VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0227-5
Math. Slovaca **64** (2014), No. 3, 571–578

AMALGAMATION BASES FOR THE CLASS OF LATTICE-ORDERED GROUPS

V. V. Bludov* — A. M. W. Glass**

Dedicated to Ján Jakubík, a pioneer of the subject, on the occasion of his 90th birthday, with gratitude for his research

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We prove:

Theorem A. *The cardinal product of two copies of the integers is an amalgamation base for the class of all lattice-ordered groups but their lexicographic product is not.*

This answers Problem 27 of [*Black Swamp Problem Book* (W. Charles Holland, ed.), Bowling Green State University, 1982].

We also prove:

Theorem B. *The cardinal product of n copies of the integers is not an amalgamation base for the class of all lattice-ordered groups if $n \geq 3$.*

1. Introduction and background

A *lattice-ordered group* G is a group and a lattice such that

$$a(x \vee y)b = axb \vee ayb \qquad \text{and} \qquad a(x \wedge y)b = axb \wedge ayb$$

for all $a, b, x, y \in G$. We will frequently use the abbreviation *ℓ-group* for lattice-ordered group. We will write *ℓ-homomorphism* as a shorthand for a map that is both a group and a lattice homomorphism, *etc.* Lattice operations $\vee$ and $\wedge$ on an ℓ-group G induce a partial order $\leq$ on G: $g \leq h$ if and only if $g \vee h = h$ (or equivalently $g \wedge h = g$). We say that g and h are *orthogonal* if $g \wedge h = 1$.

2010 Mathematics Subject Classification: Primary 06F15; Secondary 20E06.
Keywords: amalgamation, lattice-ordered group, ℓ-permutation group.

To define a lattice ordering on a group G, it suffices to give a lattice ordering of G_+, the set of elements that exceed the identity: for then $g < h$ if and ond only if $1 < g^{-1}h$.

There are two especially important ways to form new ℓ-groups from old. If G_1 and G_2 are ℓ-groups, we can partially order their direct product $G_1 \times G_2$ by: $(g_1, g_2) \geq 1$ if and only if $g_i \geq 1$ $(i = 1, 2)$. This is called *the cardinal product of* G_1 *and* G_2 and is denoted by $G_1 \oplus G_2$; it is an ℓ-group. Note that if $f_i \in G_i$ with $f_i \geq 1$ $(i = 1, 2)$, then the images of f_1 and f_2 in the cardinal product of G_1 and G_2 are orthogonal. For any positive integer n, we write $\mathbb{Z}^n$ for the cardinal product of n copies of $\mathbb{Z}$ (the additive group of integers with the natural total order). Let K be an o-*group* (*i.e.*, an ℓ-group with the order total). If G is an ℓ-group and K acts on G as a group of ℓ-automorphisms, then the splitting extension of G by K is an ℓ-group, where $1 \leq (g, k)$ if and only if $k > 1$ or both $k = 1$ and $g \geq 1$. We write $G \overleftarrow{\rtimes} K$ for this ℓ-group and call it the *lexicographic split extension of* G *by* K. In the special case that the action of K on G is the identity, the group is $G \times K$ and we write $G \overleftarrow{\times} K$ instead of $G \overleftarrow{\rtimes} K$ and call it the *lexicographic product* of G by K. For further background on lattice-ordered groups, see [4], for example.

Let $\mathcal{L}$ be a class of algebras. The algebra $A \in \mathcal{L}$ is said to be an *amalgamation base* for $\mathcal{L}$ if whenever A is embedded in algebras $G_1, G_2 \in \mathcal{L}$, there is an algebra $L \in \mathcal{L}$ and embeddings of G_1 and G_2 in L making the diagram commute.

For $\mathcal{L}$ the class of groups, every group A is an amalgamation base. K. R. Pierce [8] proved that $\mathbb{Z}$ is an amalgamation base for the class of all ℓ-groups. In contrast, he proved that $\mathbb{Z}^2 \overleftarrow{\times} \mathbb{Z}$ is not an amalgamation base (see [8]). This led to his question, [11: Problem 27]:

(i) Is $\mathbb{Z}^2$ an amalgamation base for the class of all ℓ-groups?

