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ON INVERSE LIMITS OF MONOUNARY ALGEBRAS

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ABSTRACT. We study inverse limits of monounary algebras. All monounary algebras A such that A can arise from A only by an inverse limit construction are described. We deal with an existence of an inverse limit. Some inverse limit closed classes are described. The paper ends with two problems.

Direct and inverse limits are well-known methods for building up new algebras from given families of algebras. The construction of an inverse limit is dual to the construction of a direct limit, which yields that they have a lot of common properties, cf., e.g., [1: §21]. The analogous notions are applied also in the theory of categories under names directed colimit and directed limit, cf., e.g. [12].

The paper [5] is devoted to direct limits. In the present paper we investigate inverse limits of monounary algebras. Direct limits of monounary algebras were studied in [2].

It is possible that an inverse limit of an inverse family does not exist. Examples of such families are given in Section 3. In Section 2 we will prove that there are proper classes $\mathcal{K}$ such that if an inverse family consists of algebras of $\mathcal{K}$, then this inverse family has a limit.

Theorem 1 describes all monounary algebras A such that $\{A\}$ is an inverse limit closed class, i.e. such that they satisfy the following condition:

> If an algebra B can be obtained as an inverse limit of algebras which are isomorophic to A, then B is isomorphic to A.

The class of unbounded monounary algebras from [9] is important in the proof. Another inverse limit closed classes are described in Theorems 2 and 3. All of them are proper classes.

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For a class of algebras $\mathcal{K}$ we denote by $\underset{\leftarrow}{\mathbf{L}}\mathcal{K}$ the class of all isomorphic copies of all algebras arising as inverse limits of families of elements of $\mathcal{K}$. It is obvious that $\underset{\leftarrow}{\mathbf{L}}\mathcal{K} \subseteq \underset{\leftarrow}{\mathbf{L}}\underset{\leftarrow}{\mathbf{L}}\mathcal{K}$. We will prove that there is a class of mononary algebras $\mathcal{K}$ with $\underset{\leftarrow}{\mathbf{L}}\mathcal{K} \neq \underset{\leftarrow}{\mathbf{L}}\underset{\leftarrow}{\mathbf{L}}\mathcal{K}$ in Section 5.

1. Preliminaries

The set of all positive integers is denoted by $\mathbb{N}$. If A is a set, then $|A|$ denotes the cardinality of A.

We deal with monounary algebras. For monounary algebras we use the terminology from [11] and [9]. The operation in algebras is denoted by f. The class of all monounary algebras is denoted by $\mathcal{U}$. We use the same symbol for a monounary algebra and its carrier in most of considerations in this paper.

If $A_1, \dots, A_n \in \mathcal{U}$, then the class of all isomorphic copies of $A_1, \dots, A_n$ will be denoted by $[A_1, \dots, A_n]$.

Let us denote by N the monounary algebra defined on the set $\mathbb{N}$ with the successor operation. Further, let Z be the monounary algebra defined on the set of all integers with the successor operation.

Let $A \in \mathcal{U}$. We denote

$$R_A = \{B \subseteq A : \ B \cong Z\};$$
$$R_{A,m} = \{C \subseteq A : \ C \text{ is a cycle of } A \text{ and the length of } C \text{ divides } m\}.$$

We denote

$$\begin{aligned}
\mathcal{R} &= \bigl\{A \in \mathcal{U} : \ |R_A| \in \mathbb{N}\bigr\},\\
\mathcal{R}_m &= \bigl\{A \in \mathcal{U} : \ |R_{A,m}| \in \mathbb{N}\bigr\},\\
\mathcal{T} &= \bigl\{A \in \mathcal{U} : \ \text{every component of } A \text{ is a cycle and}\\
&\qquad \text{there are no components } C, D \text{ of } A\\
&\qquad \text{such that } C \neq D \text{ and } C \in R_{A,|D|}\bigr\}.
\end{aligned}$$

The class $\mathcal{T}$ is the same as in [2]. Note that A is a cycle if and only if $A \in \mathcal{T}$ and A is connected.

Lemma 1. *It is obvious that:*

(i) $\mathcal{T} \subset \bigcup_{m \in \mathbb{N}} \mathcal{R}_m$*;*

(ii) $Z \in \mathcal{R}$*;*

(iii) $N \notin \bigcup_{m \in \mathbb{N}} \mathcal{R}_m \cup \mathcal{R}$*;*

(iv) *if a monounary algebra A is finite, then there is $m \in \mathbb{N}$ such that $A \in \mathcal{R}_m$.*

We will use the definition of an inverse limit of sets and of universal algebras from [1: §21]. We formulate it for monounary algebras.

Let $\langle I, \leq\rangle$ be an upward directed partially ordered set, $I \neq \emptyset$. For each $i \in I$ let A_i be an monounary algebra. Suppose that for each pair of elements i and j in I with $i > j$, there is defined a homomorphism φ^i_j of A_i into A_j such that $i > j > k$ implies that $\varphi^i_k = \varphi^i_j \circ \varphi^j_k$. For each $i \in I$ let φ^i_i be the identity on A_i.

The triplet (I, A_i, φ^i_j) is called an *inverse family of algebras.*

Let the set A^∞ consist of those $p \in \prod\limits_{i\in I} A_i$ for which

$$\varphi^i_j(p_i) = p_j.$$

If $A^\infty \neq \emptyset$, then A^∞ is a subalgebra of the algebra $\prod\limits_{i\in I} A_i$ and it is called the *inverse limit* of the inverse family (I, A_i, φ^i_j).

If $A^\infty = \emptyset$, then we will say that an inverse limit of the inverse family (I, A_i, φ^i_j) does not exist or that the inverse family (I, A_i, φ^i_j) has no inverse limit.

