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HÖLDER CATEGORIES

Anthony W. Hager* — Jorge Martínez**

Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. Hölder categories are invented to provide an axiomatic foundation for the study of categories of archimedean lattice-ordered algebraic structures. The basis of such a study is Hölder's Theorem (1908), stating that the archimedean totally ordered groups are precisely the subgroups of the additive real numbers $\mathbb{R}$ with the usual addition and ordering, which remains the single most consequential result in the studies of lattice-ordered algebraic systems since Birkhoff and Fuchs to the present.

This study originated with interest in $\mathbf{W}^*$, the category of all archimedean lattice-ordered groups with a designated strong order unit, and the ℓ-homomorphisms which preserve those units, and, more precisely, with interest in the epireflections on $\mathbf{W}^*$. In the course of this study, certain abstract notions jumped to the forefront. Two of these, in particular, seem to have been mostly overlooked; some notion of simplicity appears to be essential to any kind of categorical study of $\mathbf{W}^*$, as are the quasi-initial objects in a category. Once these two notions have been brought into the conversation, a Hölder category may then be defined as one which is complete, well powered, and in which

(a) the initial object I is simple, and
(b) there is a simple quasi-initial coseparator R.

In this framework it is shown that the epireflective hull of R is the least monoreflective class. And, when $I = R$ — that is, the initial element is simple and a coseparator — a theorem of Bezhanishvili, Morandi, and Olberding, for bounded archimedean f-algebras with identity, can be be generalized, as follows: for any Hölder category subject to the stipulation that the initial object is a simple coseparator, every uniformly nontrivial reflection — meaning that the reflection of each non-terminal object is non-terminal — is a monoreflection.

Also shown here is the fact that the atoms in the class of epireflective classes are the epireflective hulls of the simple quasi-initial objects. From this observation one easily deduces a converse to the result of Bezhanishvili, Morandi, and Olberding: if in a Hölder category every epireflection is a monoreflection, then the initial object is a coseparator.

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1. Introduction

The narrative of this paper is mapped out in this section, by first considering the concept of simplicity that we are about to introduce. A short catalog follows; it contains terms which are used throughout the paper.

The motivation for this study is Hölder's Theorem, one formulation of which is that the subgroups of $\mathbb{R}$, the group of additive real numbers with the usual addition and the usual ordering, are the only objects in the category **Ab** of all abelian lattice-ordered groups having no proper nonzero kernels. The place occupied by Hölder's Theorem is discussed in this section as well.

We follow with another list, containing several examples of the Hölder categories. We close the introduction with some tips on how to navigate through the paper.

1.1. Simple objects

The notion of a simple object in a category, as defined below, is made to order for the context we are interested in, although there are some references to simplicity in universal algebra — see, for example, the definition of the concept in [3: p.61]. The two definitions are different, as the interested reader can easily check. On the other hand, both lead to the fact that a simple ring is a ring which has no nonzero proper, two-sided ideals. Both lead to Schur's Lemma, in the context of unital modules over a ring with a multiplicative identity 1 ([13: Lemma 1.10, p. 419]).

Turning to the context of ordered algebraic structures, one quickly runs into Hölder's Theorem. One formulation of the theorem states that these are the only lattice-ordered abelian groups which have no proper nonzero (order) convex subobjects. Finally, and equivalently, the subgroups of $\mathbb{R}$ are the objects in **Ab** which have no proper nonzero kernels.

With all this in mind, the following definition of simplicity seems appropriate for the task at hand. Any undefined terms will be explained in short order.

Definition 1.1.1. Suppose that $\mathbf{C}$ has a terminal object T. An object S is *simple* if for every $\mathbf{C}$-morphism $h\colon S \longrightarrow B$ either h is monic or else h factors through T in a particular way: *there is a* $\mathbf{C}$*-morphism* $m\colon T \longrightarrow B$ *such that* $h = m \cdot e$*, where* $e\colon S \longrightarrow T$ *is an extremal epimorphism.*

One should observe right away that, since there is a morphism $p\colon B \longrightarrow T$, the composite $p \cdot m$ must be the identity on T. This information will be put to good use later on.

Here now is a short catalog of terminology and notation. It is limited to the terms and concepts that recur most often in the narrative below.

1.2. Catalog

We begin with some elementary concepts in the theory of categories. The ambient category is $\mathbf{C}$. We denote by $\hom(A, B)$ the set of all morphisms with domain A and codomain B.

(a) A *terminal* object T is one for which $|\hom(A, T)| = 1$, for each object A. It is well known that a terminal object is unique up to isomorphism. A terminal object is always denoted by T. It is easy to see that the terminal object is the product in the category over the empty set. Thus, if $\mathbf{C}$ is complete then it has a terminal object.

(b) The dual to a terminal object is called an *initial* object.

(c) An object S is *quasi-initial* if $|\hom(S, B)| \leq 1$ for each object B. It is straightforward to prove that S is quasi-initial if and only if the morphism $I \longrightarrow S$ is an epimorphism.

(d) The class of quasi-initial objects is denoted $\mathcal{Q}^o$. Two subclasses of $\mathcal{Q}^o$ will be distinguished:
   (i) $\mathcal{Q}^*$, the class of non-terminal quasi-initial objects, and
   (ii) $\mathcal{Q}$, the class of simple objects in $\mathcal{Q}^*$.

(e) The object A is *semisimple* if there is a monomorphism $m\colon A \longrightarrow \prod_i B_i$, with each B_i simple. In addition, if $B_i = R$, for each $i \in I$ and R is simple, we say that A is *R-semisimple*.

(f) A *coseparator* is an object C having the property that when f and g are distinct morphisms from the same domain to the same codomain B, then there is a morphism $k\colon B \longrightarrow C$ such that $k \cdot f \neq k \cdot g$.

(g) The concept of an *essential* monomorphism is used in several places, especially in §2.1: this is what one calls a monomorphism $m\colon A \longrightarrow B$ with the property, if $f\colon B \longrightarrow C$ is a morphism such that $f \cdot m$ is monic, then f itself must be monic. In the context of this paper "essential extension" and "essential monomorphism" are synonymous.

(h) This item is included here, not because of the frequency of its occurrence in the paper, but it is a subtle yet important ingredient in Theorem 3.4.6: call a reflection v *uniformly nontrivial* if for each object $A \neq T$ we also have that vA is non-terminal.

We assume that the reader has some grounding in category theory. And so, for example, we take for granted that the concepts of limits and colimits, and, specifically, particular limits, such as products, and particular colimits, such as pushouts, are familiar territory. We signal as well, without apology, that for "monomorphism" and "epimorphism" we will substitute the shorter and friendlier "mono" and "epi", respectively, both as nouns and as adjectives.

1.3. Hölder's theorem

Whether you approach an ordered algebraic structure as an algebra over the field $\mathbb{R}$ of real numbers, because you were taught that such objects are algebras of operators at heart, or whether your *modus operandi* is steeped in arguments involving convex ℓ-subgroups, perhaps because your origins are in the underlying groups, the importance of Hölder's Theorem is not easily exaggerated. We have in mind the theorem which asserts (in one of its common formulations), that *each archimedean totally ordered group is group and order-isomorphic to a subgroup of the additive group* $\mathbb{R}$ *of real numbers, with the usual ordering.*

Coming from a fairly detailed look at $\mathbf{W}^*$ (in [9]) with a special interest in its epireflections, soon enough, and perhaps predictably, it became clear that much of the mathematical landscape we wished to study was best viewed through the lenses of Category Theory. And so the resolution of problems which are particular to $\mathbf{W}^*$ will wait a while, at least, while the self-evident importance of Hölder's Theorem in this (categorical) setting is given proper expression.

It is probably not a surprise that our understanding of this landscape has not flowed in a steady progression of discovery. We discovered ([8]) that epireflective classes could alternatively and often quite effectively be studied by using the companion notion of a pushout invariant class of epimorphisms, the same notion of pushout invariance we introduced in [7]. The main result of [8] is that — under fairly routine assumptions — there is a one-to-one order-inverting correspondence between the pushout invariant classes of epimorphisms, and the epireflective classes (Theorem 4.2.3). It restricts to a correspondence between between the monoreflective classes and the pushout invariant classes of monic epimorphisms. Since there is clearly a maximum pushout invariant class of monic epimorphisms, it follows that there is a largest monoreflection. The latter point was observed in isolation, for $\mathbf{W}^*$, but there was no surprise when it turned out that the matter has not much to do with the representation of objects in $\mathbf{W}^*$ by bounded functions.

What measure of success can be claimed in this study? If what we are primarily interested in is the epireflections, then the larger category $\mathbf{W}_{\mathbb{R}}$ of all real-semisimple ℓ-groups has many of the characteristics of $\mathbf{W}^*$, except that in $\mathbf{W}^*$ the maximum monoreflection is essential, while in $\mathbf{W}_{\mathbb{R}}$ it is not. It is clear then, that a characterization of the Hölder categories in which the monoreflections are essential awaits. One is also tempted to argue that, in view of the fact that the discussion of Example 3.3.4, as well as the remarks leading up to the example require a certain fluency in general topology, that it is not clear yet how an algebraically inclined study will resolve the question posed above regarding essential reflections. A more plausible gauge of success may be found in Proposition 2.3.5, which can be seen as a categorical manifestation of Hölder's Theorem, and reassure that we are therefore on the right track.

Moreover, staying with the interpretation given to Proposition 2.3.5, Theorem 3.4.6 is not so surprising. By the same token, this theorem, viewed as characterizing the Hölder categories in which there is a unique minimal nontrivial epireflective class, will require some fine-tuning in the future, as it describes the situation in the category $\mathbf{fAlg}^*$, where we are dealing with real f-algebras, but also in **Bool**, the category of boolean algebras and boolean homomorphisms. For several obvious reasons, the characterization of Theorem 3.4.6 is not entirely satisfying.

1.4. List

This paper is written with ordered algebraic structures in mind. There are six categories of such structures that will be used throughout the paper, as examples and illustrations. However, before we launch into those descriptions, we should mention explicitly one example of what we propose to call a Hölder category. It is **Bool** again. In a sense this is the trivial example. We refer to **Bool** that way because many features which offer subtlety and nuance in this subject, get identified in boolean algebras. The reader may look ahead at the details in 3.4.2. As predicted by Theorem 3.4.6, every uniformly nontrivial reflection is a monoreflection. On the other hand, it is well known, independently, that the only such reflection in **Bool** is the identity functor.

We shall return to **Bool** in due course, but now let us proceed to the list we promised above. They are grouped in three pairs, to both contrast and suggest a pattern. Familiarity with the representation theory of the lattice-ordered groups discussed here, on their Yosida spaces, will be a great plus for the reader. We refer the reader to [10] for background on that topic. All references made here to representation as continuous functions are understood to be according to the Yosida representation, or, in the instances where the structures under study have underlying algebras, one may more conveniently appeal to the closely related Henriksen-Johnson representation.

Each of the classes listed in the itemization that follows has a pair of distinguished subclasses; in the parent category each object is an algebraic structure whose elements can be represented as continuous functions, in the same spirit as the one outlined in the preceding paragraph. Alternatively, the objects of the category under consideration are the $\mathbb{R}$-semisimple members of the parent category. If $\mathbf{C}$ is the parent category, then we denote the subclass of $\mathbb{R}$-semisimple objects by $\mathbf{C}_{\mathbb{R}}$.