(ii) Is $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ an amalgamation base for the class of all ℓ-groups?

The first part of Theorem A gives a positive answer to (i) and the second part a negative answer to (ii). The latter should be compared with [1] (see [4: pp. 265, 266]): $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ is not an amalgamation base for the class of all right-ordered groups (only one of G_1, G_2 is an ℓ-group). Also with [9: Theorem 2.2]: No non-trivial abelian o-group (including $\mathbb{Z}$ and $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$) is an amalgamation base for the class of all nilpotent ℓ-groups. This contrasts with Pierce's result for $\mathbb{Z}$ cited above; the amalgam belongs to the larger class of *all* ℓ-groups.

We will use a theorem of Holland [5]:

Theorem. *Every ℓ-group can be ℓ-embedded in the ℓ-group* $\mathrm{Aut}(\Omega, \leq)$ *for some totally ordered set* $(\Omega, \leq)$, *where* $\mathrm{Aut}(\Omega, \leq)$ *has the pointwise order.*

If $g \in \operatorname{Aut}(\Omega, \leq)$ and $\alpha \in \Omega$ with $\alpha g \neq \alpha$, then $\Delta(\alpha, g)$, the convexification of $\{\alpha g^n \mid n \in \mathbb{Z}\}$ in Ω, is called either *a supporting interval of* g or *an interval of support of* g.

2. Proof of Theorem A

Proof of the cardinal part of Theorem A. The key ideas in the proof are that $\mathbb{Z}^2$ is a one-generator ℓ-group and [2: Theorem A]: every ℓ-group can be ℓ-embedded in one with exactly 4 conjugacy classes (the identity, all elements strictly greater than the identity, all elements strictly less than the identity, and all elements incomparable to the identity (under the lattice order)). It follows easily that $\mathbb{Z}^2$ is an amalgamation base for the class of all ℓ-groups as we now show. Because $(1,-1) \vee (0,0) = (1,0)$ and $(0,1) = (1,0) - (1,-1) = (0,0) \vee -(1,-1)$, we get that $\mathbb{Z}^2$ is generated as an ℓ-group by the single generator $(1,-1)$. Let $\sigma_i \colon \mathbb{Z}^2 \to G_i$ be an ℓ-embedding of $\mathbb{Z}^2$ in G_i $(i = 1,2)$, and regard G_1 and G_2 as sublattice subgroups of $G_1 \oplus G_2$ in the natural way. By [2: Theorem A], there is an ℓ-embedding η of $G_1 \oplus G_2$ into an ℓ-group L that has exactly 4 conjugacy classes. Let $x \in L$ conjugate $(1,-1)\sigma_1\eta$ to $(1,-1)\sigma_2\eta$. Then the ℓ-embeddings $\eta\bar{x} \colon G_1 \to L$ and $\eta \colon G_2 \to L$ complete the amalgamation in L, where $\bar{x}$ is the inner ℓ-automorphism of L given by conjugation by x. This gives the affirmative answer to the first part of [11: Problem 27]. $\square$

By Holland's Theorem, for purposes of amalgamation, we may assume that $G_i = \operatorname{Aut}(\Omega_i, \leq)$ $(i = 1,2)$. Let $\mathbb{Z} \overleftarrow{\times} \mathbb{Z} = \langle a, b \rangle$, where $a = (1,0)$ and $b = (0,1)$. Suppose that either

(i) each supporting interval of the image of b in G_i contains a supporting interval of the image of a $(i = 1,2)$, or

(ii) some supporting interval of the image of b in G_i is fixed pointwise by the image of a $(i = 1,2)$.

Pierce's proof in [8] gives that $(\mathbb{Z} \overleftarrow{\times} \mathbb{Z}, G_1, G_2)$ can be amalgamated in these cases. This suggests the proof of the lexicographic part of Theorem A.