Following two lemmas follow from the definition of an inverse limit.

Lemma 2. *Let (I, A_i, φ^i_j) be an inverse family. Suppose that B_i is a subalgebra of A_i for every $i \in I$ and $(I, B_i, \varphi^i_j|B_i)$ is the inverse family.*

Then B^∞ is a subalgebra of A^∞.

Lemma 3. *Let P be a set and $\langle I, \leq\rangle$ be an upward directed partial ordered set. Let $\mathcal{A}^\alpha = (I, A^\alpha_i, \varphi^{\alpha i}_j)$ be an inverse family for every $\alpha \in P$. Suppose that if $\alpha, \beta \in P$, $\alpha \neq \beta$ and $i \in I$, then $A^\alpha_i \cap A^\beta_i = \emptyset$.*

For $i, j \in I$, $i \geq j$ denote by ξ^i_j the mapping from $\bigcup\limits_{\alpha\in P} A^\alpha_i$ into $\bigcup\limits_{\alpha\in P} A^\alpha_j$ such that if $x \in A^\alpha_i$, then $\xi^i_j(x) = \varphi^{\alpha i}_j(x)$. Denote by $A^{\alpha\infty}$ the inverse limit of $\mathcal{A}^\alpha$.

Then $\big(I, \bigcup\limits_{\alpha\in P} A^\alpha_i, \xi^i_j\big)$ is the inverse family and the inverse limit of this family is equal to $\bigcup\limits_{\alpha\in P} A^{\alpha\infty}$.

Lemma 4. *Let $A \in \mathcal{U}$. Let (I, A_i, φ^i_j) be an inverse family of algebras such that φ^i_j is an isomorphism for every $i, j \in I$, $i \geq j$. Then $A^\infty \cong A_i$ for every $i \in I$.*

Proof. Choose $i \in I$. Let ψ is a projection of A^∞ on the ith coordinate.

We will show that ψ is surjective. Let $p_i \in A_i$. Let $j \in I$. If $j \geq i$, then denote by p_j an (unique) element of A_j such that $\varphi^j_i(p_j) = p_i$. If $j \leq i$, then let $p_j = \varphi^j_i(p_i)$. Else take $k \in I$ such that $k \geq i, j$ and take $p_k \in A_k$ as in the first case and put $p_j = \varphi^k_j(p_k)$.

We have that $(p_j)_{j\in I} \in A^\infty$ and $\psi((p_j)_{j\in I}) = p_i$.

To see the injectivity of ψ let $p, q \in A^\infty$ be such that there exists $j \in I$ such that $p_j \neq q_j$. Choose $k \geq i, j$. We have $\varphi_j^k(p_k) = p_j \neq q_j = \varphi_j^k(q_k)$. Thus $p_k \neq q_k$. Further, $p_i = \varphi_i^k(p_k) \neq \varphi_i^k(q_k) = q_i$ according to injectivity of φ_i^k.

Conclude that the mapping ψ is an isomorphism from A^∞ onto A_i. □

Lemma 5. *Let (I, A_i, φ_j^i) be an inverse family of algebras such that algebras A_i are connected and homomorphisms φ_j^i are injective for all $i, j \in I$, $i \geq j$. If $A^\infty \neq \emptyset$, then the algebra A^∞ is connected.*

Proof. Let $p, q \in A^\infty$ and $i \in I$. Then there exist $m, n \in \mathbb{N}$ such that $f^m(p_i) = f^n(q_i)$. Consider $j \in I$. Choose $k \in I$ such that $k \geq i, j$. We have

$$\varphi_i^k(f^m(p_k)) = f^m(\varphi_i^k(p_k)) = f^m(p_i) = f^n(q_i) = f^n(\varphi_i^k(q_k)) = \varphi_i^k(f^n(q_k)).$$

We obtain $f^m(p_k) = f^n(q_k)$ in view of injectivity of φ_i^k. Then

$$f^m(p_j) = f^m(\varphi_j^k(p_k)) = \varphi_j^k(f^m(p_k)) = \varphi_j^k(f^n(q_k)) = f^n(\varphi_j^k(q_k)) = f^n(q_j).$$

Conclude $f^m(p) = f^n(q)$. □

Let $\mathcal{K}$ be a class of monounary algebras and $\underleftarrow{\mathbf{L}}\mathcal{K}$ be the class of all isomorphic copies of inverse limits of algebras of $\mathcal{K}$.

If every algebra from $\underleftarrow{\mathbf{L}}\mathcal{K}$ has its isomorphic copy in $\mathcal{K}$, then we will write that $\underleftarrow{\mathbf{L}}\mathcal{K} = \mathcal{K}$ or that $\mathcal{K}$ is an $\underleftarrow{\mathbf{L}}$-class or that $\mathcal{K}$ is an *inverse limit closed class.*

We remark that an intersection of $\underleftarrow{\mathbf{L}}$-classes is an $\underleftarrow{\mathbf{L}}$-class. A union of finitely many $\underleftarrow{\mathbf{L}}$-classes is an $\underleftarrow{\mathbf{L}}$-class.

A lot of $\underleftarrow{\mathbf{L}}$-classes are found in [3]. For example the class of all algebras $A \in \mathcal{U}$ such that A does not contain a cycle of the lenght which divides n, cf. [3: Ex. 5]. We will use the folowing assertion which is proved in [3: Theorems 5, 6].

Proposition 1. *The class of all algebras $A \in \mathcal{U}$ such that if B is a component of A, then B is a cycle or $B \in [N, Z]$ is an $\underleftarrow{\mathbf{L}}$-class.*

The class of all algebras $A \in \mathcal{U}$ such that if B is a component of A, then B is a cycle or $B \cong Z$ is an $\underleftarrow{\mathbf{L}}$-class.