There is a second example to consider, a full subcategory consisting of the objects being represented by bounded function; these designations are are rendered by a superscripted asterisk; e.g., $\mathbf{C}^*$.

- **W.** The category $\mathbf{W}^*$ of all archimedean ℓ-groups with a strong designated order unit and maps which are ℓ-homomorphisms that preserve the designated unit. As explained above, there is also the category $\mathbf{W}_{\mathbb{R}}$, a parent category, consisting of all the $\mathbb{R}$-semisimple ℓ-groups G with a designated unit, whose members may be represented on a dense set of some topological space $X(G)$ as real valued continuous functions. The knowledgeable reader will recognize that both of these are, in turn, full subcategories of the well documented $\mathbf{W}$, of all the archimedean ℓ-groups with a designated unit (and maps which are ℓ-homomorphisms that preserve the designated unit). This parent category, however, having no simple coseparator (see Example 3.3.2), is beyond the scope of the techniques that will be developed here. (Note that the term 'coseparator' is defined in §1.2(f).)

VL. As in the previous example, we consider the category $\mathbf{VL}_{\mathbb{R}}$, of all $\mathbb{R}$-semisimple real vector lattices with a distinguished order unit, with maps that are linear transformations as well as ℓ-homomorphisms that preserve the designated unit. We will also use the full subcategory $\mathbf{VL}^*$ of all the objects in $\mathbf{VL}_{\mathbb{R}}$ consisting of bounded functions. As in the preceding case, these are full subcategories of $\mathbf{VL}$ of all archimedean real vector lattices with a designated unit. This category has no simple coseparator.

fAlg. Let $\mathbf{fAlg}_{\mathbb{R}}$ stand for the category of all real archimedean f-algebras with identity, which are $\mathbb{R}$-semisimple, and ring homomorphisms which are simultaneously ℓ-homomorphisms which preserve the identity. It is well known that any such ring is necessarily commutative and semiprime — meaning that there are no nonzero nilpotent elements. We also take this opportunity to remind the reader that the lattice-ordered ring A is an f-ring if $a \wedge b = 0$ implies that $ca \wedge b = 0$, whenever $0 \leq c \in A$.

As with the two other pairs of categories, there is a "bounded companion" to go along with $\mathbf{fAlg}_{\mathbb{R}}$; namely, $\mathbf{fAlg}^*$, the subcategory of all f-algebras in which the identity of the algebra is a strong unit — or equivalent, the f-algebras which can be represented as bounded continuous functions.

1.5. Navigation

The rest of this paper is organized as follows.

Section 2 considers the simple quasi-initial objects in a generic category. It is proved (in §2.1), for a map m between non-terminal quasi-initial objects with simple domain, that m is an essential mono if and only if the codomain of the map is simple (Lemmas 2.1.5 and 2.1.6). As §2.2 announces in its title, a description is given of the reflection map associated with the epireflective hull of a simple quasi-initial object; see Proposition 3.1.3, as well as the closely related Proposition 2.2.4. Next, we define the "core" (§2.3) of a category, and it is shown that the core map $A \mapsto \odot A$ is a monocoreflection (Proposition 2.3.2).

Following the definition of a Hölder category at the conclusion of Section 2, Section 3 opens with a detailed discussion of the examples introduced in the

List. This section has the results on epicompleteness — Proposition 3.2.1 and Theorem 3.2.2 — that have already been mentioned in the abstract.

There is an appendix, containing background material from category theory, and enough of the referrals to [8] that the readership will likely need to successfully make their way through the present narrative.

2. Simplicity

In the three units of this section, we describe in greater detail the simple quasi-initial objects, and what happens to maps between them. Nothing is assumed about the ambient category $\mathbf{C}$ at the very outset.

In the category **CRn1** of commutative rings with identity and all ring homomorphisms which preserve the identity, the initial object — the ring of integers $\mathbb{Z}$ — is not simple. On the other hand, in $\mathbf{W}^*$, the totally ordered group of integers is simple. Our motivation thus persuades us to assume throughout the paper that the following "axiom" is satisfied:

Axiom 1. The initial object I of $\mathbf{C}$ is simple.

Nonetheless, let us persuade the reader as well, and delay the imposition of this axiom until the end of this section.

2.1. Quasi-initial objects

Recall from §1.2 that U is quasi-initial if for each object A there is at most one morphism $U \longrightarrow A$. It is also mentioned in that catalog of terms that U is quasi-initial precisely when the map $s_U : I \longrightarrow U$ is epi. We fix this notation for the unique morphism $s_{U,A} : U \longrightarrow A$ from the quasi-initial object U to an arbitrary object A, if one exists.

For the rest of this section we analyze morphisms $u \colon U \longrightarrow S$ between quasi-initial objects, the goal being to understand when simplicity of the domain is carried over to the codomain, and *vice versa*. Lemmas 2.1.5 and 2.1.6, together with Axiom 1 resolve those issues, and, in particular, establish the proposition that follows.

Proposition 2.1.1. *Suppose $e \colon S \longrightarrow U$ is an epimorphism and $S \in \mathcal{Q}$. Then e is an essential monomorphism if and only if $U \in \mathcal{Q}$.*

As stipulated in §1.2, we use $\mathcal{Q} \subseteq \mathcal{Q}^* \subseteq \mathcal{Q}^o$ for the three nested classes of quasi-initial objects which are (from smallest to largest,) both simple and non-terminal, just non-terminal, and, finally, the class of all quasi-initial objects.

Now, before proceeding with the lemmas we mentioned, there are some remarks that should be collected and recorded, for easy future reference. That is the purpose of the next lemma. Below we use the term *section* to stand for a morphism which has a left inverse. It should be clear that sections are extremal monos.

Lemma 2.1.2. *Assume now that* $\mathbf{C}$ *is an (epi, extremal mono)-category. Then*

(1) *Any morphism out of* T *(the terminal object) is a section.*

(2) *An object* A *is simple if any map* $h\colon A \longrightarrow B$ *is monic, or else the (epi, extremal mono)-factorization of* h *is through* T.

(3) *Suppose that* U *is a quasi-initial non-terminal object. There are no morphisms from* T *to* U.

(4) *If* $e\colon S \longrightarrow U$ *is a morphism between non-terminal quasi-initial objects, with* S *simple, then* e *is monic.*

Proof.

(1) is justified by the remark which concludes Definition 1.1.1.

We leave (2) for the reader to verify.

(3) If there is a map $T \longrightarrow U$, then its composition with the unique map $U \longrightarrow T$, in either order, is the identity. Thus, $T \cong U$, which contradicts the fact that U is non-terminal.

(4) Since S is simple, we may apply (3) to rule out a factorization through T. □

Remark 2.1.3. Observe now that $\mathcal{Q}^o$ has a natural quasi-order, given by:

$$U \leq V \iff \text{there exists a map } U \longrightarrow V.$$

Since maps between quasi-initial objects are unique, this does indeed define a quasi-order. Lemma 2.1.2(4) shows, in $\mathcal{Q}$, that $U \leq V$ if and only if there is a monomorphism from U to V.

Furthermore, let us note that, as a quasi-order, there is the usual partial order associated with it, which is defined on the isomorphy classes of objects in $\mathcal{Q}^o$. The reader should keep in mind that in these terms, T, which *is* quasi-initial, by (3) in Lemma 2.1.2, is the largest member of $\mathcal{Q}^o$. Likewise I is its least element, and (if there is a simple coseparator R) then R is the largest non-terminal object. To summarize, we blur the distinction between an object in $\mathbb{Q}^o$ and its isomorphy class, rather than insert the phrase "up to isomorphism" after each comparison.

Lemma 2.1.4 is easy to prove, and we leave it as an exercise for the reader. Notice that, only for the sufficiency is the assumption that $\mathbf{C}$ has (extremal epi, mono)-factorization (4.1.1) needed. In the development of this paper below, it will be assumed that the category of discourse is both an (extremal epi, mono)-category and an (epi, extremal mono)-category (4.1.3(9)). These two properties will definitely be in place for Hölder categories.

Lemma 2.1.4. *Suppose that S is a $\mathbf{C}$-object, and that $\mathbf{C}$ has (extremal epi, mono)-factorization. Then, S is simple if and only if each extremal epimorphism $e\colon S \longrightarrow B$ is either an isomorphism or else B is terminal.*

Next, here is a result which might be expected by a reader whose background lies in the classical algebra of modules over a ring with identity.

Lemma 2.1.5. *Suppose that $e\colon S \longrightarrow K$ is an epic essential monomorphism in an (epi, extremal mono)-category. Then if S is simple, K is as well.*

P r o o f. Suppose that S is simple and the morphism $e\colon S \longrightarrow K$ is essential and epic. Let $g\colon K \longrightarrow L$ be a morphism, and consider the composite $g \cdot e$, which is either monic or else its (extremal epi, extremal mono)-factorization is through T. Let $e_o\colon S \longrightarrow T$ and $m\colon T \longrightarrow L$ be that factorization.

In the first event, we have that g is monic because e is essential. In the second, consider the commutative square

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[d, "e_o"'] & K \arrow[d, "g"] \\ T \arrow[r, "m"'] & L \end{tikzcd}$$

with top and bottom maps epi and extremally mono, respectively. By the diagonalization property (4.1.1(5)), the extremally epi $u\colon K \longrightarrow T$ satisfies $m \cdot u = g$, all of which proves that K is a simple object. $\square$

The reader may well notice that Lemmas 2.1.2(4) and 2.1.6(1) below, say almost the same thing. It should also be clear that the first result assumes the codomain to be quasi-initial, whereas in the second the assumption that the map is epi that gives K that property. What the reader might find useful in having the itemized statement of Lemma 2.1.6 is that it unlinks the essential feature of the morphism.

Lemma 2.1.6. *Once more, assume that $\mathbf{C}$ is both an (epi, extremal mono)-category, as well as an (extremal epi, mono)-category. Then*

(1) *if $S \in \mathcal{Q}$ and $e\colon S \longrightarrow K$ is epi and K is non-terminal, then K is quasi-initial, and e is monic.*

(2) *If in addition, K is simple (so that both domain and codomain are in $\mathcal{Q}$), then e is essential.*

P r o o f.

(1) That K is quasi-initial is obvious. And Lemma 2.1.2(4) covers the rest.

(2) Suppose that $g\colon K \longrightarrow A$ is a morphism such that $g\cdot e$ is monic. If g is not monic, then it factors through T into an epimorphism e_o followed by extremally monic m_o — as $g = m_o \cdot e_o$. But then we have the factorization

$$g \cdot e = m_o \cdot (e_o \cdot e),$$

and it is the (extremal epi, extremal mono)-factorization of $g \cdot e$, contradicting the fact that it is mono. □

Evidently, Lemmas 2.1.5 and 2.1.6 prove Proposition 2.1.1. Now, using Axiom 1, one gets a special case of Proposition 2.1.1.

