

ON D-POSETS OF FUZZY SETS

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

(Communicated by Anatolij Dvurečenskij)

ABSTRACT. *D*-posets of fuzzy sets constitute a natural simple mathematical structure in which relevant notions of generalized probability theory can be formalized. We present a classification of *D*-posets leading to a hierarchy of distinguished subcategories of *D*-posets related to probability and study their relationships. This contributes to a better understanding of the transition from classical probability theory to fuzzy probability theory. In particular, we describe the transition from the Boolean cogenerator $\{0, 1\}$ to the fuzzy cogenerator $[0, 1]$ and prove that the generated Łukasiewicz tribes form an epireflective subcategory of the bold algebras.

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1. Introduction

D-posets have been introduced in [19] in order to model events in quantum probability. They generalize Boolean algebras, *MV*-algebras and other probability domains, and provide a category in which observables (notions dual to random variables) and states (generalized probability measures) become morphisms ([6], [11]). Recall that a *D-poset* is a partially ordered set S with the greatest element 1_S , the least element 0_S , and a partial binary operation called *difference*, such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

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(D1) $a \ominus 0_S = a$ for each $a \in S$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If $T \subseteq S$ and $0_S, 1_S \in T$, then T carrying the inherited order and closed with respect to the inherited difference is said to be a *sub- D -poset* of S .

A canonical example is the interval $[0, 1] = I$ (the usual linear order, $a \ominus b = a - b$ whenever $b \leq a$). Recall that each D -poset can be reorganized into an *effect algebra* and the two structures are equivalent (cf. [7], [6], [24]). An exposition of quantum structures (the transition to effect algebras and D -posets) can be found in [16]. Fundamental to applications ([2], [3], [23], [13], [14], [15]) are *D -posets of fuzzy sets*, i.e. systems $\mathcal{X} \subseteq I^X$ carrying the coordinatewise partial order, coordinatewise convergence of sequences, containing the top and bottom elements of I^X , and closed with respect to the partial operation difference defined coordinatewise (if $u, v \in \mathcal{X}$ and $v(x) \leq u(x)$ for all $x \in X$, then $(u \ominus v)(x) = u(x) - v(x)$). We always assume that $\mathcal{X}$ is *reduced*, i.e., for $x, y \in X$, $x \neq y$, there exists $u \in \mathcal{X}$ such that $u(x) \neq u(y)$. Denote ID the category having (reduced) D -posets of fuzzy sets (functions into $[0, 1]$) as objects and having sequentially continuous (with respect to the coordinatewise convergence) D -homomorphisms (maps preserving constants, order, and the difference) as morphisms. Objects of ID are subobjects of the powers I^X ([22]).

Concerning the undefined notions, the reader is referred to [7] and [1].

2. Motivation

In this section we provide motivation for our study of particular properties of D -posets.

Let $(X, \mathbf{A}, p)$ be a classical probability space, i.e., $\mathbf{A}$ is a σ -field of subsets of X and p is a σ -additive normed measure on $\mathbf{A}$. Denote $\mathcal{M}(\mathbf{A})$ the system of all measurable functions into $[0, 1] = I$. We identify a set A and its indicator function χ_A ($\chi_A: X \rightarrow \{0, 1\}$, $\chi_A(\omega) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise). Then $\mathbf{A}$ (as indicator functions) and $\mathcal{M}(\mathbf{A})$ are D -posets of fuzzy sets, p and the probability integral $\tilde{p}(f) = \int f \, dp$ are sequentially continuous D -homomorphisms (with respect to the coordinatewise, also pointwise, convergence, remember the Lebesgue Dominated Convergence Theorem). What is more important and surprising, the following assertions hold (cf. [11]).

THEOREM 2.1.

(i) Let p be a sequentially continuous D -homomorphism of $\mathbf{A}$ into I . Then p is a probability measure.

(ii) Let h be a sequentially continuous D -homomorphism of $\mathcal{M}(\mathbf{A})$ into I . Then h is a probability integral, i.e., there exists a probability measure p on $\mathbf{A}$ such that $h(f) = \int f \, dp$, $f \in \mathcal{M}(\mathbf{A})$.

(iii) Let $(Y, \mathbf{B})$ be a measurable space. Then each sequentially continuous D -homomorphism of $\mathbf{B}$ into $\mathcal{M}(\mathbf{A})$ can be uniquely extended to a sequentially continuous D -homomorphism of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$.

