a posemigroup and let ${}_SP$ and Q_S be respectively a left and a right S-poset. If we have an (S, S)-biposet morphism

$$\langle -, - \rangle\colon {}_SP \times Q_S \to S,$$

then it turns out that the assignment

$$(q \otimes p)(q' \otimes p') = q \otimes \langle p, q' \rangle p' \tag{3}$$

defines a posemigroup structure on $Q \otimes_S P$. It is straightforward to check that this operation is associative. Verifying that the assignment is monotone (implying it is well-defined) is a bit more involved. Assume that $q \otimes p \leq \overline{q} \otimes \overline{p}$ and $q' \otimes p' \leq \overline{q}' \otimes \overline{p}'$ in $Q \otimes_S P$. By (1) we have a scheme

$$\begin{array}{ll} q \leq q_1 s_1 & \\ q_1 t_1 \leq q_2 s_2 & s_1 p \leq t_1 p_2 \\ \vdots & \vdots \\ q_n t_n \leq \overline{q} & s_n p_n \leq t_n \overline{p}, \end{array}$$

where $q_1, \ldots, q_n \in Q$, $p_2, \ldots, p_n \in P$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$. But then also

$$\begin{array}{ll} q \leq q_1 s_1 & \\ q_1 t_1 \leq q_2 s_2 & s_1 \langle p, q' \rangle p' \leq t_1 \langle p_2, q' \rangle p' \\ \vdots & \vdots \\ q_n t_n \leq \overline{q} & s_n \langle p_n, q' \rangle p' \leq t_n \langle \overline{p}, q' \rangle p', \end{array}$$

so

$$(q \otimes p)(q' \otimes p') = q \otimes \langle p, q' \rangle p' \leq \overline{q} \otimes \langle \overline{p}, q' \rangle p' = (\overline{q} \otimes \overline{p})(q' \otimes p').$$

A symmetric argument shows that $(\overline{q} \otimes \overline{p})(q' \otimes p') \leq (\overline{q} \otimes \overline{p})(\overline{q}' \otimes \overline{p}')$, thus $(q \otimes p)(q' \otimes p') \leq (\overline{q} \otimes \overline{p})(\overline{q}' \otimes \overline{p}')$, as required.

We say that $Q \otimes_S P$ with the multiplication defined by (3) is the *Morita posemigroup over S defined by $\langle -, - \rangle$*. If Q_S and ${}_S P$ are unitary S-posets, then we call the Morita posemigroup *unitary*; if $\langle -, - \rangle$ is surjective then we say that the Morita semigroup is *surjectively defined.*

We recall an auxiliary result from [13].

Lemma 6.1. *Let A be an (S,T)-biposet and B a (T,S)-biposet. A (T,T)-act morphism $f \colon {}_T B \times A_T \to {}_T C_T$ which preserves the order in both arguments and satisfies the condition*

$$f(b \cdot s, a) = f(b, s \cdot a) \tag{4}$$

(called a balanced *morphism) yields a well-defined (T,T)-biposet morphism f' : ${}_T(B \otimes_S A)_T \to C_T$ by taking*

$$f'(b \otimes a) = f(b, a).$$

Theorem 6.1. (cf. [10: Theorem 5]) *Let S be a factorizable posemigroup, let ${}_SP$ and Q_S be unitary left and right S-posets, respectively. Furthermore, let*

$$\langle -,- \rangle \colon {}_SP \times Q_S \to S$$

be a surjective (S,S)-biposet morphism. Then the Morita semigroup $Q \otimes_S P$ defined by $\langle -,- \rangle$ is strongly Morita equivalent to S.

Proof. We can define a right $(Q \otimes_S P)$-action on P by

$$p \cdot (q' \otimes p') = \langle p, q' \rangle p'.$$

To show that such an action makes P into a unitary $(S, Q \otimes_S P)$-biposet, we need to verify that this action is monotone (which implies that it is well-defined), associative, satisfies the equality $P(Q \otimes_S P) = P$ and that the S- and $(Q \otimes_S P)$-actions on P commute.