Proof of the lexicographic part of Theorem A. Let $\operatorname{Aut}(\mathbb{R} \leq)$ be the lattice-ordered group of all order-preserving permutations of the real line under the pointwise order. Let G_1 and G_2 be ℓ-isomorphic copies of $\operatorname{Aut}(\mathbb{R}, \leq)$; say, $\eta_i \colon \operatorname{Aut}(\mathbb{R}, \leq) \cong G_i$ $(i = 1,2)$. Let $b \in \operatorname{Aut}(\mathbb{R}, \leq)_+$ have two intervals of support, $(0,2)$ and $(3,5)$. So $\Delta(1,b) = (0,2)$ and $\Delta(4,b) = (3,5)$.

Let $a_0 \in \mathrm{Aut}(\mathbb{R}, \leq)_+$ have one interval of support, $(1, 1b)$. Let $a \in \mathrm{Aut}(\mathbb{R}, \leq)$ have support contained in $(0, 2)$ and be defined on $(1b^n, 1b^{n+1})$ as $b^{-n}a_0b^n$ $(n \in \mathbb{Z})$. Let $b_1 \in \mathrm{Aut}(\mathbb{R}, \leq)$ be the identity off $(0, 2)$ and agree with b on $(0, 2)$, and $b_2 := bb_1^{-1} \in \mathrm{Aut}(\mathbb{R}, \leq)$. By construction, $b_2 > 1$ is orthogonal to $b_1 > a > 1$, $b = b_1b_2 = b_1 \vee b_2$, and $[a, b] = 1 = [a, b_1] = [a, b_2]$. Moreover, $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ is ℓ-isomorphic to the subgroup of G_1 generated by $a\eta_1, b\eta_1$ via: $(1, 0) \mapsto a\eta_1$ and $(0, 1) \mapsto b\eta_1$. Note that in any ℓ-group containing $\mathrm{Aut}(\mathbb{R}, \leq)$, if $1 < x < b$, then aa^x is orthogonal to b_2; so $(aa^x)^2 \not\geq b$.

Let $c_0 \in \mathrm{Aut}(\mathbb{R}, \leq)_+$ have one interval of support, $(4, 4b)$. Let $c \in \mathrm{Aut}(\mathbb{R}, \leq)$ have support contained in $(3, 5)$ and be defined on $(4b^n, 4b^{n+1})$ as $b^{-n}c_0b^n$ $(n \in \mathbb{Z})$. Let $d := ac \in \mathrm{Aut}(\mathbb{R}, \leq)$. By construction, $c > 1$ is orthogonal to a, $[d, b] = 1$ and $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ is also ℓ-isomorphic to the subgroup of G_2 generated $d\eta_2, b\eta_2$ via: $(1, 0) \mapsto d\eta_2$ and $(0, 1) \mapsto b\eta_2$. By a standard permutation argument, there is $x \in \mathrm{Aut}(\mathbb{R}, \leq)$ with $1 < x < b$ such that $(dd^x)^2 \geq b$. Because of the existence of $b_2\eta_1 > 1$ in G_1 and $x\eta_2 \in G_2$, we cannot identify the copies of $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ in any ℓ-group L containing ℓ-embedded copies of G_1 and G_2 (whether or not other elements of G_1 and G_2 are identified): because the image of $((d\eta_2)(d\eta_2)^{(x\eta_2)})^2$ exceeds the image of $b\eta_2$, but $((a\eta_1)(a\eta_1)^y)^2$ is othogonal to the image of $b_2\eta_1$ for any $1 < y \in L$ less than the image of $b\eta_1$. This establishes Theorem A. $\square$

Since $\mathbb{Z}$ is an amalgamation base for the class of all ℓ-groups but $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$ is not, we see that the class of amalgamation bases is not closed under lexicographic products.

3. Proof of Theorem B

In [3: Example 5.2], we showed that $\mathbb{Z}^{68}$ is not an amalgamation base for the class of all ℓ-groups. The proof can easily be adapted to show that nor is $\mathbb{Z}^4$ (see Proposition 3.4). A somewhat different proof gives that $\mathbb{Z}^3$ is not an amalgamation base for the class of all ℓ-groups. This is the main step in the proof of Theorem B. From it, we can easily deduce Theorem B.