Lemma 6. *Let (I, A_i, φ_j^i) be an inverse family of algebras such that A_i is a cycle for every $i \in I$. Then*

(i) *there exists $s \in I$ such that $A^\infty \cong A_s$ or*

(ii) *A^∞ has uncountable many components and every component of A is isomorphic to Z.*

P r o o f. We have $A^\infty \neq \emptyset$ since A_i are finite, cf. [1: Thm. 21.1]. If B is a component of A^∞, then B is a cycle or $B \cong Z$ according to the previous proposition. We have that $\varphi_j^i(A_i) = A_j$ for every $i, j \in I$, $i \geq j$. That means that the length of A_j is a divisor of $|A_i|$ and for every $b \in A_j$ there exists $(a_k)_{k\in I} \in A^\infty$ such that $a_j = b$.

Let the set $\{|A_i| : \ i \in I\}$ be finite. Denote by l the least common multiple of members of this set. The inverse limit construction yields that there exists $s \in I$ such that A_s is a cycle of length l.

Let $a = (a_i)_{i\in I} \in A^\infty$. We have $f^l(a) = a$. Further, $a, f(a), \dots, f^{l-1}(a)$ are different since $a_s, f(a_s), \dots f^{l-1}(a_s)$ are different. Consider $b = (b_i)_{i\in I} \in A^\infty$. Then there is $k \in \mathbb{N}$ such that $k \leq l$ and $b_s = f^k(a_s)$. We obtain $f^k(a) = b$. Conclude $A^\infty \cong A_s$.

Assume that the set $\{|A_i| : \ i \in I\}$ is infinite. Then A^∞ does not contain a cycle.

Let $i \in I$, $a \in A_i$. Take $j \in I$ such that $j \geq i$ and $|A_j| > |A_i|$. Therefore there are $b, c \in A_j$ such that $b \neq c$ and $\varphi_i^j(b) = \varphi_i^j(c) = a$. Therefore A^∞ contains uncountable many elements. Conclude A^∞ has uncountable many components. □

Corollary 1. *The class $\mathcal{T}$ and the class of all connected monounary algebras fail to be inverse limit closed.*

2. Inverse families with $A^\infty \neq \emptyset$

Let (I, A_i, φ_j^i) be an inverse family of algebras.

As we have already mentioned, if A_i are finite, then $A^\infty \neq \emptyset$. In this section we will see that for monounary algebras also stronger assertions are valid.

Lemma 7. *If $A_i \in \mathcal{R}_m$ for every $i \in I$, then $A^\infty \neq \emptyset$.*

P r o o f. Let $i \in I$. Suppose that $B_i = \bigcup\limits_{C \in R_{A_i,m}} C$. In view of $\varphi_j^i(B_i) \subseteq B_j$ for every $i \geq j$ we have that $(I, B_i, \varphi_j^i | B_i)$ is an inverse family of algebras. The assumption gives that B_i is finite for every $i \in I$. Thus $B^\infty \neq \emptyset$. The algebra B^∞ is a subalgebra of A^∞ according to Lemma 1. □

Lemma 8. *If $A_i \in \mathcal{R}$ for every $i \in I$ and there exists $k \in I$ such that A_k has no cycle or A_k has finitely many cycles, then $A^\infty \neq \emptyset$.*

P r o o f. Suppose that A_k has no cycle. Let $J = \{i \in I : \ k \leq i\}$. Then (J, A_j, φ_i^j) is an inverse family of algebras. In view of cofinality of J we have that an inverse limit of (I, A_i, φ_j^i) exists if and only if an inverse limit of (J, A_j, φ_i^j)

exists. Further, if an inverse limit of (I, A_i, φ_j^i) exists, then an inverse limit of (I, A_i, φ_j^i) is isomorphic to an inverse limit of (J, A_j, φ_i^j).

We will show that there are finite sets B_j such that $B_j \subseteq A_j$ for all $j \in J$ and $(J, B_j, \varphi_i^j|B_j)$ is an inverse family of sets.

First we will determine B_k. Let D be a component of A_k with $R_D \neq \emptyset$. Suppose that $R_D = \{Z_1, \dots, Z_l\}$, where $l \in \mathbb{N}$ according to $A_k \in \mathcal{R}$. Choose $z_s \in Z_s$ for every $s \in \{1, \dots, l\}$. Then there are positive integers $m_1, \dots, m_l$ such that $f^{m_1}(z_1) = f^{m_2}(z_2) = \cdots = f^{m_l}(z_l)$ in view of connectivity of D. We put the element $f^{m_1}(z_1)$ into B_k. We do the same with every component of A_k which contains a subalgebra isomorphic to Z. We obtain that B_k is finite since $A_k \in \mathcal{R}$.

Let $j \in J,\ j \neq k$. Put $B_j = \{b \in A_j :\ b$ belongs to a subalgebra of A_j which is isomorphic to Z and $\varphi_k^j(b) \in B_k\}$ for $j \neq k$. We have $B_j \neq \emptyset$ since B_k does not contain a cycle. Further, B_j is finite in view of $|B_j| \leq \left|R_{A_j}\right| \in \mathbb{N}$.

Let $i, j \in J,\ i \leq j$ and $b \in B_j$. Then $\varphi_k^i(\varphi_i^j(b)) = \varphi_k^j(b) \in B_k$, i.e., $\varphi_i^j \in B_i$. Thus $\varphi_i^j|B_j$ is a mapping from B_j into B_i.

If A_k has a cycle, then we add one element of every cycle of A_k into B_k and continue the proof by the same way. □

3. Uniform inverse families

Let $A \in \mathcal{U}$. In this section we will deal with inverse families which have all algebras isomorphic to A .