First, let $q' \otimes p' \le q'' \otimes p''$ in $Q \otimes_S P$ and $p \le \overline{p}$ in P. Then there are $q_1, \dots, q_n \in Q$, $p_2, \dots, p_n \in P$ and $s_1, \dots, s_n, t_1, \dots, t_n \in S^1$ such that

$$\begin{array}{cc} q' \le q_1 s_1 & \\ q_1 t_1 \le q_2 s_2 & s_1 p' \le t_1 p_2 \\ \vdots & \vdots \\ q_n t_n \le q'' & s_n p_n \le t_n p''. \end{array}$$

Thus

$$\begin{aligned} p \cdot (q' \otimes p') &= \langle p, q' \rangle p' \le \langle \overline{p}, q' \rangle p' \le \langle \overline{p}, q_1 s_1 \rangle p' \le \langle \overline{p}, q_1 \rangle t_1 p_2 \\ &\le \langle \overline{p}, q_2 s_2 \rangle p_2 \le \dots \le \langle \overline{p}, q_n s_n \rangle p_n \le \langle \overline{p}, q_n \rangle t_n p'' \\ &\le \langle \overline{p}, q'' \rangle p'' = \overline{p} \cdot (q'' \otimes p''). \end{aligned}$$

Now, take $p \in P$, $s \in S$, $q' \otimes p', q'' \otimes p'' \in Q \otimes_S P$. Then

$$p \cdot ((q' \otimes p')(q'' \otimes p'')) = \langle p, q' \rangle \langle p', q'' \rangle p'' = \langle \langle p, q' \rangle p', q'' \rangle p'' = (p \cdot (q' \otimes p')) \cdot (q'' \otimes p'')$$

and

$$(sp) \cdot (q' \otimes p') = \langle sp, q' \rangle p' = s \langle p, q' \rangle p' = s \cdot (p \cdot (q' \otimes p')).$$

Finally, P is unitary as a right $(Q \otimes_S P)$-poset since each $p \in P$ can be written as $p = sp'$ for some $s \in S$, $p' \in P$, and $s = \langle p'', q'' \rangle$ for some $p'' \in P$, $q'' \in Q$, so $p = \langle p'', q'' \rangle p' = p'' \cdot (q'' \otimes p')$.

Similarly, if we define a left $(Q \otimes_S P)$-action on Q by

$$(q' \otimes p') \cdot q = q' \langle p', q \rangle,$$

then Q becomes a unitary $(Q \otimes_S P, S)$-biposet.

As

$$\langle p, (q' \otimes p') \cdot q \rangle = \langle p, q' \langle p', q \rangle \rangle = \langle p, q' \rangle \langle p', q \rangle = \langle \langle p, q' \rangle p', q \rangle = \langle p \cdot (q' \otimes p'), q \rangle,$$

the mapping $\langle -, - \rangle \colon {}_S P \times Q_S \to S$ is a $Q \otimes_S P$-balanced (S,S)-biposet morphism. Therefore if we define another mapping $|-,-| \colon P \otimes_{Q \otimes_S P} Q \to S$ by $|p,q| = \langle p, q \rangle$, it will turn out to be a well-defined (S,S)-biposet morphism by Lemma 6.1. Taking $[-,-] := 1_{Q \otimes_S P}$, we get that $(S, Q \otimes_S P, P, Q, |-,-|, [-,-])$ is a unitary Morita context with surjective maps, since

$$|p,q|p' = p \cdot (q \otimes p') = p \cdot [q, p'] \quad \text{and} \quad [q,p] \cdot q' = (q \otimes p) \cdot q' = q|p, q'|.$$

□

Remark 6.2. Note that in [10: Theorem 5], Talwar claims that the required Morita context is $(R, Q \otimes_R P, P, Q, \langle -,- \rangle, [-,-])$, with $\langle -,- \rangle \colon {}_R P \times Q_R \to R$.

