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ON THE KLUVÁNEK CONSTRUCTION OF THE LEBESGUE INTEGRAL WITH RESPECT TO A VECTOR MEASURE

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Ján Borsík*)

ABSTRACT. The Kluvánek construction of the Lebesgue integral is extended in two directions. First, instead of a compact interval $[a, b]$ in the real line an abstract non-empty set X is considered, instead of the ring generated by subintervals of $[a, b]$ an arbitrary ring $\mathcal{A}$ of subsets of X. Secondly, instead of the length of intervals ($\lambda([c, d]) = d - c$) any vector measure $\lambda\colon \mathcal{A} \to V$ is considered, where V is a Riesz space.

1. Introduction

If $\mathcal{F}$ is the family of all integrable functions (in some sense) and we put for $f, g \in \mathcal{F}$

$$d(f, g) = \int_a^b |f(x) - g(x)| \,\mathrm{d}x,$$

then $(\mathcal{F}, d)$ is a pseudometric space. If $\mathcal{F}$ consists of all Lebesgue integrable functions on $[a, b]$, then the space $(\mathcal{F}, d)$ is complete. It has important applications, e.g. in the probability theory, and in functional analysis.

Usually the Lebesgue integral is defined by the help of measures $\lambda(A)$ of sets A from the family of Borel subsets of $[a, b]$, hence the length $\lambda([d, c]) = d - c$ must be extended to measures $\lambda(A)$ of sets from the σ-ring generated by intervals. And the necessity of the extension presents a didactical problem. The Kluvánek

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idea ([10], [11]) is in a construction of the Lebesgue integral by the help of lengths of intervals only.

A function $f\colon [a,b] \to [-\infty,\infty]$ is integrable by the Kluvánek construction, if there exist $\alpha_i \in \mathbb{R}$, and there exist intervals $A_i \subset [a,b]$ such that

$$\sum_{i=1}^{\infty} |\alpha_i|\lambda(A_i) < \infty$$

and

$$f(x) = \sum \alpha_i \chi_{A_i}(x)$$

whenever

$$\sum |\alpha_i| \chi_{A_i}(x) < \infty.$$

The only problem is in the proof of the independence of the integral of a function f on the presentation of f in the form

$$f(x) = \sum \alpha_i \chi_{A_i}(x).$$

Mainly, Kluvánek's proof depends on some properties of the real line. Therefore we suggested in [21] a modification of the construction considering first non-negative functions only and we proved the equality of two definitions. A non-negative function $f\colon [a,b] \to [0,\infty]$ is integrable ($f \in \mathcal{P}^+$), if there exist $\alpha_i \geq 0$ and there exist intervals $A_i \subset [a,b]$ such that

$$\sum_{i=1}^{\infty} \alpha_i \mu(A_i) < \infty$$

and

$$f(x) = \sum \alpha_i \chi_{A_i}(x)$$

whenever

$$\sum \alpha_i \chi_{A_i}(x) < \infty.$$

It is not so difficult to prove the independence of the integral on the presentation of f in the form

$$f(x) = \sum \alpha_i \chi_{A_i}(x),$$

and the independence can be realised also in a very general case. Moreover in [21] it has been proved the following result:

Theorem. *A function $f\colon [a,b] \to [\infty,\infty])$ is integrable (in the Kluvánek sense) if and only if there exist $g, h \in \mathcal{P}^+$, such that*

$$f(x) = g(x) - h(x)$$

whenever

$$g(x) < \infty, \qquad h(x) < \infty.$$

2. Riesz spaces

A Riesz space V is a special case of a linear space that is moreover an l-group $(V,+,\le)$, hence $(V,+)$ is a commutative group, $(V,\le)$ is a lattice, and $a \le b \implies a+c \le b+c$.

In the Riesz space also the following property is assumed: $a \le b$, $a,b \in V$, and $\alpha \in \mathbb{R}$, $\alpha \ge 0$ implies $\alpha a \le \alpha b$.