Lemma 2.1.7. *Suppose that* $\mathbf{C}$ *is an (extremal epi, mono)-category and an (epi, extremal mono)-category and that* Axiom 1 *is satisfied. Let* U *be a simple, quasi-initial, non-terminal object. Then the unique morphism* $s_U\colon I \longrightarrow U$ *is an essential monomorphism.*

We recall now a trio of operators on a given class $\mathfrak{A}$, as well as how one combines them to form classes with desired properties.

Definition & Remarks 2.1.8. For any class $\mathfrak{A}$ of objects, we use the symbols below:

- $\mathcal{P}\mathfrak{A}$, for the class of all products of members of $\mathfrak{A}$,
- $\mathcal{S}\mathfrak{A}$, for all subobjects of members of $\mathfrak{A}$, and
- $\mathcal{K}\mathfrak{A}$, for all extremal subobjects of members of $\mathfrak{A}$.

The above are, evidently, the smallest classes containing $\mathfrak{A}$, and closed with respect to the featured construction. We will often be concerned with classes generated by a single object A. We will then write $\mathcal{P}A$, $\mathcal{S}A$ and $\mathcal{K}A$, respectively.

Let $\mathcal{U}$ and $\mathcal{V}$ denote any of the above operators. The succession $\mathcal{V}\mathcal{U}$ simply indicates that one applies $\mathcal{U}$ first and then $\mathcal{V}$. Thus:

(a) $\mathcal{S}\mathcal{P}\mathfrak{A}$ remains closed under taking products, making it the smallest class containing $\mathfrak{A}$ and closed under the formation of products and subobjects. The reader may wish to look ahead at the Freyd-Isbell-Kennison Theorem (Theorem 4.1.2), and the precise assumptions one needs to apply it. Under such assumptions, $\mathcal{S}\mathcal{P}\mathfrak{A}$ is the smallest (extremal epi)-reflective class containing $\mathfrak{A}$.

We observe, perhaps for the reader who jumps ahead to anticipate which way the narrative is going, that we do not describe the reflection map associated with the reflection in $\mathcal{S}\mathcal{P}\mathfrak{A}$, at least not in full generality. The description of $\mathcal{S}\mathcal{P}U$, when $U \in \mathcal{Q}$, is useful for present purposes, and we proceed to describe it below.

(b) Invoking [12: Proposition 19.8], we conclude that A is a coseparator if and only if $\mathcal{S}\mathcal{P}A = \mathbf{C}$.

(c) By [12: 34.2], $\mathcal{KP}\mathfrak{A}$ remains closed under formation of products. It follows then, under assumptions of the same range as in (a), that $\mathcal{KP}\mathfrak{A}$ is the smallest epireflective class containing $\mathfrak{A}$.

Regarding the operators $\mathcal{SP}$ and $\mathcal{KP}$, there are the following elementary observations. We sketch the proofs.

Proposition 2.1.9. *Suppose that U and V are quasi-initial objects. Then the following are equivalent.*

(1) $\mathcal{SP}U = \mathcal{SP}V$.

(2) *U and V are isomorphic.*

(3) $\mathcal{KP}U = \mathcal{KP}V$.

Proof.

(1) $\Longrightarrow$ (2): there is a monomorphism $m\colon U \longrightarrow G$, with G being a product of — say — Y copies of V. However, since U is quasi-initial, $\pi_y \cdot m = u$, for each index $y \in Y$, with u being the unique morphism from U to V. By reversing the roles of U and V, we get a morphism v in the reverse direction. Then, as argued following the proof of Lemma 2.1.2, we conclude that $U \cong V$.

The rest is obvious. □

The final item in this succession of preparatory results strengthens the preceding one a bit, and it also leaves behind a new unanswered question.

Lemma 2.1.10. *Suppose that $U \in \mathcal{Q}$ and V is a non-terminal, quasi-initial object. Then $U \in \mathcal{SP}V$ if and only if there is a monomorphism $m\colon U \longrightarrow V$.*

Thus, if $R \in \mathcal{Q}$ is a coseparator, then

(1) *R is the largest object in $\mathcal{Q}$;*

(2) *R is maximal in the quasi-order of $\mathcal{Q}^*$;*

(3) *in particular, simple, quasi-initial coseparators are unique up to isomorphism.*

We leave the proof to the reader.

Remark 2.1.11. The concept of an "essential extension", or, more formally, of an essential monomorphism, can be formulated in most algebraic contexts in such a way that it has the following property: if $m\colon A \longrightarrow B$ and $n\colon B \longrightarrow C$ are morphisms, then $n \cdot m$ is an essential monomorphism if and only if both m and n are essential.

In general, the composite of two essential monos is essential, and it is easy to show that if $n \cdot m$ is an essential mono, then n is an essential mono. The problem with the remaining inference is somewhat of a mystery. The only reason to bring it up here is that, in the context of Hölder categories ahead, this property also implies that $\mathcal{Q}^* = \mathcal{Q}$.

2.2. The reflection maps of epireflective hulls

For a simple, non-terminal quasi-initial object U, we shall describe the reflection map of the epireflection in $\mathcal{SP}U$ (by Proposition 2.2.4). The corresponding account for the epireflection in $\mathcal{KP}U$ is taken up in §3.1.

Remarks 2.2.1. Let us first establish some basic notation: $\hom_{\mathbf{C}}(A, B)$ denotes the set of all morphisms in $\mathbf{C}$ with domain A and codomain B. When there is little chance for a misunderstanding, we will drop the subscript and write $\hom(A, B)$. Now, by defining, for each morphism $e\colon A \longrightarrow B$, and each $h\colon B \longrightarrow J$,

$$\hom(e, J)(h) = h \cdot e, \qquad \text{with} \quad \hom(e, J)\colon \hom(B, J) \longrightarrow \hom(A, J),$$

the construct $\hom(\cdot, J)$ is a contravariant functor.

We collect the basic facts about the construct $\hom(e, J)$ in the lemma that follows. No assumptions are made about $\mathbf{C}$, until we get to Proposition 2.2.4.

Lemma 2.2.2. *In an arbitrary category* $\mathbf{C}$*, suppose* $e\colon A \longrightarrow B$ *is a* $\mathbf{C}$*-morphism:*

- *e is epi if and only if* $\hom(e, J)$ *is one-to-one, for each object J.*
- *If R is a coseparator, then e is epi if and only if* $\hom(e, R)$ *is one-to-one.*

We make a preliminary construction. By way of a preface, let $e\colon A \longrightarrow B$ be a $\mathbf{C}$-morphism, and U be a simple, non-terminal quasi-initial object.

Definition & Remarks 2.2.3. To set up the several reflection maps that will come up in the sequel, we have to be nimble in the discussion regarding products. We note that if $A = \prod_{i \in I} A_i$, then we shall use π_i to denote the projection upon the factor A_i.

For each quasi-initial object U, and for any object A, let U_A denote the product of copies of U, indexed by $\hom(A, U)$. For each morphism $h\colon A \longrightarrow B$ we define $\widehat{h}\colon U_A \longrightarrow U_B$, by setting, for each $g \in \hom(B, U)$,

$$\pi_g \cdot \widehat{h} = \pi_{g \cdot h}.$$

Finally, we denote the *diagonal morphism* by $\Delta_A\colon A \longrightarrow U_A$. It is the map defined by the condition $\pi_h \cdot \Delta_A = h$, for each $h \in \hom(A, U)$.

The following facts are then easily verified. In (i) and (ii) below, but not in (iii), it is assumed as well that U is a coseparator.

(i) *The induced* $\hom(e, U)$ *is a section.* (Apply Lemma 2.2.2.)

(ii) Δ_A *is monic.* If $g, h \in \hom(X, A)$ and $\Delta_A \cdot g = \Delta_A \cdot h$, then for each $w \in \hom(A, U)$,

$$\pi_w \cdot \Delta_A \cdot g = \pi_w \cdot \Delta_A \cdot h,$$

that is $w \cdot g = w \cdot h$, for all $w \in \hom(A, U)$, and so $g = h$, since U is assumed to be a coseparator.

(iii) *With the notation as above, the following square commutes:*

$$\begin{tikzcd} A \arrow[r, "\Delta_A"'] \arrow[d, "h"'] & U_A \arrow[d, "\widehat{h}"] \\ B \arrow[r, "\Delta_B"'] & U_B \end{tikzcd}$$

Now we have the characterization we were looking for.

Proposition 2.2.4. *Suppose that* $\mathbf{C}$ *is an (extremal epi, mono)-category which is complete and co-well-powered. Then the diagonal* Δ_A *factors as*

$$\Delta_A = x_A \cdot \mu_A,$$

with $x_A \colon \mu A \longrightarrow U_A$ *monic, and* $\mu_A \colon A \longrightarrow \mu$ *extremally epic.*

Moreover, for each morphism $h \colon A \longrightarrow B$*,*

(1) μA *is in* $\mathcal{SPU}$*.*

(2) *There is a unique morphism* $\mu(h) \colon \mu A \longrightarrow \mu B$ *such that the diagram below commutes:*

$$\begin{tikzcd} A \arrow[r, "\mu_A"'] \arrow[d, "h"'] & \mu A \arrow[r, "x_A"'] \arrow[d, "\mu(h)"'] & U_A \arrow[d, "\widehat{h}"'] \\ B \arrow[r, "\mu_B"'] & \mu B \arrow[r, "x_B"'] & U_B \end{tikzcd} \tag{2.2.4}$$

(3) μ_A *is the reflection of* A *in* $\mathcal{SPU}$*.*

Proof. The first claim is obvious, and an application of the diagonalization property (4.1.1(5), or [12: 34.3]) settles the second claim. (An almost identical argument is supplied, with greater detail, when we prove Proposition 3.1.3.) □

The reader has probably already conceded that Axiom 1 is a reasonable assumption to make, and so we do assume, henceforth, that Axiom 1 holds, unless the contrary is explicitly stated.

2.3. The initial core

The initial core of an object in a category $\mathbf{C}$ can best be seen as a classification device. We have come to view it as a label that is put on an object, and said label, at some point in the process of becoming a familiar part of the landscape, also serves notice that it defines a monocoreflection with values in the subcategory

$\mathcal{Q}^o$, and thus the quasi-initial objects are integrated more intimately into the parent category.

In order to define the initial core of an object in our generic category $\mathbf{C}$, all that is required is that the category have an initial object, and that $\mathbf{C}$ be an (epi, extremal mono)-category. For each $\mathbf{C}$-object A, we examine the unique (epi, extremal mono)-factorization of the unique morphism $s_A\colon I \longrightarrow A$ from the initial object I, written

$$I \xrightarrow{s_B} B \xrightarrow{h} A, \tag{$\diamond$}$$

with $s_A = h \cdot s_B$, and h extremally mono. Since B is unique up to isomorphism, we may set $B \equiv \odot A$.

We remind the reader that Axiom 1 is in place. Moreover, along with the assumption that the underlying category is an (epi, extremal mono)-category, we also assume, henceforth, that it is an (extremal epi, mono)-category.

With these preliminary comments in place, some of the properties of the assignment $A \mapsto \odot A$ are summarized in the following proposition.