Proposition 6.1. (cf. [10: Proposition 4]) *Let $(S, T, P, Q, \langle -,- \rangle, [-,-])$ be a unitary Morita context of posemigroups. Then the (S,S)-biposet $P \otimes_T Q$ together with multiplication $(p \otimes q)(p' \otimes q') = p[q,p'] \otimes q'$ and the (T,T)-biposet $Q \otimes_S P$ with multiplication $(q \otimes p)(q' \otimes p') = q \otimes \langle p, q' \rangle p'$ are posemigroups and the maps $\langle -,- \rangle \colon P \otimes_T Q \to S$ and $[-,-] \colon Q \otimes_S P \to T$ are posemigroup morphisms. If the latter maps are also surjective, then all the posemigroups $P \otimes_T Q$, $Q \otimes_S P$, S and T are strongly Morita equivalent.*

Proof. The proof is essentially the same as in [10: Proposition 4]. □

Corollary 6.1. (cf. [10: Corollary 6]) *Let S be a posemigroup with local units, ${}_S P$ a unitary left S-poset, Q_S a unitary right S-poset and $\langle -,- \rangle \colon {}_S P \times Q_S \to S$ a surjective (S,S)-biposet morphism. Then $Q \otimes_S P$ is a sandwich posemigroup.*

Proof. This is a direct consequence of [10: Corollary 6]. □

We now provide links between Morita posemigroups and Rees matrix posemigroups.

Proposition 6.2. (cf. [6: Proposition 10]) *Let S be a posemigroup with ordered weak local units and let $\mathcal{M} = \mathcal{M}(S, U, V, M)$ be a Rees matrix posemigroup over S. Then $\mathcal{M}$ is isomorphic to a unitary Morita posemigroup. If $S = S\Im(M)S$, then that Morita semigroup is surjectively defined.*

Proof. Again, the verifications of algebraic details can be found in the proof of [6: Proposition 10]. Let ${}_S P := {}_S(S \times V)$ and $Q_S := (U \times S)_S$ be the free S-posets with bases V and U. Define a map $\langle -,- \rangle \colon {}_S P \times Q_S \to S$ by

$$\langle (s,v), (u,z) \rangle = sM(v,u)z.$$

Since this map is clearly monotone, we can consider the Morita semigroup $Q \otimes_S P$ over S defined by $\langle -, - \rangle$. Because S has weak local units, P and Q are unitary, so $Q \otimes_S P$ is also unitary. Moreover, $Q \otimes_S P$ is surjectively defined iff $S = S\Im(M)S$.

We can now define another map $\varphi \colon \mathcal{M} \to Q \otimes_S P$ by

$$\varphi(u, s, v) = (u, e) \otimes (s, v),$$

where $s = es$, $e \in S$.

To see that φ is monotone, take $(u, s, v) \leq (u, s', v)$ in $\mathcal{M}$, whence $s \leq s'$ in S. Then the existence of ordered weak local units in S provides $e, e' \in S$ such that $es = s$, $e's' = s'$ and $e \leq e'$. So

$$\varphi(u, s, v) = (u, e) \otimes (s, v) \leq (u, e') \otimes (s', v) = \varphi(u, s', v).$$

We now show that φ also reflects order. Take $\varphi(u, s, v) \leq \varphi(u', s', v')$, where $s = es$ and $s' = e's'$. Then $(u, e) \otimes (s, v) \leq (u', e') \otimes (s', v')$, whence $u = u'$, $v = v'$ and $e \otimes s \leq e' \otimes s'$ in $S \otimes_S S$. By Lemma 4.1, $s = es \leq e's' = s'$, implying $(u, s, v) \leq (u', s', v')$ in $\mathcal{M}$. □

Proposition 6.3. (cf. [6: Proposition 11]) *Let S be an arbitrary posemigroup and let $Q \otimes_S P$ be a unitary Morita posemigroup defined by $\langle -, - \rangle \colon {}_S P \times Q_S \to S$. Then there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S, U, V, M)$ over S and a surjective strict local isomorphism $\tau \colon \mathcal{M} \to Q \otimes_S P$. If the mapping $\langle -, - \rangle$ is surjective, then $S = S\Im(M)S$.*