Recall that in any Riesz space there holds the following identities

$$\begin{aligned} a+(b\vee c) &= (a+b)\vee(a+c),\\ a+(b\wedge c) &= (a+b)\wedge(a+c),\\ a-(b\vee c) &= (a-b)\wedge(a-c),\\ a-(b\wedge c) &= (a-b)\vee(a-c),\\ a\vee b + a\wedge b &= a+b. \end{aligned}$$

A very well known example of a Riesz space is the vector space $\mathbb{R}$ of real numbers with usual operations and ordering. Also in the general case we define

$$a^+ = a\vee 0, \qquad a^- = (-a)\vee 0, \qquad |a| = a^+ + a^-.$$

Evidently

$$a = a^+ - a^-$$

for every $a \in V$. We shall work with complete spaces, i.e. any upper bounded subset $A \subset V$ has the supremum = the least upper bound $\sup A = \bigvee A$. It follows that also any lower bounded subset $A \subset V$ has the infimum. In complete Riesz spaces the identities stated above can be generalized:

$$\begin{aligned} a+\bigvee a_i &= \bigvee(a+a_i),\\ a+\bigwedge a_i &= \bigwedge(a+a_i). \end{aligned}$$

Every complete Riesz space is Archimedean, i.e. if $a,b \in V$, and

$$na \le b$$

for any $n \in \mathbb{R}$, then

$$a \le 0.$$

Indeed, put $c = \bigvee\limits_{n=1}^{\infty} na$. Then

$$c + a = \bigvee_{n=1}^{\infty}(n+1)a \le c,$$

hence $a \le 0$. Recall that some integration theories with respect to Riesz space valued measures have been studied e.g. in [1], [2], [4], some probability applications in [13], [16]. Of course, some role in these investigations played so-called weak σ-distributivity of the considered vector space V ([6],[7],[8],[9]). Using the Kluvanek approach the assumption can be ommited.

3. Integration

Definition 1. Let V be a Riesz space, $\mathcal{A}$ be a ring of subsets of a set X. A mapping $\mathcal{A} \to \{v \in V: \ v \geq 0\}$ is a measure, if

$$A \in \mathcal{A}, \quad A = \bigcup_{i=1}^{\infty} A_i, \qquad A_i \in \mathcal{A} \ (i = 1, 2, \dots), \quad A_i \cap A_j = \emptyset \ (i \neq j)$$

implies

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(A_i).$$

A nonnegative function $f\colon X \to \mathbb{R}$ is integrable ($f \in \mathcal{P}^+$) if and only if

$$\exists \alpha_i \geq 0, \ \exists A_i \in \mathcal{A}: \quad \sum_{i=1}^{\infty} \alpha_i \mu(A_i) < \infty$$

and

$$f(x) = \sum \alpha_i \chi_{A_i}(x)$$

We want to define the integral of a function $f \in \mathcal{P}^+$

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}, \qquad \alpha_i \geq 0, \ \ A_i \in \mathcal{A}$$

by the equality

$$\int f \, \mathrm{d}\mu = \sum_{i=1}^{\infty} \alpha_i \mu(A_i).$$

Of course, it is first necessary to prove the independence of the sum on the representation of f in the form

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}, \qquad \alpha_i \geq 0, \ \ A_i \in \mathcal{A}.$$

It is realized in Theorem 1 what is the main result of the chapter.

Definition 2. Let $\mathcal{A}$ be a ring of subsets of a set X, $\mu\colon \mathcal{A} \to [0, \infty)$ be a measure. A function $f\colon X \to \mathbb{R}$ belongs to $\mathcal{P}_0$, if there exist $n \in \mathbb{R}$, $\alpha_1, \dots, \alpha_n \geq 0$, $A_1, \dots, A_n \in \mathcal{A}$ such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Lemma 1. *To any $f \in \mathcal{P}_0$, $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$ there exist $m \in \mathbb{R}$, $\beta_1, \ldots, \beta_m \geq 0$, $B_1, \ldots, B_m \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ $(i \neq j)$ such that*

$$f = \sum_{j=1}^{m} \beta_j \chi_{B_j},$$

and

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

P r o o f. By induction. The idea:

$$\alpha_1 \chi_{A_1} + \alpha_2 \chi A_2 = \alpha_1 \chi_{A_1 \setminus A_2} + (\alpha_1 + \alpha_2) \chi_{A_1 \cap A_2} + \alpha_2 \chi_{A_2 \setminus A_1},$$
$$\alpha_1 \mu(A_1) + \alpha_2 \mu(A_2) = \alpha_1 \mu(A_1 \setminus A_2) + (\alpha_1 + \alpha_2) \mu(A_1 \cap A_2) + \alpha_2 \mu(A_2 \setminus A_1),$$