Proposition 2.3.1. *Assume that the underlying category is both an (epi, extremal monic)-category and (extremal epi, mono)-category. Suppose that A is a non-terminal object of $\mathbf{C}$, and ($\diamond$), above, represents the (epi, extremal mono)-factorization of the monomorphism $s_A\colon I \longrightarrow A$. Then we have that*

1. *$\odot A$ is quasi-initial.*
2. *If $U \in \mathcal{Q}^*$ and s_A factors as $s_A = m \cdot s_U$, with s_U epic and $m\colon U \longrightarrow A$ monic, then there is a morphism $k\colon U \longrightarrow \odot A$, which is both monic and epic, such that $k \cdot s_U = s_{\odot A}$ and $h \cdot k = m$. Thus, U is quasi-initial, and $U \leq \odot A$, which makes $\odot A$ the supremum in $\mathcal{Q}^*$ of the quasi-initial objects which are subobjects of A.*

P r o o f.

(1) Observe that since ($\diamond$) is a unique factorization that defines $\odot A$ above, and the first factor is epi, then $\odot A$ quasi-initial.

(2) Applying the diagonalization property (4.1.1(5), or [12: 34.3]) to the square

$$\begin{tikzcd}
I \arrow[r, "s_U"] \arrow[d, "s_{\odot A}"'] & U \arrow[d, "m"] \\
\odot A \arrow[r, "h"] & A
\end{tikzcd}$$

produces the desired k, satisfying the equations $k \cdot s_U = s_{\odot A}$ and $h \cdot k = m$, which imply that k is monic and epic. The rest is straightforward. $\square$

We will refer to $\odot A$ as the *initial core* of A, or simply as the *core* of A.

The following result suffices to make the initial core interesting. The class $\mathcal{Q}^o$ is viewed as a full subcategory. Since the objects are quasi-initial, there is at most one map $A \longrightarrow B$, and there is a map if and only if $A \leq B$. It is understood that the core of T is T itself.

Proposition 2.3.2. *The assignment $A \mapsto \odot A$ defines a functor on $\mathbf{C}$ which is a monocoreflection in the subcategory $\mathcal{Q}^o$ of all quasi-initial objects in $\mathbf{C}$.*

It will be useful to first tend to the following lemma.

Lemma 2.3.3. *Suppose that $h\colon A \longrightarrow B$ is a $\mathbf{C}$-morphism. Then $\odot A \leq \odot B$.*

Proof of the lemma. Consider the diagram below. It gives the (epi, extremal mono)-factorizations of $s_A\colon I \longrightarrow A$ and $s_B\colon I \longrightarrow B$, the unique maps from I to A and B, respectively.

$$\begin{tikzcd}
I \arrow[dr, "s_A"] \arrow[drrr, "s_B"] \arrow[dddr, dotted, bend right] \arrow[dddrrr, dotted, bend left] & & & \\
& \odot A \arrow[dd] \arrow[dr, "e"] & & \odot B \arrow[dd] \\
& & Q \arrow[dr, "m"] & \\
& A \arrow[rr, "h"'] & & B
\end{tikzcd}$$

The vertical morphisms in this diagram are extremally monic. We think of them as "inclusions", though the ambient category need not be concrete. By extension of this kind of intuition, the use of one of these $\odot G \longrightarrow G$ is regarded as restriction of a morphism to the core of the domain.

Now factorize the restriction of h to the core $\odot A$, as $m \cdot e$, with m extremally monic, and e epic, as is also indicated in the diagram. Next, observe: since e is epi, it follows that $e \cdot s_A$ is also epic, and also that Q is quasi-initial. Thus, we have that $e \cdot s_A$ and m are an (epi, extremal mono)-factorization of the morphism $I \longrightarrow B$, which shows that $Q \cong \odot B$, and we are done. $\square$

Very little else needs to be said now to establish the proposition.

Proof. What the preceding lemma demonstrates is that, for any morphism $h\colon A \longrightarrow B$, one gets that $\odot A \leq \odot B$ in the quasi-ordered $\mathcal{Q}$. Since the latter is a quasi-ordered class, $\odot$ is a functor, and that it is a coreflection is then obvious. $\square$

The following assertions are consequences of the fact that $\odot$ is a coreflection, and that (i) any coreflection preserves limits and (ii) the class of objects that is coreflected upon is closed under formation of colimits. And, finally, observe that in any quasi-ordered class the infimum is the product, while the supremum is the coproduct.

Corollary 2.3.4.

(1)

$$\odot\Big(\prod_{\lambda} A_\lambda\Big) = \bigcap_{\lambda} \odot A_\lambda, \qquad \textit{for each set } A_\lambda \textit{ of } \mathbf{C}\textit{-objects.}$$

(2) *The supremum in $\mathcal{Q}$ and the coproduct, taken in $\mathbf{C}$, coincide.*

Before putting aside this first look at the core, we should record a number of observations about the epireflective hulls $\mathcal{KP}Q$ (with Q ranging over all the non-terminal quasi-initials of our category) in a special light. They are the atoms of the class of all epireflective classes in $\mathbf{C}$.

Proposition 2.3.5. *For each nontrivial epireflective class $\mathfrak{X}$, and each non-terminal object $A \in \mathfrak{X}$, $\mathcal{KP}(\odot A) \subseteq \mathfrak{X}$.*

Proof. Simply observe that $\odot A$, being an extremal subobject of A, must therefore lie in $\mathfrak{X}$ as well. □

We are now in a position to confirm a number of facts about epireflective hulls $\mathcal{KP}U$, and, more generally, about the epireflective class generated by any class of objects. First, a very general and easy lemma.

Lemma 2.3.6. *If $m\colon A \longrightarrow B$ is any extremal monomorphism, then $\odot A = \odot B$.*

Proposition 2.3.7. *Suppose $\mathcal{F}$ generates the epireflective class $\mathfrak{X}$. Then for each $A \in \mathfrak{X}$, $\odot A$ is in the complete meet subsemilattice generated by the cores of members of $\mathcal{F}$.*

With this proposition it is easy to see what happens when $\mathbf{Y}$ and $\mathbf{Y}'$ are distinct atoms in the lattice of epireflective classes.

Corollary 2.3.8. *Suppose U_1 and U_2 are distinct non-terminal quasi-initial objects. Then*

$$\mathcal{KP}U_1 \cap \mathcal{KP}U_2 = \{T\}.$$

Thus, the class of all $\mathcal{KP}U$, with U ranging over the representatives of all the isomorphy classes of minimal nontrivial epireflective classes, is the class of all atoms in the lattice of epireflective subcategories.

2.4. The main fare

It seems reasonable to look at the state of affairs, and conclude that the case has been made for a class of categories having the following properties. We say that the category **C** is a *Hölder category* if

- it is complete, well-powered,
- Axiom 1 holds, and
- it has a simple, quasi-initial coseparator R.

The reader should now observe — using [12: 23.14], and supplementing with the various items in 4.1.3 — that because a Hölder category is complete and well-powered, and it has a coseparator, then it is also cocomplete and co-well-powered. Moreover, a Hölder category is necessarily both an (epi, extremal mono)- and an (extremal epi, mono)-category (4.1.3(9), or [12: 34.1,34.5]). By Corollary 2.3.8 the collection of atoms of the lattice of epireflective classes is a set.

In short order we shall proceed to examine Hölder categories, highlighting the examples in §1.4. However, we think it is appropriate in light of the definition of a Hölder category, to give the reader, right away and without much fanfare, two examples of these categories: the first one is reasonably well known and well understood, whereas the second might surprise a little, but also enlighten the reader.

Example 2.4.1. Consider the category **Bool** of all boolean algebras and the boolean homomorphisms. It is important to add that we include the boolean algebra of one element; the one-element boolean algebra is, in fact, the terminal object. The initial algebra is the two-element algebra **2**; it is clearly simple. It is also well known that in **Bool** every epimorphism is extremal and surjective. Thus, **2** is the only non-terminal, quasi-initial object.

Finally, it is standard information that **2** is a coseparator in **Bool**, which is also known to be complete. That is enough to see that **Bool** is a Hölder category.

Example 2.4.2. Consider a poset X as a category; its elements are the objects, and there is an arrow $x \longrightarrow y$ if and only if $x \leq y$, and in that case there is only one arrow. Here is now a list of properties of the category X, together with the translation from lattice-theoretic conditions. All of these are obvious, or else easy to verify.

(1) All morphisms are both epi and mono.
(2) The extremal epis are the identity maps; that is to say, the identities on each object. The same is true about the extremal monos.
(3) Thus, trivially, the category X is both an (extremal epi, mono)- and an (epi, extremal mono)-category.

(4) Categorical products are the infima, and coproducts are the suprema. Thus, to say that X is complete (resp. cocomplete) is tantamount to requiring it to be a complete meet-semilattice (resp. join-semilattice). X is both complete and cocomplete precisely when it is a complete lattice.

(5) The least element of X (if there is one) is the initial element; the largest, the terminal element.

(6) Note that each element of X is simple.

(7) Each element of X is quasi-initial; the terminal element (if X has one) is the largest member of X.

The reader will easily notice that X exhibits the following features:

- For each $x \in X$, $\odot x = x$.
- To say that $r \in X$ is a coseparator, implies that r is the largest element of X. One checks without difficulty, that $r = 1$, the largest element of X, is indeed the coseparator of X.

Thus, if a poset X is a complete lattice, then, as a category, X is a complete category, obviously well-powered, having 1 (the top) as the simple, quasi-initial object, and 0 (the bottom) being the simple initial object. Therefore, X is a Hölder category.

The examples of this type are the only examples we know of Hölder categories in which the terminal object and the coseparator coincide. They are also the only known Hölder categories in which every object equals its core.

It is easily verified that if $\mathbf{H}$ is a Hölder category in which the coseparator is terminal, then each object of $\mathbf{H}$ is a subobject of T.

3. Hölder categories

Throughout, from this point forward, $\mathbf{H}$ stands for a Hölder category. In any category, if it has a simple, quasi-initial coseparator, such an object will be denoted throughout by R; as we have seen, it is unique up to isomorphism. Both of these conventions are implicit, and the contrary will be explicitly noted.

Starting right away, we shall refer to the material reviewed in Section 4.2. Following the notation of that section, let $\mathcal{E}_{\mathrm{pi}}$ be the class of all pushout invariant epic monomorphisms. The reader will likely benefit from a review of the two correspondences described in that section that use the concept of injectivity with respect to a class of epimorphisms to move from classes of epimorphisms to epireflective classes and back. We will invoke Theorem 4.2.3 in the following.

3.1. Monoreflections in Hölder categories

The reader should recall the "hom-set" comments, leading up to Proposition 2.2.4. We record two new items; the first is a restatement of an earlier claim, while the second brings injectivity into the conversation.

Suppose X is any complete lattice. Then, viewed as a category, we have by the remarks in 2.4.2 that 1 (the top) is a simple quasi-initial object, and also a coseparator. It is therefore a Hölder category.

Lemma 3.1.1. *In an arbitrary category $\mathbf{C}$, suppose $e\colon A \longrightarrow B$ is a $\mathbf{C}$-morphism:*

- *If S is a coseparator, then e is epi if and only if* $\hom(e, S)$ *is one-to-one.*
- *J is e-injective if and only if* $\hom(e, J)$ *is surjective.*

The next theorem gives us several alternative descriptions of the epireflective hull of R, and reminds us (in the proof) that the associated reflection is the maximum monoreflection.