Proof. Once more, the purely algebraic details of the proof are the same as in [6: Proposition 11]. Take $U = Q$, $V = P$ and $M = \langle -, - \rangle$. Then we can define $\tau \colon \mathcal{M} \to Q \otimes_S P$ by

$$\tau(q, s, p) = q \otimes sp.$$

As ${}_S P$ is unitary, τ is surjective and trivially satisfies (LI3). Also, τ is a posemigroup homomorphism as it is clearly monotone in s.

To demonstrate (LI1), we only need to show that τ reflects order on certain subsets. Fix $(q_1, s_1, p_1), (q_2, s_2, p_2) \in \mathcal{M}$ and $s_1', s_2' \in S$, $p_2' \in P$, $q_2' \in Q$ such that

$$(q_1, s_1', p_2')(q_1, s_1, p_1) = (q_1, s_1, p_1) \quad \text{and} \quad (q_2, s_2, p_2)(q_2', s_2', p_2) = (q_2, s_2, p_2).$$

Take $(q_1, s, p_2), (q_1, z, p_2) \in (q_1, s_1, p_1)\mathcal{M}(q_2, s_2, p_2)$ for which

$$q_1 \otimes sp_2 = \tau(q_1, s, p_2) \leq \tau(q_1, z, p_2) = q_1 \otimes zp_2$$

in $Q \otimes_S P$. Then

$$(q_1, s_1', p_2')(q_1, s, p_2) = (q_1, s, p_2) = (q_1, s, p_2)(q_2', s_2', p_2)$$

and

$$(q_1, s_1', p_2')(q_1, z, p_2) = (q_1, z, p_2) = (q_1, z, p_2)(q_2', s_2', p_2),$$

implying $s'_1\langle p'_2, q_1\rangle s = s = s\langle p_2, q'_2\rangle s'_2$ and $s'_1\langle p'_2, q_1\rangle z = z = z\langle p_2, q'_2\rangle s'_2$. The inequality $q_1 \otimes sp_2 \leq q_1 \otimes zp_2$ means that there are $\overline{q}_1, \ldots, \overline{q}_n \in Q$, $\overline{p}_2, \ldots, \overline{p}_n \in P$ and $u_1, \ldots, u_n, v_1, \ldots, v_n \in S^1$ such that

$$\begin{array}{rlrl} q_1 & \leq \overline{q}_1 u_1 & & \\ \overline{q}_1 v_1 & \leq \overline{q}_2 u_2 & \qquad u_1 s p_2 & \leq v_1 \overline{p}_2 \\ & \vdots & & \vdots \\ \overline{q}_n v_n & \leq q_1 & \qquad u_n \overline{p}_n & \leq v_n z p_2. \end{array}$$

Thus

$$\begin{aligned} s &= s'_1\langle p'_2, q_1\rangle s\langle p_2, q'_2\rangle s'_2 \leq s'_1\langle p'_2, \overline{q}_1 u_1\rangle s\langle p_2, q'_2\rangle s'_2 \\ &\leq s'_1\langle p'_2, \overline{q}_1\rangle\langle v_1\overline{p}_2, q'_2\rangle s'_2 \leq s'_1\langle p'_2, \overline{q}_2 u_2\rangle\langle \overline{p}_2, q'_2\rangle s'_2 \\ &\leq \ldots \leq s'_1\langle p'_2, \overline{q}_n u_n\rangle\langle \overline{p}_n, q'_2\rangle s'_2 \\ &\leq s'_1\langle p'_2, \overline{q}_n\rangle\langle v_n z p_2, q'_2\rangle s'_2 \leq s'_1\langle p'_2, q_1\rangle z\langle p_2, q'_2\rangle s'_2 = z, \end{aligned}$$

whence $(q_1, s, p_2) \leq (q_1, z, p_2)$.