Let the assertion hold for some $n \in \mathbb{R}$. Then

$$\begin{aligned}\sum_{i=1}^{n+1} \alpha_i \chi_{A_i} &= \sum_{i=1}^{n} \alpha_i \chi_{A_i} + \alpha_{n+1} \chi_{A_{n+1}} \\ &= \sum_{j=1}^{m} \beta_j \chi_{B_j} + \alpha_{n+1} \chi_{A_{n+1}}\end{aligned}$$

Put $B = \bigcup\limits_{j=1}^{m} B_j$. Then

$$\sum_{i=1}^{n+1} \alpha_i \chi_{A_i} = \sum_{j=1}^{m} \beta_j \chi_{B_j \setminus A_{n+1}} + \sum_{j=1}^{m} (\beta_j + \alpha_{n+1}) \chi_{A_{n+1} \cap B_j} + \alpha_{n+1} \chi_{A_{n+1} \setminus B}.$$

Similarly

$$\begin{aligned}&\sum_{i=1}^{n+1} \alpha_i \mu(A_i) \\ &= \sum_{j=1}^{m} \beta_j \mu(B_j \setminus A_{n+1}) + \sum_{j=1}^{m} (\beta_j + \alpha_{n+1}) \mu(A_{n+1} \cap B_j) + \alpha_{n+1} \mu(A_{n+1} \setminus B).\end{aligned}$$

□

Lemma 2. *Let $f \in \mathcal{P}_0$, $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i} = \sum\limits_{j=1}^{m} \beta_j \chi_{B_j}$. Then*

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

P r o o f. By Lemma 1 there exist nonnegative η_k, δ_l and pairwise disjoint C_k, or D_l resp. such that

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{k=1}^{u} \eta_k \mu(C_k),$$
$$\sum_{j=1}^{m} \beta_j \mu(B_j) = \sum_{l=1}^{v} \delta_l \mu(D_l).$$

Since

$$\eta_k \chi_{C_k \cap D_l} = \delta_l \chi_{C_k \cap D_l},$$

we have

$$\eta_k \mu(C_k \cap D_l) = \delta_l \mu(C_k \cap D_l).$$

Therefore

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \mu(A_i) &= \sum_{k=1}^{u} \eta_k \mu(C_k) = \sum_{k=1}^{u} \eta_k \sum_{l=1}^{v} \mu(C_k \cap D_l) \\ &= \sum_{l=1}^{v} \delta_l \sum_{k=1}^{u} \mu(C_k \cap D_l) = \sum_{j=1}^{m} \beta_j \mu(B_j).\end{aligned}$$

□

Theorem 1. *Let* $f \in \mathcal{P}^+$

$$f = \sum_i \alpha_i \chi_{A_i} = \sum_j \beta_j \chi_{B_j}.$$

Then

$$\sum_i \alpha_i \mu(A_i) = \sum_i \beta_j \mu(B_j)).$$

P r o o f. For $f \in \mathcal{P}_0$, $f = \sum\limits_i \alpha_i \chi_{A_i}$ put

$$J_0(f) = \sum_i \alpha_i \mu(A_i).$$

It is possible by Lemma 2. □

Assertion 1. *If* $f_n \in \mathcal{P}_0$, $f_n \searrow 0$, *then* $J_0(f_n) \searrow 0$.