Theorem 3.1.2. *The following are equivalent for a $\mathbf{C}$-object J:*

(1) *For each $e\colon A \longrightarrow B$ in $\mathcal{E}_{\mathrm{pi}}$, and each $g\colon A \longrightarrow J$ there is a morphism $f\colon B \longrightarrow J$ such that $f \cdot e = g$.*

(2) *J lies in the epireflective hull of R.*

(3) *J is an extremal subobject of a product of copies of R.*

P r o o f. Using the machinery of Theorem 4.2.3, (2) expresses that $J \in R^{\perp\perp}$. So does (3). As for (1), it asserts that $J \in \mathcal{E}_{\mathrm{pi}}^{\perp}$. Furthermore, Theorem 4.2.3 ensures that, since $\mathcal{E}_{\mathrm{pi}}$ is clearly the largest pushout invariant class consisting of monomorphisms, $\mathcal{E}_{\mathrm{pi}}^{\perp}$ is the smallest monoreflective class. Therefore, the following two claims will, when proved, show that the class in (1) is also that of (3).

(a) R is injective for all the maps in $\mathcal{E}_{\mathrm{pi}}$.

(b) The epireflective hull is monoreflective.

For (a), consider a pushout invariant monic epimorphism $e\colon A \longrightarrow B$ and a map $g\colon A \longrightarrow R$, and form the pushout of e and g:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ R \arrow[r, "f"'] & P \end{tikzcd} \tag{3.1.2.1}$$

Note that $f\colon R \longrightarrow P$ is again monic and epic, which entails that P is a member of $\mathcal{Q}$, which cannot be, since R is the largest one.

And (b) follows immediately from the presence of the coseparator. This concludes the proof. □

To give this epireflective hull its due, we shall describe the reflection. Begin as in 2.2.3, using the same notation.

(a) Recall that R_A is the product of copies of R, indexed by $\hom(A,R)$. For each morphism $e\colon A \longrightarrow B$ we define $\widehat{e}\colon R_A \longrightarrow R_B$, by setting, for each $g \in \hom(B,R)$

$$\pi_g \cdot \widehat{e} = \pi_{g\cdot e}.$$

(b) With the respective diagonal maps vertically, we once again have a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "e"'] \arrow[d] & B \arrow[d] \\ R_A \arrow[r, "\widehat{e}"'] & R_B \end{tikzcd}$$

The novelty here is that, in order to get the maximum monoreflection, we must use, not the (extremal epi, mono)-factorization of the diagonal map, as in Proposition 2.2.4, but the (epi, extremal mono)-factorization.

Proposition 3.1.3. *For each* $\mathbf{C}$*-object* A*, factorize* $\Delta_A\colon A \longrightarrow R_A$*, as* $\Delta_A = x_A \cdot \mu_A$*, now into an epi first factor* $\mu_A\colon A \longrightarrow \mu A$*, followed by a extremal monomorphism* $x_A\colon \mu A \longrightarrow R_A$*. Then there is a unique morphism* $m\colon \mu A \longrightarrow \mu B$ *so that the ensuing diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\mu_A"'] \arrow[d, "e"'] & \mu A \arrow[r, "x_A"'] \arrow[d, "m"'] & R_A \arrow[d, "\widehat{e}"'] \\ B \arrow[r, "\mu_B"'] & \mu B \arrow[r, "x_B"'] & R_B \end{tikzcd} \tag{3.1.3.1}$$

P r o o f. We apply the diagonalization property (4.1.1(5)) to the commuting square

$$\begin{tikzcd} A \arrow[r, "\mu_A"] \arrow[d, "\mu_B\cdot e"'] & \mu A \arrow[d, "\widehat{e}\cdot x_A"] \\ \mu B \arrow[r, "x_B"'] & R_B \end{tikzcd} \tag{3.1.3.2}$$

using the hypotheses that μ_A is epic and x_B is extremally monic. Then carefully redraw (3.1.3.2) to recover the diagram (3.1.3.1) Now, defining $m \equiv \mu(e)$, μ is a functor which conforms to all the requirements. □

A closer look — in the next proposition — and the reader will see that in proving that the third condition is equivalent to the other two, much of what goes into the proof of [8: Theorem 3.6(1)] can be used, *mutatis mutandis.*

Proposition 3.1.4. *Suppose that* $e\colon A \longrightarrow B$ *is an epimorphism. Then, the following are equivalent:*

(1) R *is* e*-injective.*

(2) *The map* $\hom(e,R)\colon \hom(B,R) \longrightarrow \hom(A,R)$ *defined by* $\hom(e,R)(h) = h\cdot e$ *is a one-to-one correspondence.*

(3) $\mu(e)$ *is an isomorphism.*

In the next section we consider epicomplete objects in Hölder categories. We find the role of the special pushouts intriguing. On the other hand, one ought to be able to improve on the present contribution.

3.2. Epicompleteness

Recall that an object E, in any category, is *epicomplete* if each epi mono $m\colon E \longrightarrow Y$ is an isomorphism. We begin with the following observation.

Proposition 3.2.1. *Any simple, quasi-initial coseparator is epicomplete.*

P r o o f. Suppose $m\colon R \longrightarrow J$ is a monic epimorphism. Since R is quasi-initial, J must be one too. But R is maximal in the quasi-order of $\mathcal{Q}^*$, and so m is an isomorphism. □

Thanks to Proposition 4.2.5, we have the following theorem. In preparation for that, we recall from this proposition, the following conventions: let $\mathcal{E}_M$ stand for the class of all epic monomorphisms. If the reader will refer to the string of inclusions in 4.2.5,

$$\mathcal{E}_M^{\perp} \subseteq \mathcal{E}_M^{\Gamma} \subseteq \mathcal{E}_{\mathrm{pi}}^{\perp} = \mathcal{E}_{\mathrm{pi}}^{\Gamma},$$

the second inclusion expresses the fact that each epicomplete object A is injective with respect to each pushout invariant epic monomorphism. According to 4.2.4(b), the injectives with respect to all the pushout invariant epic monomorphisms are precisely the members of the least monoreflective class, which — per Theorem 3.1.2 — is $\mathcal{KP}R$.

Thus, we have *almost* proved:

Theorem 3.2.2. *In any Hölder category* $\mathbf{H}$*, the following are equivalent:*

(1) $\mathcal{KP}R \subseteq \mathcal{E}_M^{\perp}$.

(2) *Any object which is injective, for each pushout invariant epic monomorphism, is, in fact, injective, for each epic monomorphism.*

(3) $\mathcal{E}_M^{\perp} = \mathcal{E}_M^{\Gamma}$.

(4) *The object J is epicomplete precisely when it is injective for each epic monomorphism.*

(5) *Each object of the epireflective hull of R is epicomplete.*

Proof. The first thing to notice is that (2) merely translates into words what (1) says in symbols, and (4) fulfills the same task for (3).

(1) forces equality throughout the array of inclusions from Proposition 4.2.5; in particular, one has (3). Next, suppose (3) is true; this identity says that the class $\mathcal{E}_M^{\Gamma}$ of epicomplete objects of $\mathbf{H}$ is an epireflective class, which contains R, by Proposition 3.2.1. $\mathcal{E}_M^{\Gamma}$ is contained in the epireflective hull of R, which it then equals, proving (5). An argument very much like this one will demonstrate that (5) — that is to say, the hypothesis that $\mathcal{KP}R \subseteq \mathcal{E}_M^{\Gamma}$ — implies (1). We leave the details to the reader. □

The reader might take notice of the fact that in a Hölder category that satisfies the conditions of the preceding theorem, the epicomplete objects are the members of the least monoreflective subclass. In general, each epicomplete object must be in every monoreflective class, but, in general, if a category $\mathbf{C}$ has a least monoreflective class $\mathbf{M}$, its members need not be epicomplete.

By considering certain pushouts one can add at least one more condition to the ones in Theorem 3.2.2; coupled with Example 3.2.4, we have an instance of the situation described in the preceding paragraph.

Definition & Remarks 3.2.3. In the square below, $e\colon A \longrightarrow B$ is an epi monomorphism and $g\colon A \longrightarrow R$. The discussion that follows takes place in $\mathbf{H}$; R is the simple, quasi-initial non-terminal coseparator.

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ R & \xrightarrow[f]{} & P \end{array} \tag{3.2.3.1}$$

Since e is epic, it follows that $f\colon R \longrightarrow P$ is also epic, and using the (extremal epi, mono)-factorization of f, it is not hard to see that either $P = T$ (i.e., P is terminal), or else that f is mono. We may therefore conclude that P is terminal or else in $\mathcal{Q}^*$, and in the latter case Lemma 2.1.10 once more implies that f is an isomorphism.

As to the label we have applied to this issue, the pushout itself may be interpreted in terms of lifting suitable "kernels" of morphisms of A to R, to kernels of B. The upshot of these remarks is that the lifting pushout either lifts the given morphism g from A to B, or else P is terminal.

If the outcome of the lifting pushout above is non-terminal in $\mathbf{H}$, we will say that it is a *true* lifting pushout.

We give an example of the latter (terminal) outcome.

Example 3.2.4. Let us now consider lifting pushouts in $\mathbf{fAlg}_{\mathbb{R}}$. In particular, look at the convex ℓ-subring $C^*(\mathbb{N})$ of all bounded real valued sequences in $C(\mathbb{N})$, the ring of all real valued sequences. This inclusion is also epi, because the latter is a ring of fractions of the former.

However, the reader should observe the following:

> for $\mathfrak{m}$ is any maximal ideal of $C^*(\mathbb{N})$ corresponding to a free ultrafilter on the natural numbers, there is no maximal ideal $\mathfrak{p}$ of $C(\mathbb{N})$ for which $\mathfrak{m} = \mathfrak{p} \cap C^*(\mathbb{N})$, and $C(\mathbb{N})/\mathfrak{p}$ is archimedean.

This means that the homomorphism $C^*(\mathbb{N}) \longrightarrow \mathbb{R}$ with kernel $\mathfrak{m}$, does not lift to a character of $C(\mathbb{N})$. This is a situation in which the lifting pushout is terminal.

We have the following result concerning epicompleteness and lifting pushouts.

Proposition 3.2.5. *In the Hölder category* $\mathbf{H}$, *every lifting pushout is true if and only if every member of the epireflective hull of* R *is epicomplete.*

Proof. Once the reader realizes that the assumption that every lifting pushout is true is equivalent to R being $\mathcal{E}_M$-injective, then as in previous arguments, $R \in \mathcal{E}_M^{\perp}$ precisely when $\mathcal{KPR} \subseteq \mathcal{E}_M^{\perp}$. By Theorem 3.2.2, we are done. $\square$

3.3. Examples

We have already given the reader some worthy examples of Hölder categories (in Examples 2.4.1 and 2.4.2). However, those were not the examples that motivated and drove this industry. We turn now to highlight them — these categories of ordered algebraic structures, starting with the example which *was* the driving force for this investigation. For the record once more, $\mathbf{W}^*$ is the category of all archimedean lattice-ordered groups with a designated strong unit, and all the ℓ-homomorphisms which preserve the designated units.