Now we check that τ lifts idempotents, i.e. it satisfies condition (LI2). Take an idempotent $q \otimes p = (q \otimes p)(q \otimes p) = q \otimes \langle p, q\rangle p \in E(Q \otimes_S P)$. The equality $q \otimes p = q \otimes \langle p, q\rangle p$ is equivalent to the two inequalities $q \otimes p \leq q \otimes \langle p, q\rangle p$ and $q \otimes p \geq q \otimes \langle p, q\rangle p$. The latter means that there exist $\overline{q}_1, \ldots, \overline{q}_n \in Q$, $\overline{p}_2, \ldots, \overline{p}_n \in P$ and $u_1, \ldots, u_n, v_1, \ldots, v_n \in S^1$ such that

$$\begin{array}{rlrl} q & \leq \overline{q}_1 u_1 & & \\ \overline{q}_1 v_1 & \leq \overline{q}_2 u_2 & \qquad u_1 \langle p, q\rangle p & \leq v_1 \overline{p}_2 \\ & \vdots & & \vdots \\ \overline{q}_n v_n & \leq q & \qquad u_n \overline{p}_n & \leq v_n p. \end{array}$$

Thus

$$\begin{aligned} q\langle p, q\rangle\langle p, q\rangle &\leq \overline{q}_1 u_1\langle p, q\rangle\langle p, q\rangle \leq \overline{q}_1\langle v_1\overline{p}_2, q\rangle \leq \overline{q}_2 u_2\langle \overline{p}_2, q\rangle \\ &\leq \ldots \leq \overline{q}_n u_n\langle \overline{p}_n, q\rangle \leq \overline{q}_n\langle v_n p, q\rangle \leq q\langle p, q\rangle. \end{aligned}$$

Analogously, $q \otimes p \leq q \otimes \langle p, q\rangle p$ implies that $q\langle p, q\rangle \leq q\langle p, q\rangle\langle p, q\rangle$, so

$$q\langle p, q\rangle = q\langle p, q\rangle\langle p, q\rangle.$$

Using this equality, we calculate in $\mathcal{M}$ that

$$\begin{aligned} (q, \langle p, q\rangle^2, p)(q, \langle p, q\rangle^2, p) &= (q, \langle p, q\rangle^5, p) = (q, \langle p, q\langle p, q\rangle^4\rangle, p) \\ &= (q, \langle p, q\langle p, q\rangle\rangle, p) = (q, \langle p, q\rangle^2, p). \end{aligned}$$

Therefore $(q, \langle p,q\rangle^2, p) \in E(\mathcal{M})$ and since

$$\tau(q, \langle p,q\rangle^2, p) = q \otimes \langle p,q\rangle^2 p = (q \otimes p)^3 = q \otimes p,$$

(LI2) holds. □

Corollary 6.2. (cf. [6: Corollary 12]) *Let S and T be posemigroups with ordered weak local units. Then the following are equivalent:*

(1) *there exist a Rees matrix posemigroup $\mathcal{M} = \mathcal{M}(S,U,V,M)$ that satisfies $S\Im(M)S = S$, and a surjective strict local isomorphism $\tau\colon \mathcal{M} \to T$;*

(2) *there exist a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $\tau\colon Q \otimes_S P \to T$.*

Proof.

(1) $\Longrightarrow$ (2) by Proposition 6.2.

(2) $\Longrightarrow$ (1) by precomposing with the surjective strict local isomorphism from Proposition 6.3 (see also Lemma 3.2(1)). □

As a consequence of Corollary 6.2 and Theorem 4.2, we have

Theorem 6.3. (cf. [6: Theorem 13]) *Let S and T be posemigoups with ordered local units. Then S is strongly Morita equivalent to T if and only if there exist a surjectively defined unitary Morita posemigroup $Q \otimes_S P$ and a surjective strict local isomorphism $\tau\colon Q \otimes_S P \to T$.*

Since unordered (trivially ordered) semigroups have local units iff they have ordered local units, our additional assumption of ordered local units is not too restrictive.

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