Put $\alpha = \max f_1$, $Y = \{x \in X : \ f_1(x) > 0\}$. Let $\varepsilon > 0$ be arbitrary. Put

$$A_n = \{x \in X : \ f_n(x) \geq \varepsilon\}.$$

Then

$$A_n \searrow \emptyset,$$

hence

$$\mu(A_n) \searrow 0.$$

Let $f_n = \sum\limits_{i=1}^{n} \alpha_i \chi_{C_i}$, C_i disjoint. Then

$$J_0(f_n) = \sum_{i=1}^{n} \alpha_i \mu(C_i) = \sum_{i=1}^{n} \alpha_i \mu(C_i \cap A_n) + \sum_{i=1}^{n} \alpha_i \mu(C_i \cap A_n{}').$$

If $x \in C_i \cap A_n'$, then $f_n(x) \leq \alpha_i < \varepsilon$. Therefore

$$\begin{aligned} J_0(f_n) &\leq \sum \alpha \mu(C_i \cap A_n) + \varepsilon \sum \mu(C_i \cap A_m^{'}) \\ &= \alpha \mu\Big(A_n \cap \bigcup C_i\Big) + \varepsilon \mu\Big(\bigcup C_i\Big) \\ &\leq \alpha \mu(A_n) + \varepsilon \mu(Y). \end{aligned}$$

Therefore

$$0 \leq \lim_{n\to\infty} J_0(f_n) \leq \alpha \lim_{n\to\infty} \mu(A_n) + \varepsilon \mu(Y) = \varepsilon \mu(Y).$$

Since the previous inequality holds for every $\varepsilon > 0$, and V is Archimedean, we have

$$\lim_{n\to\infty} J_0(f_n) = 0.$$

Assertion 2. *If $f_n \nearrow f$, $f_n \in \mathcal{P}_0$, $f \in \mathcal{P}_0$, then*

$$J_0(f_n) \nearrow J_0(f).$$

Put $g_n = f - f_n$. Then $g_n \searrow 0$, hence

$$J_0(f) - J_0(f_n) = J_0(g_n) \searrow 0.$$

Proof of Theorem 1. Let $f \in \mathcal{P}^+$

$$f = \sum_i \alpha_i \chi_{A_i} = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \chi_{A_i} = \bigvee f_n,$$

$$f = \sum_i \beta_j \chi_{B_j} = \lim_{n\to\infty} \sum_{j=1}^{m} \beta_j \chi_{B_j} = \bigvee g_m.$$

Then

$$f_n \wedge g_m \nearrow f \wedge g_m = g_m,$$

hence

$$\sum_i \alpha_i \mu(A_i) = \lim_{n\to\infty} J_0(f_n) \geq \lim_{n\to\infty} J_0(f_n \wedge g_m) = J_0(g_m) = \sum_{j=1}^{m} \beta_j \mu(B_j),$$

for any $m \in \mathbb{R}$, and therefore

$$\sum_i \alpha_i \mu(A_i) \geq \sum_i \beta_j \mu(B_j).$$

□

Definition 3. Let $f \in \mathcal{P}^+$,

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}, \qquad \alpha_i \in \mathbb{R},\ \alpha_i \geq 0, \quad A_i \in \mathcal{A}\ (i = 1, 2, \dots).$$

Then we define

$$\int f \, \mathrm{d}\mu = \sum_{i=1}^{\infty} \alpha_i \mu(A_i).$$

Definition 4. A function $f \colon X \to \mathbb{R}$ is integrable ($f \in \mathcal{P}$) if there exist $g, h \in \mathcal{P}^+$, such that

$$g(x) < \infty,\ h(x) < \infty \implies f(x) = g(x) - h(x).$$

Theorem 2. *Let $g, h, k, l \in \mathcal{P}$ and*

$$f(x) = g(x) - h(x), \quad \text{or} \quad f(x) = k(x) - l(x)$$

whenever $g(x) < \infty$, and $h(x) < \infty$, or $k(x) < \infty$, and $l(x) < \infty$, resp. Then

$$\int g \, \mathrm{d}\mu - \int h \, \mathrm{d}\mu = \int k \, \mathrm{d}\mu - \int l \, \mathrm{d}\mu.$$