If $A \in \bigcup\limits_{m \in \mathbb{N}} \mathcal{R}_m$, then $A^\infty \neq \emptyset$ according to Lemma 7. If $A \in \mathcal{R}$ and A contains no cycle, then $A^\infty \neq \emptyset$ according to Lemma 8. If A is finite, then A^∞ is a retract of A according to [3: Cor. 2]. We will describe all algebras A such that $\{A\}$ is an $\underleftarrow{\mathbf{L}}$-class.

The following three examples describe some inverse families which have no inverse limit. We consider $\mathbb{N}$ with a usual linear order and $A_i = A$ for all $i \in \mathbb{N}$ in every example.

Example 1. Let A be a monounary algebra which contains no cycle and no subalgebra isomorphic to Z.

Consider the inverse family $(\mathbb{N}, A_i, \varphi_j^i)$ such that $\varphi_j^i = f^{i-j}$ for all $i \geq j$, $i, j \in \mathbb{N}$, see Fig. 1.

Let $p \in A^\infty$. Then $p_i = \varphi_i^{i+1}(p_{i+1}) = f(p_{i+1})$. Thus the set

$$\{p_i :\ i \in I\} \cup \{f^k(p_1) :\ k \in \mathbb{N}\}$$

creates a cycle of A_1 or a subalgebra of A_1 which is isomorphic to Z, a contradiction.

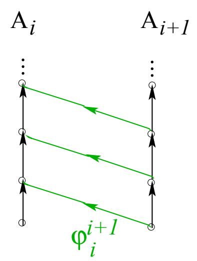

FIGURE 1.

Example 2. Let A be a monounary algebra and $\{B_k :\ k \in \mathbb{N}\}$ be the set of all components of A. Suppose that for every $k \in \mathbb{N}$ there exists a homomorphism ψ_k from B_k into B_{k+1}.

Consider the inverse family $(\mathbb{N}, A_i, \varphi_j^i)$ such that

$$\varphi_i^{i+1} \,|\, B_k = \psi_k \circ \psi_{k+1} \circ \cdots \circ \psi_{2k-1}$$

for all $i \in \mathbb{N},\ k \in \mathbb{N}$, see Fig. 2.

To show that $A^\infty = \emptyset$ we will prove that for every $x \in A_1$ there exists $i \in \mathbb{N}$ such that $x \notin \varphi_1^i(A_i)$.

Assume that $x \in B_k$ for $k \in \mathbb{N}$.

If k is odd, then we put $i = 2$.

Let k be even. Consider i such that $k = 2^{i-2} \cdot j$ where j is odd. We have $x \in \varphi_1^{i-1}(B_j)$ and $B_j \not\subseteq \varphi_{i-1}^i(A_i)$. Therefore $x \notin \varphi_1^i(A_i)$.

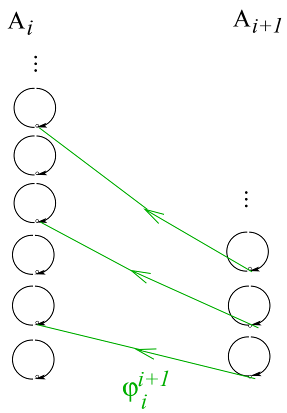

FIGURE 2.

Example 3. Let $A = \mathbb{N} \cup \{m_n : \ m, n \in \mathbb{N}\}$. Further, let $f(i) = i + 1$ for all $i \in \mathbb{N}$, $f(1_n) = 1$ for all $n \in \mathbb{N}$ and $f(m_n) = (m-1)_n$ for all $m > 1$, $m, n \in \mathbb{N}$.

Consider the inverse family $(\mathbb{N}, A_i, \varphi^i_j)$ such that $\varphi^i_{i-1}(m) = m + 1$, $\varphi^i_{i-1}(1_n) = 1$, $\varphi^i_{i-1}(m_n) = (m-1)_{n+1}$ for all $i, m, n \in \mathbb{N},\ i > 1$, see Fig. 3.

We have $\varphi^i_j(m) = m + i - j$, $\varphi^i_j(1_n) = i - j$ and
$\varphi^i_j(m_n) = (m - i + j)_{n+i-j}$ for $m > 1$. Further, $\varphi^i_j(A) \cap \{m_1 : \ m \in \mathbb{N}\} = \emptyset$ for $i > j$.

Let $a \in A^\infty$, $j \in \mathbb{N}$. Suppose that $a_j = m_n$. Then

$$\varphi^{j+n-1}_j((m+n-1)_1) = (m+n-1-(n-1))_{1+n-1} = m_n.$$

Therefore $a_{j+n-1} = (m+n-1)_1$. This contradicts to $a_{j+n-1} \in \varphi^{j+n}_{j+n-1}(A)$.

If $a_1 = m$, then $a_m = 1$ and $a_{m+1} = 1_n$.

We conclude $A^\infty = \emptyset$.

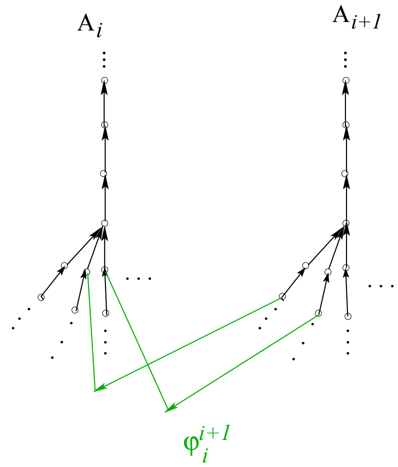

FIGURE 3.