Example 3.3.1. It is easily seen that $\mathbf{W}^*$ has the following features:

(a) It satisfies Axiom 1.

(b) The quasi-initial objects are the groups that are isomorphic to subgroups of the reals $\mathbb{R}$. In fact, the quasi-initial objects and the simple ones coincide here.

(c) $\mathbb{R}$ itself is a simple coseparator.

(d) Obviously, $\mathbf{W}^*$ is well-powered, and, finally, it is a complete category, although the product is not the direct product; instead, it is most effectively described as follows: Let G_i $(i \in I)$ be a set of $\mathbf{W}^*$-objects, with designated unit u_i. Form the direct product G of the G_i; that is to say, the usual group-theoretic product, with coordinate-wise ordering. Denote by $u \in G$ the element whose ith coordinate is u_i.

Finally, let $\prod\limits_{i \in I} G_i$ denote the convex ℓ-subgroup of G generated by u. It is easy to verify that this construct is the categorical product in $\mathbf{W}^*$.

Thus, $\mathbf{W}^*$ is a Hölder category, the prototypical one. A detailed study of $\mathbf{W}^*$ is being prepared ([9]). There it is easily shown that the epicomplete objects are precisely the groups of the form $C(X)$, with X ranging over all compact Hausdorff spaces.

Now an example — namely, the amply documented category $\mathbf{W}$ of all archimedean ℓ-groups with a designated weak unit, and the ℓ-homomorphisms that preserve the designated unit — which satisfies all the conditions needed to make it a Hölder category, but one.

Example 3.3.2. There is no simple coseparator in $\mathbf{W}$. (But we do not know if there is a coseparator which is not simple.)

To see the claim, observe that any simple object in $\mathbf{W}$ is an archimedean totally ordered group, and, therefore, according to Hölder's Theorem, a subgroup of $\mathbb{R}$. But it is clear that not every group in $\mathbf{W}$ is a subobject of a product of subgroups of $\mathbb{R}$.

In between the two preceding examples, there is $\mathbf{W}_{\mathbb{R}}$, the class of all the $\mathbb{R}$-semisimple members of $\mathbf{W}$. We shall also refer to these groups as the **real-semisimple** ℓ-groups. We review some of the aspects relevant to the discussion of this class first. Then we indicate how to use those resources to reach our goal: namely, to show that $\mathbf{W}_{\mathbb{R}}$ does admit nonessential monoreflections.

Remarks 3.3.3. There are several convenient ways to discuss the most important features of $\mathbf{W}_{\mathbb{R}}$, regarded as a full subcategory of $\mathbf{W}$. Next, the reader should check (a) through (d) of Example 3.3.1, and thus show that $\mathbf{W}_{\mathbb{R}}$ is a Hölder category.

Now we may regard the objects in this category in one of the following equivalent ways. The reader should keep in mind that all groups under discussion here are abelian.

(1) A typical object in $\mathbf{W}_{\mathbb{R}}$ is an ℓ-group A, which contains a set of maximal convex ℓ-subgroups $\{M_i : \ i \in I\}$ such that $\cap_i M_i = 0$.

(2) For our purposes it is more productive to view these objects with reference to their Yosida space. For a thorough discussion of this subject, the reader is referred to [10]. We limit ourselves here to sketching an account of the highlights.

With each $\mathbf{W}$-object G (with designated unit u) one associates the set YG of all convex ℓ-subgroups which are the *values* of u; that is, YG consists of all the convex ℓ-subgroups which are maximal with respect to not containing u.

By endowing YG with the hull-kernel topology, it may be regarded as a compact Hausdorff space. The central theorem of [10] shows that any G in $\mathbf{W}$ may be embedded in a lattice of functions, denoted $D(YG)$, and defined as follows. First, we let $\mathbb{R} \cup \{\pm\infty\}$ denote the extended real line with the usual topology. Now,

$f \in D(YG) \iff$

- f is a function on YG with values in the extended reals $\mathbb{R} \cup \{\pm\infty\}$.
- f is continuous, and $f^{-1}(\mathbb{R})$ is dense in YG.

$D(Y)$ is a lattice relative to pointwise inequality.

(3) If $x \in YG$ such that $f(x) \in \mathbb{R}$, for each $f \in G$, we say that x is a *real point*. Then it follows that G is an $\mathbb{R}$-semisimple ℓ-group if and only if YG contains a dense subset of real points. We use $\mathfrak{R}G$ to designate the set of real points of YG. We restrict the Yosida representation of G to the real points, which results in G being embedded in $C(\mathfrak{R}G)$.

If $\psi \colon G \longrightarrow H$ is a morphism of $\mathbf{W}_{\mathbb{R}}$-objects, then there is a dual continuous function $Y(\psi) \colon YH \longrightarrow YG$ and we have that, for each point $x \in YH$,

$$g(\psi x) = h(g)(x), \qquad \text{for each} \quad g \in G.$$

This identity makes it clear that, by restriction, ψ maps $\mathfrak{R}H$ to $\mathfrak{R}G$. Further, we have the reflection $G \mapsto C(\mathfrak{R}G)$; let us denote the reflection map by $\mathfrak{r}_G$.

Example 3.3.4. We construct a monoreflection on $\mathbf{W}_{\mathbb{R}}$ that is not essential. It is probable that the reader will find a method below the surface, suggested by our example, but this simply does not seem to be the place to develop any generalizations.

(a) To begin, recall that a topological space in which any countable intersection of open sets is open, is called a *P-space*. There are many important and interesting properties of this condition, starting with the development in [4: Chapter 14]. What needs to be highlighted here is the following. Suppose that X is any Hausdorff space.

- Let X^P be the space defined on X for which the countable intersections of open sets form a base. Then it is clear that X^P is a P-space, and since the new topology on X is finer than the given one, the identity map $X^P \longrightarrow X$ is continuous.
- Moreover, if $g\colon Y \longrightarrow X$ is a continuous map from a P-space Y, then, since the inverse image of a countable intersection of X-open sets is open, it follows that g is continuous when viewed as a map $Y \longrightarrow X^P$.

Thus, the map $X \mapsto X^P$ is a coreflection. Note that the inclusion map $C(X) \longrightarrow C(X^P)$ is a $\mathbf{W}_{\mathbb{R}}$-morphism (where the designated units are the constants 1).

(b) It then follows without any further trouble that the composite

$$G \mapsto C(\mathfrak{R}G) \mapsto C((\mathfrak{R}G)^P),$$

is a monoreflection, which is not essential. For suppose we take $G = C[0,1]$; since every point is a countable intersection of open intervals, we see that $[0,1]^P$ is discrete. Thus, the reflection enlarges G to the $\mathbb{R}$-algebra of all real valued functions, which is not an essential extension.

3.4. When $R = I$

There is a collection of examples, which at first blush would appear to be the most straightforward examples of Hölder categories: the categories for which $R = I$. Equivalently, these are the categories in which the initial object I is simple and also a coseparator.

Let us now collect the above remarks, and some additional comments about such categories, in a single proposition.

Proposition 3.4.1. *Suppose that* $\mathbf{C}$ *is a complete category which is well powered. Then the following are equivalent.*

(a) $\mathbf{C}$ *is a Hölder category in which* $R = I$.

(b) *The initial object* I *is simple and a coseparator.*

(c) *For each non-terminal object* A, I *is an extremal subobject of* A, *and* I *is a coseparator.*

Furthermore, if the above are true, then I is a section of every non-terminal object. Finally, for any two objects A and B, with B non-terminal, the coprojection $A \longrightarrow A \coprod B$ is a section.

P r o o f. The equivalence of (a), (b) and (c) is an easy exercise with all that has gone before, with the cautionary note that in order to show that (c) is enough to make I simple, and thereby prove (b), the reader may require condition (2) in Theorem 3.4.4. In any case, these details will be left to the reader. For the rest, simply note that, because I is a coseparator, every non-terminal object A has sufficiently many morphisms $A \longrightarrow I$. Pick one; the composite of it with the unique morphism $I \longrightarrow A$ must be the identity on I, proving that I is, indeed, a section of A.

Finally, observe that $A \coprod I = A$, always, with I initial, and that the application $A \coprod (\cdot)$ to a section, is again a section. □

Remark 3.4.2. The reader should recall Example 2.4.1, which identifies **Bool**, the category of boolean algebras and boolean homomorphisms, as a Hölder category. It is obvious from the discussion there that this category fits the theme of this section, because $R = I = \mathbf{2}$.

Remark 3.4.3. From a category $\mathbf{A}$, for which we we assume only that it is complete and well-powered, and also that it has an initial object; we will "take away" what we need to get a Hölder category for which $R = I$.

Consider the subclass $\mathcal{SP}I$; that is, the class of all $\mathbf{A}$-objects which admit an embedding into a product of copies of I. Ahead, we associate this subclass with the category in which it lies by writing $\mathcal{SP}(I, \mathbf{A})$.

Recall now the discussion leading up to Proposition 2.2.4, and later to Theorem 3.1.2 and descriptions of the maximum monoreflection. Let us sketch it in $\mathcal{SP}(I, \mathbf{A})$. The product of copies of I, indexed by $\hom(A, I)$. For each morphism $e\colon A \longrightarrow B$, again define $\widehat{e}\colon I_A \longrightarrow I_B$, by

$$\pi_g \cdot \widehat{e} = \pi_{g \cdot e}, \qquad \text{for each} \quad g \in \hom(B, I).$$

This time the relevant commutative diagram is

$$\begin{tikzcd} A \arrow[r, "e"'] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ I_A \arrow[r, "\widehat{e}"'] & I_B \end{tikzcd}$$

Our goal is clear:

Theorem 3.4.4. *Suppose that* $\mathbf{A}$ *is a category which is complete, well-powered, and has an initial object. Then* $\mathcal{SP}(I, \mathbf{A})$ *is extremally epireflective, the reflection map being* $\nu_A \colon A \longrightarrow \nu A$*, the first factor of the (extremal epi, mono)-factorization of the diagonal map* $\Delta \colon A \longrightarrow I_A$*.*

Furthermore,

(1) *The* $\mathbf{A}$*-object* A *is in the class* $\mathcal{SP}(I, \mathbf{A})$ *precisely when* ν_A *is monic, and*

(2) I *is a simple initial object in* $\mathcal{SP}(I, \mathbf{A})$*.*

Thus, $\mathcal{SP}(I, \mathbf{A})$ *is a Hölder category.*

Proof. We merely sketch the arguments. First, it is once again the diagonalization property, this time applied to extremal epis and monomorphisms, that establishes the needed functorial properties.

The only nontrivial claim being made here is that of the simplicity of I in $\mathcal{SP}(I, \mathbf{A})$. As to that, suppose that $h \colon I \longrightarrow B$, with B a non-terminal subobject of a product of copies of I. Then, as argued before, h is a section, and therefore monic. $\square$

Among the Hölder categories satisfying $R = I$, there are examples having a richer structure than **Bool**; for instance, the category $\mathbf{VL}_{\mathbb{R}}$ and the subcategory $\mathbf{VL}^*$; the category $\mathbf{fAlg}_{\mathbb{R}}$ and the subcategory $\mathbf{fAlg}^*$. For a description of these categories, the reader is referred to §1.4.