Proof. Evidently $g + l \in \mathcal{P}^+$, $k + h \in \mathcal{P}^+$, and

$$\int (g + l) \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu + \int l \, \mathrm{d}\mu, \int (k + h) \, \mathrm{d}\mu = \int k \, \mathrm{d}\mu + \int h \, \mathrm{d}\mu.$$

Moreover,

$$g(x) + l(x) = k(x) + h(x).$$

□

Definition 5. If $f \in \mathcal{P}$ and $g, h \in \mathcal{P}^+$ are such that

$$f(x) = g(x) - h(x)$$

whenever $g(x) < \infty$, and $h(x) < \infty$, then we define

$$\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu - \int h \, \mathrm{d}\mu.$$

4. Linearity and positivity

We want to prove the identity

$$\int (\alpha f + \beta g) \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu + \beta \int g \, \mathrm{d}\mu.$$

It will be useful to have the following characterization of functions from $\mathcal{P}^+$.

Lemma 3. *A mapping* $f\colon X \to [0,\infty]$ *belongs to* $\mathcal{P}^+$ *if and only if there exist* $f_n \in \mathcal{P}_0$, $f_n \geq 0$ $(n = 1,2,3,\dots)$ *such that* $f_n(x) \nearrow f(x)$ *for any* $x \in X$. *In the case*

$$\int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu.$$

P r o o f. Let $f \in \mathcal{P}^+$,

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x),$$

$\alpha_i \geq 0$ $(i = 1,2,\dots)$. Then it suffices to put

$$f_n(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x).$$

On the other hand, if $f_n \in \mathcal{P}_0$, $f_n \geq 0$ $(n = 1,2,3,\dots)$, $f_n \nearrow f$, then

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x)) + f_1(x) \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} (f_{i+1}(x) - f_i(x)) + f_1(x) \right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x). \end{aligned}$$

Of course,

$$\begin{aligned} \int f \,\mathrm{d}\mu &= \sum_{i=1}^{\infty} \alpha_i \mu(A_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i) = \lim_{i\to\infty} \int f_n \,\mathrm{d}\mu. \end{aligned}$$

□

Theorem 3. *If* f, g *are integrable, then* $f+g$ *is integrable, too, and*

$$\int (f+g) \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu.$$

P r o o f. If $f, g \in \mathcal{P}^+$ then by Lemma 3 there are $f_n, g_n \in \mathcal{P}_0$ such that $f_n \nearrow f$, $g_n \nearrow g$, hence $f_n + g_n \nearrow f + g$, and

$$\begin{aligned} \int (f+g) \,\mathrm{d}\mu &= \lim_{n\to\infty} \int (f_n + g_n) \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu + \lim_{n\to\infty} \int g_n \,\mathrm{d}\mu \\ &= \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu. \end{aligned}$$

Generally let $f_1, f_2, g_1, g_2 \in \mathcal{P}^+$ be such that

$$f(x) = f_1(x) - f_2(x), \qquad g(x) = g_1(x) - g_2(x),$$

whenever $f_1(x) < \infty$, $f_2(x) < \infty$ or $g_1(x) < \infty$, $g_2(x) < \infty$, resp. Then

$$f_1 + g_1 \in \mathcal{P}^+, \qquad f_2 + g_2 \in \mathcal{P}^+,$$

and

$$f(x) + g(x) = \big(f_1(x) + g_1(x)\big) - \big(f_2(x) + g_2(x)\big),$$

whenever $f_1(x) + g_1(x) < \infty$, and $f_2(x) + g_2(x) < \infty$. Therefore

$$\begin{aligned} \int (f+g)\,\mathrm{d}\mu &= \int (f_1 + g_1)\,\mathrm{d}\mu - \int (f_2 + g_2)\,\mathrm{d}\mu \\ &= \int f_1\,\mathrm{d}\mu - \int f_2\,\mathrm{d}\mu + \int g_1\,\mathrm{d}\mu - \int g_2\,\mathrm{d}\mu \\ &= \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu. \end{aligned}$$