The rest of this section we will study $\underleftarrow{\mathbf{L}}\{A\}$-classes. It is easy to see that every retract of A belongs to $\underleftarrow{\mathbf{L}}\{A\}$. The notion of a retract of an algebra is substantial. For retracts of monounary algebras see [6]–[10].

LEMMA 9. *Let $A \in \mathcal{U}$.*

(i) *If A contains a cycle, then $\underleftarrow{\mathbf{L}}\{A\} \cap \mathcal{T} \neq \emptyset$.*

(ii) *If A has no cycle and it contains a subalgebra isomorphic to Z, then $Z \in \underleftarrow{\mathbf{L}}\{A\}$.*

P r o o f.

(i) The algebra A has a retract from $\mathcal{T}$, cf. [2: Lemma 20].

(ii) An algebra isomorphic to Z is a retract of A. □

The following notion is used in [9: Df. 6.2.1, Thm. 6.2.5].

Definition. Let A be a connected monounary algebra. We say that A is *unbounded*, if

(i) A does not contain a cycle and a subalgebra isomorphic to Z;

(ii) A does not contain a retract isomorphic to N.

Lemma 10. *Let A be a connected monounary algebra. If A is unbounded, then A contains at least two nonisomorphic retracts.*

P r o o f. We have that if $a \in A$ and $n \in \mathbb{N}$, then there is $m \in \mathbb{N}$ such that

$$f^{-(m+n)}(f^m(a)) \neq \emptyset$$

according to [9: Df. 6.2.1, Thm. 6.2.5].

We construct a sequence $\{a_k\}_{k=0}^{\infty}$ by the folowing way:

Let $a_0 \in A$. Let $k \in \mathbb{N}$ and a_k be determined. If $f^{-1}(a_k) \neq \emptyset$, then put a_{k+1} an element of $f^{-1}(a_k)$. If $f^{-1}(a_k) = \emptyset$, then we take the least possible m such that $f^{-(m+1)}(f^m(a_k)) \neq \emptyset$ and put a_{k+1} an element of $f^{-(m+1)}(f^m(a_k))$.

Let B be a subalgebra of A generated by $\{a_k\}_{k=0}^{\infty}$. We have that B is a retract of A since for every element $b \in B$ we have that the degree of b in the algebra B equals to the degree of b in the algebra A. If B is not isomorphic to A, then the proof is finished.

Let $B \cong A$. Let i be the least such that $f^{-1}(a_i) = \emptyset$. Let j be the least such that $j > i$ and $f^{-1}(a_j) = \emptyset$. Take the least possible $m, n \in \mathbb{N}$ such that $f^m(a_i) = f^n(a_j)$. We have $m < n$. Further, if $k, l \in \mathbb{N}$ are such that $k \neq l$ and $f^{-1}(a_k) = f^{-1}(a_l) = \emptyset$ and $k > i$ or $l > i$, then $f^m(a_k) \neq f^n(a_l)$ according to the construction. Consider B' a subalgebra of A which is generated by $\{a_k\}_{k=j}^{\infty}$. We obtain B' a retract of B. Therefore B' is a retract of A.

Assume that $B' \cong B$. Let φ be an isomorphism from B onto B'. We have $\varphi(a_i), \varphi(a_j) \in \{a_k\}_{k=j}^{\infty}$ and $f^{-1}(\varphi(a_i)) = f^{-1}(\varphi(a_j)) = \emptyset$. Further, $f^m(\varphi(a_i)) = \varphi(f^m(a_i)) = \varphi(f^n(a_j)) = f^n(\varphi(a_j))$, a contradiction. Thus B' is not isomorphic to B. □

Corollary 2. *Let A be a connected monounary algebra. If A is unbounded, then $\underset{\leftarrow}{\mathbf{L}}\{A\}$ contains infinitely many nonisomorphic algebras.*

P r o o f. The above lemma gives a retract of A which is not isomorphic to A and which is an unbounded algebra again. □

Theorem 1. *Let $A \in \mathcal{U}$. Then $\{A\}$ is an $\underset{\leftarrow}{\mathbf{L}}$-class if and only if $A \in \mathcal{T} \cup [N, Z]$.*

P r o o f. Suppose that $A \notin \mathcal{T} \cup [N, Z]$.

If A contains a cycle or A contains a retract isomorphic to Z or N, then $\underleftarrow{\mathbf{L}}\{A\}$ possesses an algebra which is not isomorphic to A. Else A contains a retract B such that every component of B is unbounded. If B is not isomorphic to A, then $\underleftarrow{\mathbf{L}}\{A\} \neq \{A\}$.

Assume that $B \cong A$. Choose E a component of B. Let $\{E_\alpha: \ \alpha \in K\}$ be the set of all components of B isomorphic to E. Take D a retract of E which is not isomorphic to E by the previous lemma. Put D_α a retract of E_α which is isomorphic to D for every $\alpha \in K$. Then $\big(A - \bigcup\limits_{\alpha\in K} B_\alpha\big) \cup \big(\bigcup\limits_{\alpha\in K} D_\alpha\big)$ is a retract of A which is not isomorphic to A.

Conclude $\{A\}$ fails to be an $\underleftarrow{\mathbf{L}}$-class.

Let $A \in \mathcal{T} \cup [Z]$. Then $\{A\}$ is an $\underleftarrow{\mathbf{L}}$-class according to Lemma 4.

Let (I, A_i, φ_j^i) be an inverse family of algebras and $A_i \cong N$ for every $i \in I$. Suppose that $A^\infty \neq \emptyset$. Then A^∞ is connected according to Lemma 5. We obtain that A^∞ is isomorphic to N or Z or it is a cycle in view of Theorem. 1.