Observe that if $\{a\}$ is a singleton and $\mathbf{A} = \{\emptyset, \{a\}\}$ is the corresponding trivial field of all subsets of $\{a\}$, then I and $\mathcal{M}(\mathbf{A})$ coincide. So, probability measures on $\mathbf{A}$ and sequentially continuous D -homomorphisms of $\mathbf{A}$ into I can be identified. Since there is a one-to-one correspondence between σ -fields of sets and measurable $[0, 1]$ -valued functions (indeed, $\mathbf{A}$ is the set of all $\{0, 1\}$ -valued elements of $\mathcal{M}(\mathbf{A})$) and a one-to-one correspondence between probability measures on $\mathbf{A}$ and probability integrals on $\mathcal{M}(\mathbf{A})$, we can consider $(X, \mathcal{M}(\mathbf{A}), \tilde{p})$ as a fuzzified model of $(X, \mathbf{A}, p)$ ([25]). The elements of $\mathbf{A}$ are called *random events* and the elements of $\mathcal{M}(\mathbf{A})$ are called *fuzzy random events*.

Now, let us turn to sequentially continuous D -homomorphisms of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$; such maps are called *fuzzy observables*. According to (iii) in Theorem 2.1, there is a one-to-one correspondence between sequentially continuous D -homomorphisms of $\mathbf{B}$ into $\mathbf{A}$ and fuzzy observables on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ for which all elements of $\mathbf{B}$ are mapped into $\mathbf{A}$; such fuzzy observables are called *conservative*. It is known (cf. [3], [17], [13]) that a fuzzy observable on $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ can map some $B \in \mathbf{B}$ into $\mathcal{M}(\mathbf{A}) \setminus \mathbf{A}$; such fuzzy observables are called *genuine*.

Generalizations of random events are modeled by various algebraic structures (cf. [28], [7], [19], [26], [27]). In the next section we present a system of properties of D -posets related to such generalizations. In turn, this leads to a classification scheme of D -posets and a better understanding of the transition from the classical probability theory, CPT for short, to the fuzzy probability theory, FPT for short.

3. Properties

D -poset is a simple mathematical structure in which basic probabilistic notions can be modeled. Let $(S, \leq, 1_S, 0_S, \ominus)$ be a D -poset; it will be condensed to S . The elements of S model (generalized) random events, 1_S models the *sure event*, 0_S model the *impossible event*, $1 \ominus a$ models the *negation* of a . If $a \leq c$, then a and $b = c \ominus a$ are “disjoint events covering c ” and if s a D -homomorphism of S into I , then “additivity” of s amounts to the “subtracitivity”: $s(c \ominus a) = s(c) - s(a) = s(b)$.

To model additional properties of random events, we have to assume additional properties of D -posets. Denote D the category having D -posets as objects and having D -homomorphisms (maps preserving constants, order, and difference) as morphisms. First of all, we will restrict ourselves to the objects of ID , i.e. (reduced) D -posets of fuzzy sets, ID -posets, for short.

Let $\mathcal{X} \subseteq I^X$ be an *ID*-poset. Then each projection $\text{pr}_x: \mathcal{X} \rightarrow I$, $\text{pr}_x(u) = u(x)$, $x \in X$, is a sequentially continuous *D*-homomorphism of $\mathcal{X}$ into I and $\mathcal{X}$ carries the *weak* (also initial) *D*-poset structure with respect to the projections: $u \neq v$ iff there exists $x \in X$ such that $\text{pr}_x(u) \neq \text{pr}_x(v)$, $v \leq u$ iff $\text{pr}_x(v) \leq \text{pr}_x(u)$ for all $x \in X$ and then $(u \ominus v)(x) = \text{pr}_x(u) - \text{pr}_x(v)$. Moreover, a sequence $\{u_n\}_{n=1}^\infty$ converges in $\mathcal{X}$ to u iff $\lim_{n \rightarrow \infty} \text{pr}_x(u_n) = \text{pr}_x(u)$ for all $x \in X$. Denote $\text{Proj}(\mathcal{X})$ the set of all projections $\text{pr}_x: \mathcal{X} \rightarrow I$, $x \in X$ and denote $\mathcal{H}(\mathcal{X})$ the set of all sequentially continuous *D*-homomorphisms of $\mathcal{X}$ into I . Clearly, $\mathcal{X}$ carries the weak *D*-poset structure with respect to $\mathcal{H}(\mathcal{X})$.