□

Theorem 4. *If $f\colon X \to [-\infty,\infty]$ is integrable, and $c \in \mathbb{R}$, then cf is integrable, too, and*

$$\int cf\,\mathrm{d}\mu = c\int f\,\mathrm{d}\mu.$$

P r o o f. The assertion holds evidently for $f \in \mathcal{P}^+$ and $c \geq 0$. Generally

$$\int cf\,\mathrm{d}\mu = \int (cg - ch)\,\mathrm{d}\mu = c\int g\,\mathrm{d}\mu - c\int h\,\mathrm{d}\mu = c\int f\,\mathrm{d}\mu$$

if $c \geq 0$, and

$$\int cf\,\mathrm{d}\mu = \int ((-c)h - (-c)g)\,\mathrm{d}\mu = -c\int h\,\mathrm{d}\mu + c\int g\,\mathrm{d}\mu = c\int f\,\mathrm{d}\mu,$$

if $c < 0$. □

Theorem 5. *If f, g are integrable, and $f \leq g$, then*

$$\int f\,\mathrm{d}\mu \leq \int g\,\mathrm{d}\mu.$$

P r o o f. If $f, g \in \mathcal{P}^+$, then there exist $f_n, g_n \in \mathcal{P}_0$ such that $f_n \nearrow f$, $g_n \nearrow g$. Put $h_n = \max(f_n, g_n)$. Then $f_n \leq h_n$, $h_n \nearrow g$, hence

$$\int f\,\mathrm{d}\mu = \lim \int f_n\,\mathrm{d}\mu \leq \lim \int h_n\,\mathrm{d}\mu = \int g\,\mathrm{d}\mu.$$

In the general case there exist $f_1, f_2, g_1, g_2 \in \mathcal{P}^+$ such that $f(x) = f_1(x) - f_2(x)$, $g(x) = g_1(x) - g_2(x)$, whenever the differences are defined. Then $f_1 + g_2 \le g_1 + f_2$, and

$$\int f_1 \, \mathrm{d}\mu + \int g_2 \, \mathrm{d}\mu \le \int g_1 \, \mathrm{d}\mu + \int f_2 \, \mathrm{d}\mu.$$

□

5. Continuity

The basic property of the Lebesgue integral is obtained in the following Beppo Levi theorem.

Theorem 6. *Let f_n be integrable functions such that $f_n \nearrow f$ and the sequence $\left(\int f_n \, \mathrm{d}\mu\right)_n$ is bounded. Then f is integrable, and*

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$

The proof of the theorem will be realized by a sequence of the following propositions.

Proposition 1. *A non-negative function $f\colon X \to [0,\infty]$ is integrable if and only if $f \in \mathcal{P}^+$.*

Proof. If $f \in \mathcal{P}^+$, then $f = f - 0$, and $f \in \mathcal{P}$, since $0 \in \mathcal{P}^+$. Let f be integrable, and $g, h \in \mathcal{P}^+$ be such that

$$f(x) = g(x) - h(x)$$

whenever $g(x) < \infty$, $h(x) < \infty$. If $f \ge 0$, then $g \ge h$. Let $g_n \in \mathcal{P}_0$, $h_n \in \mathcal{P}_0$, $g_n \ge h_n$, $g_n \nearrow g, h_n \nearrow h$. We have $g_n - h_n \ge 0$, and $g_n - h_n \in \mathcal{P}_0$. Moreover

$$g_n(x) - h_n(x) \nearrow g(x) - h(x) = f(x),$$

whenever $g(x) < \infty$, hence $f \in \mathcal{P}^+$. □

Proposition 2. *Let $f_n \in \mathcal{P}^+$, $f_n \nearrow f$, the sequence $\left(\int f_n \, \mathrm{d}\mu\right)_n$ is bounded. Then $f \in \mathcal{P}^+$, and*

$$\int f \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$

Proof. Since $f_n \in \mathcal{P}^+$, there exists a sequence $(g_{nm})_{m=1}^{\infty}$ of functions of $\mathcal{P}_0$ such that $g_{nm} \nearrow f_n$ ($m \to \infty$, $n = 1, 2, \ldots$). Put