Let $p \in A^\infty$ be such that $p_i = 1$ for some $i \in I$. (If $p_k \neq 1$ for all $k \in I$, then $(p_k - 1)_{k\in I} \in A^\infty$.) If $p' \in A^\infty$ is such that $f(p') = p$, then $f(p'_i) = p_i = 1$, a contradiction. Conclude $A^\infty \cong N$. □

4. Inverse limit closed classes

First, some inverse limit closed classes in monounary case from [3] will be studied in this section. Further, some new inverse limit closed classes will be recognized.

We use the symbol $\subset$ for proper subsets.

We denote by T the smallest set of operations such that T contains all projections, $f \in T$ and T is closed under compositions. Elements of T are called terms, cf. e.g. [4].

It is clear that terms of monounary algebras have very special and easy form:

Lemma 11. *If $h \in T$ is n-ary, then there exist $j \in \{1, \dots, n\}$ and a nonnegative integer k such that $h(x_1, \dots, x_n) = f^k(x_j)$.*

Let $h \in T$. Let $A \in \mathcal{U}$ and h be unary.

The following notation is used in the paper [3].

Notation. For $a \in A$ we denote

$$h^{-n}(a) = \{b \in A: \ h^n(b) = a\}.$$

Denote

$$\mathcal{S}_h = \big\{A \in \mathcal{U} :\ h^{-1}(a) \neq \emptyset \ \text{ for each } \ a \in A\big\};$$
$$\mathcal{E}_{h,m} = \big\{A \in \mathcal{U} :\ |h^{-1}(a)| \leq m \ \text{ for each } \ a \in A\big\}.$$

If $h = f^k$ for any nonnegative integer k, then we will write $\mathcal{S}_k$ instead of $\mathcal{S}_h$ and $\mathcal{E}_{k,m}$ instead of $\mathcal{E}_{h,m}$.

A component description of $\mathcal{E}_{1,1}$ and $\mathcal{E}_{1,1} \cap \mathcal{S}_1$ is given in Proposition 1.

Example 4. Let $A = \{a, b, c, d\}$ and $f(a) = f(b) = c$, $f(c) = f(d) = d$. Then $A \in \mathcal{E}_{1,2} - \mathcal{E}_{2,2}$.

Lemma 12. *Let* $k \in \mathbb{N}$. *Then*

(i) $\mathcal{S}_0 = \mathcal{E}_{0,m} = \mathcal{U}$;

(ii) $\mathcal{S}_k = \mathcal{S}_1$ *for* $k > 1$;

(iii) $\mathcal{E}_{k,1} = \mathcal{E}_{1,1}$ *for* $k > 1$;

(iv) $\mathcal{E}_{k,m} \subset \mathcal{E}_{k,m+1}$;

(v) $\mathcal{E}_{k+1,m} \subset \mathcal{E}_{k,m}$ *for* $m > 1$;

(vi) $\mathcal{E}_{k,m} \neq \mathcal{E}_{l,n}$ *for* $l \in \mathbb{N}$, $m > n$.

Proof.

(i) We handle an identity operation.

(ii) Suppose that $(A, f) \in \mathcal{S}_1$. Let $u \in A$. We denote by $a_1, a_2, \ldots, a_k$ the following elements of A: $f(a_1) = u$, $f(a_2) = a_1$, $\ldots$, $f(a_k) = a_{k-1}$. Then $f^k(a_k) = u$. Therefore $A \in \mathcal{S}_k$.

Suppose that $A \in \mathcal{S}_k$. Let $v \in A$. Let $a \in A$ be such that $f^k(a) = v$. Then $f(f^{k-1}(a)) = v$. Conclude $A \in \mathcal{S}_1$.

(iii) It is obvious that $\mathcal{E}_{k,1} \subseteq \mathcal{E}_{1,1}$.

Let $A \notin \mathcal{E}_{k,1}$. Then there exist $a, b \in A$ such that $a \neq b$ and $f^k(a) = f^k(b)$. We can assume that $f^{k-1}(a) \neq f^{k-1}(b)$. Put $c = f^{k-1}(a)$, $d = f^{k-1}(b)$. We have $c \neq d$ and $f(c) = f(d)$. Thus $A \notin \mathcal{E}_{1,1}$

(iv) The inclusion follows from the definition $\mathcal{E}_{k,m}$. Let $B = \{1, 2, \ldots, m+1\}$ and $f(x) = 1$ for all $x \in B$. We have $B \in \mathcal{E}_{k,m+1} - \mathcal{E}_{k,m}$.

(v) The relation $\mathcal{E}_{k+1,m} \subseteq \mathcal{E}_{k,m}$ follows from the fact that an equality $f^k(a) = f^k(b)$ implies $f^{k+1}(a) = f^{k+1}(b)$.

Let $D = \{1, 2, \ldots, m + 2k\}$ and D be a connected monounary algebra from Fig. 4. We have $D \in \mathcal{E}_{k,m} - \mathcal{E}_{k+1,m}$, see Fig. 5.

(vi) Let A be an infinite connected monounary algebra from Fig. 6. Then $A \in \mathcal{E}_{k,m} - \mathcal{E}_{l,n}$.

□

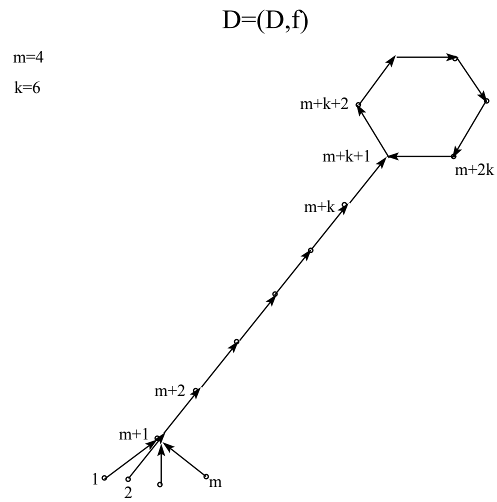

FIGURE 4.