Proposition 3.1. *$\mathbb{Z}^3$ is not an amalgamation base for the class of all lattice-ordered groups.*

Proof. Let $a := (1, 0, 0)$, $b := (0, 1, 0)$, $c := (0, 0, 1)$ be the standard generators of $\mathbb{Z}^3$ with $a, b, c > (0, 0, 0)$. Let

$$G_1 := (\langle a, b\rangle \overleftarrow{\rtimes} \langle x\rangle) \oplus \langle c\rangle$$

with $x > 1$, the splitting extension being defined by

$$a^x = b, \quad b^x = a, \qquad \text{and}$$

$$G_2 := \langle a, b, c\rangle \overleftarrow{\rtimes} \langle y\rangle$$

with $y > 1$, the splitting extension being defined by

$$a^y = b, \quad b^y = c, \quad c^y = a.$$

If $(\mathbb{Z}^3, G_1, G_2)$ could be amalgamated, it would follow by Holland's Theorem cited above that G_1 and G_2 could be ℓ-embedded in $\mathrm{Aut}(\Omega, \leq)$ for some totally ordered set $(\Omega, \leq)$. For ease of notation, identify G_1 and G_2 with their images in $\mathrm{Aut}(\Omega, \leq)$. Note that G_1, G_2 are metabelian, indeed "ℓ-metabelian".

Let a_0 be any bump of a; that is, a_0 is the restriction of a to one of its supporting intervals. Since $a^x = b$ and $x \wedge c = 1$, it follows that if $b_0 := a_0^x$, $a_1 := b_0^x$, *etc.*, then Γ, the convexification of the union of the supporting intervals of $\{a_n \mid n \in \mathbb{Z}\}$ is also the convexification of the union of the supporting intervals of $\{b_n \mid n \in \mathbb{Z}\}$, and Γ is disjoint from the support of c (since c and x are orthogonal). Moreover, for every supporting interval $\Delta \subseteq \Gamma$ of a, there is a supporting interval of b contained in Γ that is less than Δ (pointwise) and also one that is greater than Δ (pointwise). *Mutatis mutandis*, with b in place of a. This holds for any bump of a or b, and implies that every interval of Ω that contains supporting intervals of a and c must also contain a supporting interval of b. But for every Γ as above, Γ^{y^2} contains supporting intervals of c and a but not of b. This contradiction establishes that $\mathbb{Z}^3$ is not an amalgamation base for the class of all ℓ-groups. $\square$

Theorem B immediately follows from Proposition 3.1 and the trivial fact:

Lemma 3.2. *If a lattice-ordered group A is not an amalgamation base for the class of all lattice-ordered groups, then nor is the cardinal product $A \oplus B$ for any lattice-ordered group B.*

Proof. Let G_1, G_2 be ℓ-groups with ℓ-embeddings of A in G_1 and G_2 witnessing that A is not an amalgamation base for the class. Then there are naturally induced ℓ-embeddings of $A \oplus B$ in the cardinal products $G_1 \oplus B$ and $G_2 \oplus B$. If $A \oplus B$ were an amlgamation base, there would be an ℓ-group L and ℓ-embeddings of $G_1 \oplus B$ and $G_2 \oplus B$ into L which would agree on the images of $A \oplus B$. They would then agree on A contradicting that (A, G_1, G_2) witnesses that A is not an amalgamation base for the class of all ℓ-groups. $\square$

Since $\mathbb{Z}$ and $\mathbb{Z}^2$ are amalgamation bases but $\mathbb{Z}^3$ is not, we see that the property of being an amalgamation base is not closed under cardinal products.

In our example in the proof of Proposition 3.1, we had G_1, G_2 "ℓ-metabelian". Hence:

Corollary 3.3. *For any $n \geq 3$, $\mathbb{Z}^n$ is not an amalgamation base for any class of lattice-ordered groups containing all ℓ-metabelian lattice-ordered groups.*

For fun and completeness, we use the ideas in [3] to prove Theorem B directly if $n = 4$.