$$h_n = \max(g_{1n}, g_{2n}, \ldots, g_{nn}).$$

Then $h_n \in \mathcal{P}_0$, $h_n \le h_{n+1}$ ($n = 1, 2, \ldots$). For arbitrary $m, n \in \mathbb{R}$ put $k = \max(m, n)$. Then $n \le k$, hence

$$g_{nk} \le \max\bigl(g_{1k}, g_{2k}, \ldots, g_{kk}\bigr) = h_k.$$

Since $m \le k$, and $(g_{nm})_m$ is non-decreasing, we have

$$g_{nm} \le g_{nk} \le h_k \le \lim_{n\to\infty} h_k,$$

hence

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \lim_{m\to\infty} g_{nm} \le \lim_{k\to\infty} h_k.$$

On the other hand $h_n \le f_n$ $(n = 1, 2, \dots)$, hence

$$f = \lim_{k\to\infty} h_k \in \mathcal{P}^+,$$

and

$$\int f \, \mathrm{d}\mu = \lim_{k\to\infty} \int h_k \, \mathrm{d}\mu \le \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu \le \int f \, \mathrm{d}\mu.$$

□

Proposition 3. *Let $g_n \in \mathcal{P}^+$ $(n = 1, 2, \dots)$ and $\sum\limits_{n=1}^{\infty} \int g_n \, \mathrm{d}\mu \in V$. Then $\sum\limits_{n=1}^{\infty} g_n \in \mathcal{P}^+$, and*

$$\int \sum_{n=1}^{\infty} g_n \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int g_n \, \mathrm{d}\mu.$$

Proof. Put $f_n = \sum\limits_{i=1}^{n} g_i$ $(n = 1, 2, \dots)$. Then $f_n \in \mathcal{P}^+$, and $f_n \nearrow \sum\limits_{n=1}^{\infty} g_n$. Therefore by Proposition 2

$$\int \sum_{n=1}^{\infty} g_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^{n} \int g_i \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int g_n \, \mathrm{d}\mu.$$

□

Proof of Theorem 6. Put

$$h_1(x) = f_1(x), \qquad h_n(x) = f_n(x) - f_{n-1}(x) \quad (n = 1, 2, \dots)$$

if $f_n(x) < \infty$, and $h_n(x) = \infty$ in the opposite case. Then $h_n \in \mathcal{P}^+$, and

$$\begin{aligned} \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu &= \lim_{n\to\infty} \left(\int \left(\sum_{i=1}^{n} (f_i(x) - f_{i-1}(x)) + f_1(x) \right) \mathrm{d}\mu \right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \int h_i \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int h_n \, \mathrm{d}\mu = \int \left(\sum_{n=1}^{\infty} h_n \right) \mathrm{d}\mu = \int f \, \mathrm{d}\mu. \end{aligned}$$

□

6. Concluding remarks

Theorem 6 has many very important corollaries, e.g. the Lebesgue dominated convergence theorem: If $f_n \colon X \to \mathbb{R}$ are integrable, $f_n(x) \to f(x)$ for every $x \in X$, and there exists an integrable function $g \colon X \to \mathbb{R}$ such that $|f_n| \leq g$, $(n = 1, 2, \dots)$, then f is integrable, and

$$\int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu,$$

of course, the convergence must be taken with respect to the ordering in V. Moreover, if $V = \mathbb{R}$, then the pseudometric $d \colon \mathcal{L} \times \mathcal{L} \to [0, \infty)$ defined by the equality

$$d(f, g) = \int |f - g| \,\mathrm{d}\mu$$

leads to the complete pseudometric space $(\mathcal{L}, d)$.

As we mentioned above some applications were useful in the study of probability measures and vector valued random variables. Another area for using the ideas presented above is in the fuzzy sets theory, multivalued logic ([14], [20], [23]) and the corresponding algebraic structures ([3], [5], [12], [15], [17], [18], [22]).

Recall that in [24] the title A note on the Kluvánek has been used, of course the definition doesn't coincide with the main Kluvánek idea. The Kluvánek construction was presented in [11], of course, in Slovak only. The construction is possible to use only in the one dimensional case. Therefore we modified it in [21] and the modification can be use also in the Riesz space valued case.

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