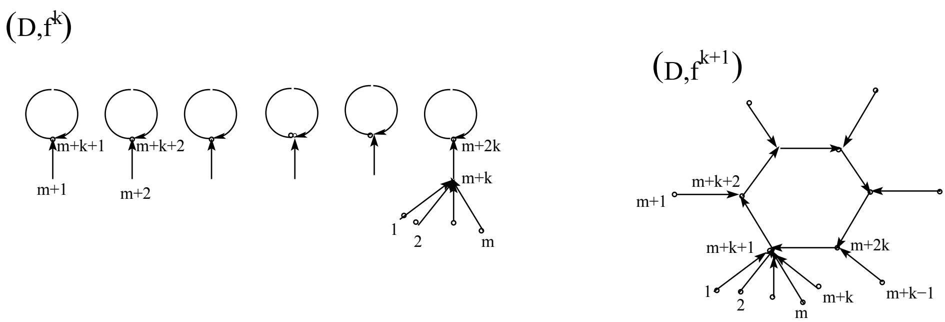

FIGURE 5.

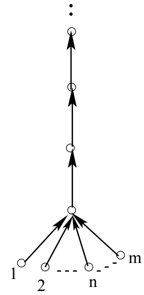

FIGURE 6.

A component description of $\mathcal{E}_{1,1}$ and $\mathcal{E}_{k,m} \cap \mathcal{S}_1$ is given in Theorem. 1.

Theorem 2. ([3: Thm. 5, 6]) *Classes $\mathcal{E}_{k,m}$ and $\mathcal{S}_1 \cap \mathcal{E}_{k,m}$ are $\underleftarrow{\mathbf{L}}$-classes.*

It is obvious that:

Lemma 13. *Let k be a nonnegative integer, $A \in \mathcal{U}$ and B be a component of A. If $A \in \mathcal{S}_k$, then $B \in \mathcal{S}_k$. If $A \in \mathcal{E}_{k,m}$, then $B \in \mathcal{E}_{k,m}$.*

The following lemma is an immediate consequence of [3: Lemma 1] . The set $R_{A,m}$ is defined in preliminaries of this paper.

Lemma 14. *Let (I, A_i, φ_j^i) be an inverse system of algebras with a nonempty inverse limit.*

- (i) *If A^∞ contains cycles C, D such that $C \neq D$ and $C \in R_{A^\infty,|D|}$, then there exists $i \in I$ such that A_i contains cycles C', D' such that $C' \neq D'$ and $C' \in R_{A_i,|D'|}$.*
- (ii) *If there exist $p, q \in A^\infty$ such that $p \neq q$, $f(p) = f(q)$ and q belongs to a cycle, then there exists $i \in I$ such that $p_i \neq q_i$, $f(p_i) = f(q_i)$ and q_i belongs to a cycle.*

Notation. We denote

$$\mathcal{E}'_{k,m} = \{A \in \mathcal{E}_{k,m} : \ A \text{ has no cycle}\}.$$

$$\mathcal{T}_{k,m} = \mathcal{T} \cup \mathcal{E}'_{k,m} \cup \{A \in \mathcal{U} : \ A = B \cup C, \ \ B \in \mathcal{T}, \ \ C \in \mathcal{E}'_{k,m}\}.$$

Analogously as in Lemma 12 we can see that:

Lemma 15. *Let $k \in \mathbb{N}$. Then*

- (i) $\mathcal{E}'_{k,1} = \mathcal{E}'_{1,1}$ *for* $k > 1$;
- (ii) $\mathcal{E}'_{k,m} \subset \mathcal{E}'_{k,m+1}$;
- (iii) $\mathcal{E}'_{k+1,m} \subset \mathcal{E}'_{k,m}$ *for* $m > 1$;
- (iv) $\mathcal{E}'_{k,m} \neq \mathcal{E}'_{l,n}$ *for* $l \in \mathbb{N}$, $m > n$.

Lemma 16. *Let $k \in \mathbb{N}$. Then*

- (i) $\mathcal{T}_{k,1} = \mathcal{T}_{1,1}$ *for* $k > 1$;
- (ii) $\mathcal{T}_{k,m} \subset \mathcal{T}_{k,m+1}$;
- (iii) $\mathcal{T}_{k+1,m} \subset \mathcal{T}_{k,m}$ *for* $m > 1$;
- (iv) $\mathcal{T}_{k,m} \neq \mathcal{T}_{l,n}$ *for* $l \in \mathbb{N}$, $m > n$.

Lemma 17. *Let $k \in \mathbb{N}$.*

Then $\mathcal{T}_{1,1} \subset \mathcal{E}_{k,m}$ for $m \geq 1$ and if $m > 1$, then $\mathcal{T}_{1,1} \subset \mathcal{T}_{k,m} \subset \mathcal{E}_{k,m}$.

Theorem 3. *Classes $\mathcal{E}'_{k,m}$, $\mathcal{E}'_{k,m} \cap \mathcal{S}_1$, $\mathcal{T}_{k,m}$ and $\mathcal{T}_{k,m} \cap \mathcal{S}_1$ are $\underleftarrow{\mathbf{L}}$-classes.*

P r o o f. Let (I, A_i, φ_j^i) be an inverse family. Let $A^\infty \neq \emptyset$.

Suppose that $A_i \in \mathcal{E}'_{k,m}$. The algebra A^∞ is a subalgebra of a direct product of algebras without cycles. Therefore A^∞ has no cycle. Further, $A^\infty \in \mathcal{E}_{k,m}$ since $A_i \in \mathcal{E}_{k,m}$. Conclude $A^\infty \in \mathcal{E}'_{k,m}$.