Recently, through conversation with the authors of [2] they pointed out that $\mathbf{fAlg}^*$ — whose objects they refer to as bounded f-algebras — has a very interesting property. Note that to say that a reflection v is *non-trivial* means that for some non-terminal object A, $vA \neq T$. There is an ostensibly stronger notion, namely, that $vA \neq T$, whenever A is non-terminal; such a reflection will be called *uniformly non-trivial.* The theorem we want may be formulated as follows: *Every reflection* v *on* $\mathbf{fAlg}^*$ *for which the accompanying reflective class contains* $\mathbb{R}$*, is uniformly nontrivial, and in fact a monoreflection.*

Let us carry this over to the level of Hölder categories. The following lemma carries the technical information that we will need. We use the term *proper* epimorphism for an epi which is not an isomorphism.

Lemma 3.4.5. *Suppose that* $\mathbf{H}$ *is a Hölder category with coseparator* R*. For each non-terminal object* A *and each proper extremal epimorphism* $e \colon A \longrightarrow B$*, there is a morphism* $g \colon A \longrightarrow R$ *so that the pushout of* g *and* e *is terminal.*

Proof. To the contrary, suppose there is an $A \in \mathcal{KP}R$ so that, for some proper extremal epi $e \colon A \longrightarrow B$ we have for each morphism $g \colon A \longrightarrow R$ a non-terminal outcome to forming the pushout for g and e. We consider two cases:

(a) *A lies in $\mathcal{KP}R$, the epireflective hull of R, and we only assume that e is epic.* Now the pushout condition says that R is e-injective, which entails that each object of $\mathcal{KP}R$ is e-injective. Thus, $1_A = h \cdot e$, for a suitable $h \colon B \longrightarrow A$, which makes e a section, and therefore an isomorphism. This is a contradiction.

(b) In general, let μ denote the maximum monoreflection on $\mathbf{C}$, and once again suppose $e\colon A \longrightarrow B$ is an proper extremal epi. Then it is easy to see that $\mu(e)$ is a proper epi, and so by (a) we have a morphism $g\colon \mu A \longrightarrow R$ such that the pushout of g and $\mu(e)$ is terminal. And now we leave it for the reader to check that this also yields a pushout (of e and $g \cdot \mu_A$) which is terminal. $\square$

Last, in the preparation for Theorem 3.4.6, we draw the reader's attention to a small issue, which is easy to get around, yet has to be observed. Note that any morphism $A \longrightarrow T$ is extremally epi, which makes the functor that sends everything to the terminal object (trivially) an epireflection which is only a monoreflection if the category on which the functor is defined itself is trivial.

In any event, the following theorem is the generalization of the theorem in [2], discussed in the preamble to Lemma 3.4.5.

Theorem 3.4.6. *Suppose that* $\mathbf{H}$ *is a non-trivial Hölder category in which the initial object* I *is a simple coseparator. Then for every reflection* v *such that* $vI \neq T$,

(1) $vI = I$, *and*

(2) v *is uniformly non-trivial and, in fact, a monoreflection.*

Proof. The proof is subdivided into several assertions, which are items to be recalled, but also a few assertions which are quite straightforward.

(1) There are no morphisms out of the terminal object, except for the identity $T \longrightarrow T$. This has been noted before.

(2) As is shown in [2: Lemma 3.6], one shows, almost verbatim, that there is exactly one morphism $vR \longrightarrow vR$.

(3) Suppose v is a reflection on $\mathbf{C}$ with $vR \neq T$, and use the factorization of $v_R = u \cdot n$ into an epic first factor n and an extremal mono second factor u, to conclude, first, that n is an isomorphism, because R is the largest simple, quasi-initial object in this category.

(4) Also, since R is a coseparator, there is at least one map $f\colon vR \longrightarrow R$. By the preceding observation, if $g\colon vR \longrightarrow R$, then by item (2) in this catalog, we have that $v_R \cdot f = v_R \cdot g$. As v_R is monic, and we conclude in fact that $|\,\hom(vR, R)\,| = 1$.

(5) Thus, the single morphism $vR \longrightarrow R$ is the inverse of v_R, and we have proved (1).

(6) Next, if A is non-terminal let $e\colon A \longrightarrow B$ and $m\colon B \longrightarrow vA$ be the factors of the (extremal epi, mono)-factorization of the reflection map $v_A\colon A \longrightarrow vA$. Then $vA = vB$, with m as reflection map.

(7) With the notation of the previous item, here and in subsequent items, then
$$|\,\hom(A, R)| = |\,\hom(B, R)| = |\,\hom(vA, R)|.$$
This uses the fact that $vR = R$.

(8) Thus, v is uniformly non-trivial.
(9) Now consider the first factor of v_R above. According to Lemma 3.4.5, there is a morphism $f\colon A \longrightarrow R$ such that the pushout of f and e is T. On the other, R is an e-injective, which when squared with the pushout, implies that there is a map from T to R.

This contradiction implies that e is an isomorphism, whence v is a monoreflection. $\square$

Remark 3.4.7. The proof of the theorem of [2], now generalized by Theorem 3.4.6, is a clever patchwork of elements of Category Theory with ring-theoretic resources. Yet, the spirit of their proof is the Stone-Weierstrass Theorem.

Notice, on the other hand, that one need not require that the objects be groups or f-algebras with a strong designated unit. In sum, Theorem 3.4.6, applies to all of the following Hölder categories:

- $\mathbf{fAlg}_{\mathbb{R}}$, the category of all real-valued f-algebras. It is assumed, in all the examples involving rings that the morphisms preserve the identity.
- The categories $\mathbf{VL}_{\mathbb{R}}$ and the full subcategory $\mathbf{VL}^*$.

To conclude this section, we have a converse of sorts for Theorem 3.4.6. Owing to Proposition 2.3.5 and Corollary 2.3.8, if U and V are distinct, simple quasi-initial objects, then $\mathcal{KP}U \cap \mathcal{KP}V = \{T\}$. We therefore have the following.

Proposition 3.4.8. *Suppose that in the Hölder category* $\mathbf{H}$, $I < R$. *Then*

(1) *$\mathcal{KP}R$ is monoreflective, while $\mathcal{KP}I$ is not, and*
(2) *$\mathcal{SP}U \subset \mathcal{SP}V$, with $\mathcal{SP}U \cap \mathcal{KP}V = \{T\}$.*

In particular, none of the epireflective hulls is monoreflective, except for $\mathcal{KP}R$.

3.5. Work points

Let us start this list of questions with an example of the situation described in Theorem 3.4.4. In the category **CRn1** of commutative rings with identity and all ring homomorphisms which preserve the identity, the initial object is the ring of integers. For the problem we have in mind, we use the notation of Theorem 3.4.4.

Question 3.5.1. *$\mathcal{SP}(\mathbb{Z}, \mathbf{CRn1})$ is a Hölder category, in which the initial object is simple. What can be said about this category?*

We supply a short list of properties, all more-or-less easy to verify;

(1) The rings in $\mathcal{SP}(\mathbb{Z}, \mathbf{CRn1})$ are all semiprime.
(2) The characteristic of all the rings in this category is zero.

Now for some of the dicey questions:

Question 3.5.2. *Let* $\mathbf{H}$ *be a Hölder category. Which additional conditions must be assumed to characterize when all monoreflections in* $\mathbf{H}$ *are essential?*

As pointed out, specifically, in the pairing of $\mathbf{W}^*$ with $\mathbf{W}_{\mathbb{R}}$, the first has the property we wish to characterize here, but $\mathbf{W}_{\mathbb{R}}$ does not; see Example 3.3.4(b). The obvious distinction in $\mathbf{W}^*$, which is absent in $\mathbf{W}_{\mathbb{R}}$, is the boundedness of the functions in each object of the category. For good measure, this distinction is repeated in the pairing of $\mathbf{VL}^*$ with $\mathbf{VL}_{\mathbb{R}}$, and again with $\mathbf{fAlg}^*$ and $\mathbf{fAlg}_{\mathbb{R}}$.

The apparent insight is tested by **Bool**. Since "monic" means the same as "extremally mono", and again the same as "one-to-one", while "epi" and "extremally epi" are the same as "surjective", the only non-trivial monoreflection is the identity. To see this, note that the coseparator in question is $\mathbf{2}$; the techniques of Theorem 3.1.2, which first embeds a boolean algebra B in a product of copies of $\mathbf{2}$, and subsequently the image-factorization comes around to declare that B is its own reflection, and that the identity functor is the only non-trivial epireflection. In particular, all monoreflections are (trivially) essential. And on top of that, the epireflective hull of R is $\mathcal{KP}R = \mathbf{H}$.

We will need the concept of a singular element of a lattice-ordered group G in the discussion below. The element $0 \leq s \in G$ is *singular* if $a \wedge (s - a) = 0$, for each $0 \leq a \leq s$.

Now — seeking a parallel for the pair $\mathbf{W}^*$ and $\mathbf{W}_{\mathbb{R}}$ — consider the category $\mathbf{S}_{\mathbb{R}}$, whose objects are all $\mathbb{Z}$-semisimple object of $\mathbf{W}_{\mathbb{R}}$ with a singular designated unit u. The reader will find an extensive study of these groups in [6], to which the reader is referred. From [6] we extract the following information:

- First, observe that for each object A in $\mathbf{S}_{\mathbb{R}}$, there is a representation of A in which its elements are represented by integer-valued functions.
- Let $\mathbf{S}^*$ stand for the subcategory of all the objects in which the above-mentioned representation is by bounded functions. In the discussion of [6: p. 131] it is shown that $\mathbf{Bool} \cong \mathbf{S}^*$.
- The reader will also easily verify that the argument of 3.3.4, showing that there are non-essential reflections in $\mathbf{W}_{\mathbb{R}}$, fits the current session. That is to say, **Bool**, disguised as the category $\mathbf{S}^*$, reappears paired with its parent category, $\mathbf{S}_{\mathbb{R}}$, yet separated by the same distinction: *the known Hölder categories which only admit essential monoreflections are also the ones whose objects are representable by bounded real valued functions.*

As we have already noted, **Bool** has just one non-trivial monoreflection, namely, the identity functor. And so now we ask:

Question 3.5.3. *Suppose that* $\mathbf{H}$ *is a Hölder category. Classify the ones in which* $\mathcal{KP}R$*, the least monoreflective class, and* $\mathbf{H}$ *coincide.*

Here's what we know about this problem, apart from the fact that **Bool** has this feature: either one of the following two equivalent conditions implies that $\mathbf{H} = \mathcal{KP}R$.

- Every monomorphism is an extremal monomorphism.
- Every epimorphism is an extremal epimorphism.

We do not know whether these morphism conditions are necessary for $\mathbf{H} = \mathcal{KPR}$ to hold.

If each monomorphism is extremally so, then since R coseparates, the desired property is now obvious.

4. Appendix

Here is that review, first, of epireflections and, second, the essential material from [8]. The centerpiece of the first part is the Freyd-Isbell-Kennison Theorem. The main theorem in [8] (4.2.3 describes the correspondence between pushout invariant classes of epimorphisms and epireflective classes.