Proposition 3.4. *$\mathbb{Z}^4$ is not an amalgamation base for the class of lattice-ordered groups.*

We will use:

Lemma 3.5. (Holland [6]) *Let G be a lattice-ordered group and $f_1, f_2, h_1, h_2 \in G$. If $f_1 \wedge f_2 = 1$ and $h_1 \wedge h_2 = 1$, then*

$$g := (f_1 f_2^{-1}) h_1^{-1} f_2 h_1 f_1^{-1} h_2^{-1} f_1 (h_2 h_1^{-1}) f_2^{-1} h_1 f_2 h_2^{-1} f_1^{-1} h_2 = 1.$$

Proof of Proposition 3.4. Let A be the cardinal product of 4 copies of $\mathbb{Z}$, say $A := A_0 \oplus A_1 \oplus A_2 \oplus A_3$, where $A_j := \langle a_j \rangle$ with $a_j > 0$ ($j = 0, 1, 2, 3$).

Let $B_1 := A_0 \oplus A_1$, the subgroup of A generated by $\{a_0, a_1\}$ and $B_2 := A_2 \oplus A_3$, the subgroup generated by $\{a_2, a_3\}$. So $B_1 \oplus B_2 = A$. Let $H_1 := B_1 \overset{\leftarrow}{\rtimes} \langle \beta_1 \rangle$, where the action of β_1 on B_1 is induced by the permutation $(0, 1)$ on the subscripts of a. That is:

$$a_0^{\beta_1} := a_1 \qquad \text{and} \qquad a_1^{\beta_1} := a_0.$$

Similarly, let $H_2 := B_2 \overset{\leftarrow}{\rtimes} \langle \beta_2 \rangle$, where the action of β_2 on B_2 is induced by the permutation $(2, 3)$ on the subscripts of a. Let $G_1 := H_1 \oplus H_2$, the cardinal product.

Let $C_1 := A_1 \oplus A_3$, the subgroup of A generated by $\{a_1, a_3\}$ and $C_2 := A_0 \oplus A_2$, the subgroup generated by $\{a_0, a_2\}$. So $C_1 \oplus C_2 = A$. Let $K_1 := C_1 \overset{\leftarrow}{\rtimes} \langle \gamma_1 \rangle$, where the action of γ_1 on C_1 is induced by the permutation $(1, 3)$ on the subscripts of a. Let $K_2 := C_2 \overset{\leftarrow}{\rtimes} \langle \gamma_2 \rangle$, where the action of γ_2 on C_2 is induced by the permutation $(0, 2)$ on the subscripts of a. Let $G_2 := K_1 \oplus K_2$, the cardinal product.

Suppose that there were an ℓ-group L and ℓ-embeddings $\varepsilon_1 \colon G_1 \to L$ and $\varepsilon_2 \colon G_2 \to L$ coinciding on A (*i.e.*, $a^{\varepsilon_1} = a^{\varepsilon_2}$ in L for all $a \in A$). For ease of reading, identify the images in L with their preimages in G_1 and G_2, respectively. Since $\beta_1 \wedge \beta_2 = 1$ and $\gamma_1 \wedge \gamma_2 = 1$,

$$g := (\beta_1 \beta_2^{-1}) \gamma_1^{-1} \beta_2 \gamma_1 \beta_1^{-1} \gamma_2^{-1} \beta_1 (\gamma_2 \gamma_1^{-1}) \beta_2^{-1} \gamma_1 \beta_2 \gamma_2^{-1} \beta_1^{-1} \gamma_2 = 1,$$

by Lemma 3.5. By the above equations, $a_0^g = a_3 \neq a_0$; so $g \neq 1$. This contradiction establishes the proposition. $\square$

4. A consequence concerning orderings

There are essentially three ways to make $\mathbb{Z} \times \mathbb{Z}$ an ℓ-group:

(i) the cardinal order, $\mathbb{Z}^2$;

(ii) the lexicographic order $\mathbb{Z} \overleftarrow{\times} \mathbb{Z}$; and

(iii) archimedean orders:
let r be a positive real number and
let $(m_1, n_1) < (m_2, n_2)$ if and only if $(m_2 - m_1) + r(n_2 - n_1) > 0$.