Analogously we can prove that if $A_i \in \mathcal{E}'_{k,m} \cap \mathcal{S}_1$, then $A^\infty \in \mathcal{E}'_{k,m} \cap \mathcal{S}_1$.

Suppose that $A_i \in \mathcal{T}_{k,m}$. Then $A_i \in \mathcal{E}_{k,m}$ according to Lemma 17. That means $A^\infty \in \mathcal{E}_{k,m}$. We have that every component of A^∞ belongs to $\mathcal{E}_{k,m}$ according to Lemma 13.

Let D be a component of A^∞ with a cycle. Then D is a cycle according to Lemma 14(ii). If D_1, D_2 are two different cycles of A^∞, then $D_1 \notin R_{A^\infty,|D_2|}$ according to Lemma 14(i).

Analogously we can prove that if $A_i \in \mathcal{T}_{k,m} \cap \mathcal{S}_1$, then $A^\infty \in \mathcal{T}_{k,m} \cap \mathcal{S}_1$. □

5. About inverse limits of $\mathcal{T}$

The class $\mathcal{T}$ is not an inverse limit closed according to Cor.1. In the previous section we described countable many inverse limit closed classes which contain $\mathcal{T}$. The question is: How $\underset{\leftarrow}{\mathbf{L}}\mathcal{T}$ looks like? It is not solved in this section, but we give some observations. Further, we will prove that $\underset{\leftarrow}{\mathbf{L}}\underset{\leftarrow}{\mathbf{L}}\mathcal{T} \neq \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$.

Lemma 18. *Let $A \in \mathcal{U}$ contain a proper subalgebra B such that $B \in \mathcal{T}$ and for every prime $p \in \mathbb{N}$ there is $b \in B$ such that $f^p(b) = b$. Then $A \notin \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$.*

P r o o f. Let (I, A_i, φ_j^i) be an inverse family of algebras from $\mathcal{T}$. Let A^∞ contain a subalgebra B such that $B \in \mathcal{T}$ and for every prime $p \in \mathbb{N}$ there is $b \in B$ such that $f^p(b) = b$. Consider $J = \{j \in I : |A_j| > 1\}$. The set J is cofinal with I, since B is a subalgebra of A^∞. Therefore (J, A_j, φ_k^J) is the inverse system with the limit isomorphic to A^∞.

Suppose that $p \in \mathbb{N}$ is prime and $(b_i)_{i \in I} \in B$ is such that $f^p((b_i)_{i \in I}) = (b_i)_{i \in I}$. Then $f^p(b_i) = b_i$ for every $i \in I$. That means that every algebra A_j, $j \in J$ has a cycle of the length p. Conclude $A_j \cong B$ according to $A_j \in \mathcal{T}$ for every $j \in J$. We obtain $A^\infty \cong B$ according to Theorem. 1. □

Corollary 3.

$$\underset{\leftarrow}{\mathbf{L}}\mathcal{T} \subset \mathcal{T}_{1,1} \cap \mathcal{S}_1$$

We will use the set R_A from preliminaries in the following lemma.

Lemma 19. *Let $A \in \mathcal{U}$ be such that $1 \leq |R_A| \leq |\mathbb{N}|$. Then $A \notin \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$.*

P r o o f. Let (I, A_i, φ_j^i) be an inverse family of algebras from $\mathcal{T}$. Then $A^\infty \neq \emptyset$ according to Lemma 7. Let B be a subalgebra of A^∞ which is isomorphic to Z and $(b_i)_{i\in I} \in B$. Denote by B_i a component of A_i such that $b_i \in A_i$ for every $i \in I$. Then $(I, B_i, \varphi_j^i|B_i)$ is an inverse family of cycles and B is a subalgebra of its inverse limit B^∞. We have that B^∞ consists of uncountable many components isomorphic to Z according to Lemma 6. The fact that $B^\infty \subseteq A^\infty$ completes the proof. □

Corollary 4.

$$\underset{\leftarrow}{\mathbf{L}}\underset{\leftarrow}{\mathbf{L}}\mathcal{T} \neq \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$$

Proposition 2.

$$\underset{\leftarrow}{\mathbf{L}}(\mathcal{T} \cup [N, Z]) = [N, Z] \cup \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$$

P r o o f. Let (I, A_i, φ_j^i) be an inverse family of algebras from $\mathcal{T} \cup [N, Z]$.

Let exist $i \in I$ such that $A_i \cong N$. Then $A_j \cong N$ for every $j \geq i$, $j \in I$. Put $J = \{j \in I : j \geq i\}$. Then J is cofinal with I. We obtain that $A^\infty \cong N$ according to Theorem. 1.

Let our inverse system contain no algebra isomorphic to N and let there exist $i \in I$ such that $A_i \cong Z$. Then $A_j \cong Z$ for every $j \geq i$, $j \in I$. Put $J = \{j \in I : j \geq i\}$. Then J is cofinal with I. We obtain that $A^\infty \cong Z$ according to Theorem. 1 .

Therefore $\underset{\leftarrow}{\mathbf{L}}(\mathcal{T} \cup [N, Z]) \subseteq [N, Z] \cup \underset{\leftarrow}{\mathbf{L}}\mathcal{T}$.

The opposite inclusion is trivial. □

6. Problems

1. Let A be an unbounded monounary algebra. Does the algebra N, failing to be unbounded, belong to $\underset{\leftarrow}{\mathbf{L}}\{A\}$?

2. Describe $\underset{\leftarrow}{\mathbf{L}}\mathcal{T}$.

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