This part of the paper is intended as a sufficient review for those who have a little more than a working relationship with category theory. It is adequately annotated and referenced for the reader who feels either like a stranger in this theory, or else like a beginner. The expert will doubtless have little use for the first part of this review, and probably can safely move on to the second.

4.1. Epireflections, on the one hand

In a few brief catalogs, we outline the basic categorical principles used in this article, beginning with factorization in categories.

Remarks 4.1.1. Let $\mathbf{C}$ be a category. Throughout this commentary $\mathcal{E}$ stands for a class of epimorphisms and $\mathcal{M}$ for a class of monos. It is assumed that such classes always are closed under composition and also that they contain all isomorphisms.

(1) The morphism f in $\mathbf{C}$ has *unique* $(\mathcal{E}, \mathcal{M})$*-factorization*: f can be factored as $f = m \cdot e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, such that if $f = m_1 \cdot e_1$ with $e_1 \in \mathcal{E}$ and $m_1 \in \mathcal{M}$, then there is an isomorphism k making the diagram below commute.

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e_1"'] & \cdot \arrow[d, "m"] \arrow[dl, "k"] \\ \cdot \arrow[r, "m_1"'] & \cdot \end{tikzcd} \tag{4.1.1.1}$$

(2) $\mathbf{C}$ is an $(\mathcal{E}, \mathcal{M})$*-category*: every morphism of $\mathbf{C}$ has unique $(\mathcal{E}, \mathcal{M})$-factorization.

(3) If $\mathbf{C}$ is an $(\mathcal{E}, \mathcal{M})$-category, then the classes $\mathcal{M}$ and $\mathcal{E}$ determine one another, in the following sense: ([12: Theorem 33.6]) *Suppose that* $\mathbf{C}$ *is an* $(\mathcal{E}, \mathcal{M})$*-category. Then* $f \in \mathcal{M}$ *if and only if* $f = m \cdot e$*, with* $e \in \mathcal{E}$*, implies that* e *is an isomorphism. Likewise,* $f \in \mathcal{E}$ *if and only if* $f = m \cdot e$*, with* $m \in \mathcal{M}$*, implies that* m *is an isomorphism.*

(4) Per the above theorem, the members of $\mathcal{M}$ are the monomorphisms which are *$\mathcal{E}$-extremal monos.* If $\mathcal{E}$ consists of all the epimorphisms, then $\mathcal{M}$ is the class of all *extremal monomorphisms.* Interchanging the roles of $\mathcal{E}$ and $\mathcal{M}$ gives us a definition of an *extremal epimorphism.*

(5) The pair $(\mathcal{E}, \mathcal{M})$ has the *diagonalization property:* suppose that the outer square below commutes, and that $e \in \mathcal{E}$, and $m' \in \mathcal{M}$.

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "e'"'] & \cdot \arrow[d, "m"] \arrow[dl, "k"] \\ \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd} \tag{4.1.1.2}$$

Then there is a morphism k, as indicated in the diagram, making the two triangles commute. Each *$(\mathcal{E}, \mathcal{M})$-category* has this property ([12: 33.2]).

Finally, we have the notion of *pushout invariance (of a class $\mathcal{E}$ of morphisms):* For each pushout diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ C \arrow[r, "n"'] & P \end{tikzcd} \tag{$\dagger$}$$

in which $e \in \mathcal{E}$, then it follows that $n \in \mathcal{E}$.

It is well known that the class of all epis $\mathcal{E}_{\mathrm{all}}$ is pushout invariant. $\mathcal{E}_{\mathrm{pi}}$ stands for the class of all pushout invariant epic monomorphisms, while $\mathcal{E}_M$ denotes the class of all epic monomorphisms.

Here is a formulation of the Freyd-Isbell-Kennison Theorem. It is followed by a series of notes which should provide sufficient illustration.

Theorem 4.1.2. *Suppose that* $\mathbf{C}$ *is an $(\mathcal{E}, \mathcal{M})$-category, and also that* $\mathbf{C}$ *has products and is $\mathcal{E}$-co-well-powered. The subcategory* $\mathbf{R}$ *is $\mathcal{E}$-reflective if and only if* $\mathbf{R}$ *is*

(1) *closed under formation of products, and*

(2) *whenever $m \colon B \longrightarrow A$ is in $\mathcal{M}$, and A is an* $\mathbf{R}$*-object, then B too is an* $\mathbf{R}$*-object.*

As promised, here are some comments concerning epireflections. Only epireflections in full subcategories are considered here, and, consequently, there is no difference here in the labels "epireflective subclass" and "epireflective subcategory".

Definition & Remarks 4.1.3. In the background we have a full subcategory $\mathbf{R}$ of $\mathbf{C}$. Ahead, $\mathcal{E}$ denotes a class of epis and $\mathcal{M}$ a class of monics, both having the properties mentioned in the preamble to 4.1.1.

(1) A *reflection of* $\mathbf{C}$ *in* $\mathbf{R}$ is a left adjoint to the inclusion functor; see [12: §36]. That is, for each $\mathbf{C}$-object A there is a $\mathbf{C}$-morphism $r_A\colon A \longrightarrow RA$, with RA an object in $\mathbf{R}$, such that for each morphism $g\colon A \longrightarrow B$ into the $\mathbf{R}$-object B, there is a unique $g'\colon RA \longrightarrow B$ such that $g' \cdot r_A = g$.
(2) $\mathcal{E}$-*Reflection*: a reflection for which each $r_A \in \mathcal{E}$. We also say that $\mathbf{R}$ is $\mathcal{E}$-*reflective.*
(3) *Epireflection*: when $\mathcal{E} = \mathcal{E}_{\mathrm{all}}$, the class of all epis.
(4) *Monoreflection*: a reflection r for which each r_A is monic.
(5) Every monoreflection is necessarily an epireflection ([12: 36.3]).
(6) A category is $\mathcal{M}$-*well-powered* if for each object A there is a *set* S of members of $\mathcal{M}$ with codomain A such that if $m\colon M \longrightarrow A$ is any mono in $\mathcal{M}$, then there is $m'\colon M' \longrightarrow A$ in S and an isomorphism $k\colon M \longrightarrow M'$ such that $m' \cdot k = m$.
(7) *Well-powered*: $\mathcal{M}$-well-powered, when $\mathcal{M}$ is the class of all monomorphisms.
(8) The dual property is called $\mathcal{E}$-*co-well-powered.*
(9) Being well- and/or co-well-powered is a strong attribute, but also a reasonably natural one. Perhaps the reader will come to the same conclusion after reading this paper. We refer to results in [12] which bear out these comments:
    (a) ([12: 34.1,34.5]) If $\mathbf{C}$ is complete and well-powered, then it is both an (epi, extremal mono)-category and an (extremal epi, mono)-category.
    (b) ([12: 23.14]) If $\mathbf{C}$ is complete and well-powered *and has a coseparator*, then it is also cocomplete and co-well-powered.

4.2. And pushout invariance, on the other

Here we recall the fundamental correspondence between epireflective classes of the category $\mathbf{C}$ and the pushout invariant classes of epimorphisms. It is the main theorem of [8], to which we refer the reader. We precede that by a notion of injectivity, which seems tailor-made for these purposes.

Definition 4.2.1. If for each morphism $g\colon A \longrightarrow J$ there is a morphism $\bar{g}\colon B \longrightarrow J$ such that the diagram below commutes, then we say that J an *e-injective.*

$$\begin{tikzcd} A \arrow[rr, "e"] \arrow[dr, "g"'] & & B \arrow[dl, "\bar{g}"] \\ & J & \end{tikzcd}$$

Definition 4.2.2. Suppose $\mathfrak{X}$ is any class of objects and $\mathcal{E}$ a class of epimorphisms.

- Denote by $\mathfrak{X}^{\perp}$ the class of all epimorphisms e for which each $J \in \mathfrak{X}$ is e-injective.
- Reversing, consider all $\mathbf{C}$-objects J which are e-injective, for each $e \in \mathcal{E}$; denote it by $\mathcal{E}^{\perp}$. When $\mathfrak{X} = \{A\}$, we write $A^{\perp}$, and if $\mathcal{E}$ consists of the single morphism e, then, likewise, write $e^{\perp}$, for the associated class of injectives.

Theorem 4.2.3. *Let $\mathfrak{X}$ denote a class of $\mathbf{C}$-objects and $\mathcal{E}$ a class of epimorphisms. Assume that $\mathbf{C}$ has products and pushouts and that it is co-well-powered. Then*

(1) $\mathfrak{X} = \mathfrak{X}^{\perp\perp}$ *if and only if $\mathfrak{X}$ is epireflective.*

(2) $\mathcal{E}^{\perp\perp} = \mathcal{E}$ *precisely when $\mathcal{E}$ is pushout invariant.*

(3) $\mathcal{E}_{\mathrm{push}} = \mathcal{E}^{\perp\perp}$ *and $\mathfrak{X}^{\perp\perp}$ is the epireflective hull of $\mathfrak{X}$.*

In particular, $\mathfrak{X} \mapsto \mathfrak{X}^{\perp}$ and $\mathcal{E} \mapsto \mathcal{E}^{\perp}$ are mutually inverse, order-reversing correspondences between the p. o. class of epireflective classes and the class of pushout invariant classes.

Some remarks follow, to round out the review of resources from [8].

Remark 4.2.4.

(a) The *epireflective hull* of a class $\mathfrak{X}$: the smallest epireflective class that contains $\mathfrak{X}$. By Theorem 4.2.3, this hull is $\mathfrak{X}^{\perp\perp}$.

(b) ([8: Corollary 3.8]) Since $\mathcal{E}_{\mathrm{pi}}$ is the largest pushout invariant class of epic monomorphisms, then $\mathcal{E}_{\mathrm{pi}}^{\perp}$ is the smallest monoreflective subcategory.

(c) By the proof of [8: Theorem 3.6]:

$$\mathcal{E}^{\Gamma} = \{A \in \mathbf{C} : \ e\colon A \longrightarrow B, \ \ e \in \mathcal{E} \implies e \ \text{ is an isomorphism}\}.$$

If $\mathcal{E}$ is pushout invariant, then $\mathcal{E}^{\Gamma} = \mathcal{E}^{\perp}$, but this identity not true, in general. For example, the class $\mathcal{E}_M^{\Gamma}$ consists of the epicomplete objects of $\mathbf{C}$, a class which need not be reflective.

The preliminaries we have just reviewed lead in [8] to the following result, subject to the standing assumptions of this section.

Proposition 4.2.5. ([8: 4.2, et al.])

$$\mathcal{E}_M^{\perp} \subseteq \mathcal{E}_M^{\Gamma} \subseteq \mathcal{E}_{\mathrm{pi}}^{\perp} = \mathcal{E}_{\mathrm{pi}}^{\Gamma}. \tag{4.2.5.1}$$

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* *Department of Mathematics*
and Computer Science
Wesleyan University
Middletown, CT 06459-0128
USA
E-mail: ahager@wesleyan.edu

** *Department of Mathematics*
University of Florida
Box 118105
Gainesville, FL 32611-8105
USA
E-mail: gnocchiverdi@gmail.com