By [8], $\mathbb{Z} \times \mathbb{Z}$ with any archimedean order is an amalgamation base for the class of all ℓ-groups. Hence, by Theorem A and either Lemma 3.2 or [8], we obtain:

Corollary 4.1. *Let n be a positive integer and A_n be the free abelian group of rank n. Then A_1 results in an amalgamation base for the class of all lattice-ordered groups for the trivial partial order and both lattice orders if $n = 1$, and A_2 is an amalgamation base for any non-lexicographic lattice order. If $n \geq 2$, there are lattice orderings of A_n for which the resulting lattice-ordered group is not an amalgamation base for the class of all lattice-ordered groups.*

We can compare this result with ones in [7] and [10]. In [7], it is shown that A_2 is not an amalgamation base for the class of all right-orderable groups. Since torsion-free nilpotent groups are orderable, we can also compare Corollary 4.1 with a result in [10]: Let G_1, G_2 be copies of the free nilpotent class 2 group on two generators. If A_2 is appropriately embedded in each, then (A_2, G_1, G_2) cannot be amalgamated in the class of all nilpotent groups (torsion-free or otherwise); cf. [7: Theorem 2.2].

REFERENCES

[1] BLUDOV, V. V.: *On free products of right-ordered groups with amalgamated subgroup.* In: Problems of Modern Mathematics, Proc. Scientific Works, V.II, NII MIOO NGU, Novosibirsk, 1966, pp. 30–35 (Russian).

[2] BLUDOV, V. V.—GLASS, A. M. W.: *Conjugacy in lattice-ordered groups and right-ordered groups*, J. Group Theory **11** (2008), 623–633.

[3] BLUDOV, V. V.—GLASS, A. M. W.: *Right orders and amalgamation for lattice-ordered groups*, Math. Slovaca **61** (2011), 355–372.

[4] GLASS, A. M. W.: *Partially Ordered Groups.* Series in Algebra 7, World Scientific Publ. Co., Singapore, 1999.

[5] HOLLAND, W. C.: *The lattice-ordered group of automorphisms of an ordered set*, Michigan Math. J. **10** (1963), 399–408.

[6] HOLLAND, W. C.: *Group equations which hold in lattice-ordered groups*, Symposia Math. **XXI** (1977), 365–378.

[7] MEDVEDEV, N. Ya.: *Quasivarieties of ℓ-groups and groups*, Sib. Math. J. **26** (1985), 717–722.

[8] PIERCE, K. R.: *Amalgamations of lattice-ordered groups*, Trans. Amer. Math. Soc. **172** (1972), 249–260.

[9] POWELL, W. B.—TSINAKIS, C.: *Amalgamations of lattice-ordered groups.* In: Ordered Algebraic Structures (W. B. Powell, C. Tsinakis, eds.). Lecture Notes in Pure and Appl. Algebra, Vol. 99, Marcel Dekker, New York-Basel, 1982, pp. 171–178.

[10] WIEGOLD, J.: *Nilpotent products of groups with amalgamations*, Publ. Math. Debrecen **6** (1959), 131–168.

[11] *Black Swamp Problem Book* (W. Charles Holland, ed.), Bowling Green State University, 1982.

Received 5. 4. 2013
Accepted 6. 4. 2013

* *Department of Mathematics*
Physics, and Informatics
Irkutsk State Teachers
Training University
Irkutsk 664011
RUSSIA
E-mail: vasily-bludov@yandex.ru

** *Queens' College*
Cambridge CB3 9ET
ENGLAND
Postal address:
Auchengassel
Thornhill
Dumfriesshire DG3 4AS
SCOTLAND
E-mail: amwg@dpmms.cam.ac.uk