In general, we start with a distinguished *D*-poset C (possibly having some additional properties, e.g., C is a lattice) and consider the class of *D*-posets “living in powers C^X ”, i.e., carrying the weak *D*-poset structure with respect to the projections. We call C the *cogenerator* of the corresponding class of *D*-posets. *D*-posets of the class are called *domains of probability* and model generalized random events. Sequentially continuous *D*-homomorphisms into C model *generalized probability measures* ([14]).

The two traditional cogenerators are $\{0, 1\}$ and $I = [0, 1]$. Indeed, $\{0, 1\}$ cogenerates random events (fields of sets) in CPT, while I cogenerates random events (measurable I -valued functions) in FPT.

Both in CPT and in FPT, sequentially continuous *D*-homomorphisms into I model probability measures on random events. The weak structure means that “everything is determined by probability measures”. This is in accordance with “a requirement that positive measures determine the partial order is the sine qua non of algebraic measure theory” (cf. [8]).

Recall ([4], [7], [9]) that a *bold algebra* (also T_L -clan, see [21]) is a system $\mathcal{X} \subseteq [0, 1]^X$ containing the constant functions $0_X, 1_X$ and closed with respect to the usual (Łukasiewicz) operations: for $u, v \in \mathcal{X}$ put $(u \oplus v)(x) = u(x) \oplus v(x) = \min\{1, u(x) + v(x)\}$, $u^*(x) = 1 - u(x)$, $x \in X$. Bold algebras are *MV*-algebras representable as $[0, 1]$ -valued functions, *MV*-algebras generalize Boolean algebras and bold algebras generalize in a natural way fields of sets (viewed as indicator functions). More information concerning *MV*-algebras and probability on *MV*-algebras can be found in [26]. If a bold algebra $\mathcal{X} \subseteq [0, 1]^X$ is sequentially closed in $[0, 1]^X$ (with respect to the coordinatewise sequential convergence), then $\mathcal{X}$ is a *Łukasiewicz tribe* ($\mathcal{X}$ is closed not only with respect to finite, but also with respect to countable Łukasiewicz sums, cf. [9: Corollary 2.8]). It is known (cf. [20], [7]) that if $\mathcal{X} \subseteq [0, 1]^X$ is a Łukasiewicz tribe, then there exists a unique σ -field $\mathbf{A}_\mathcal{X}$ of subsets of X such that $\mathbf{A}_\mathcal{X} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_\mathcal{X})$ and $\mathcal{X} = \mathcal{M}(\mathbf{A}_\mathcal{X})$ iff $\mathcal{X}$ contains all constant functions r_X , $r \in [0, 1]$. Let $\mathcal{X} \subseteq [0, 1]^X$ be a bold algebra. Then $[0, 1]^X$ is a Łukasiewicz tribe containing $\mathcal{X}$ and the intersection of all Łukasiewicz tribes $\mathcal{Y} \subseteq [0, 1]^X$ such that $\mathcal{X} \subseteq \mathcal{Y}$ is a Łukasiewicz tribe; it will be called the *induced* Łukasiewicz tribe and denoted by $\sigma(\mathcal{X})$. Each bold algebra

$\mathcal{X} \subseteq [0, 1]^X$ is a lattice, where for $u, v \in \mathcal{X}$ we have $(u \vee v)(x) = u(x) \vee v(x)$, $(u \wedge v)(x) = u(x) \wedge v(x)$, $x \in X$, and each ID -poset $\mathcal{X} \subseteq [0, 1]^X$ which is a lattice (with respect to the coordinatewise order) can be reorganized into a bold algebra (the Lukasiewicz operations can be redefined via the difference and order, cf. [14], [25]). Finally, each bold algebra can be considered as an object of ID . Recall that two elements u and v of a D -poset S are *compatible* if there exist elements $c, d \in S$ such that $d < u < c$, $d < v < c$ and $c \ominus u = v \ominus d$. In a bold algebra $\mathcal{X} \subseteq [0, 1]^X$ each two elements are “compatible” and, in particular, if $u < v$, then $(v \vee u) - u = v - (v \wedge u) = v - u$ is defined and $u \oplus (v - u) = v$.

To understand the transition from CPT to FPT, we should identify characteristic properties (in terms of D -posets) of random events in CPT (σ -fields of sets) and additional properties of random events in FPT (measurable functions into $[0, 1]$). Roughly, the transition is analogous to the transition from integers to real numbers: divisibility and completeness. First, we characterize the transition from the Boolean cogenerator $\{0, 1\}$ to the fuzzy cogenerator $[0, 1]$.

DEFINITION 3.1. Let S be a D -poset and let n be a natural number, $n > 1$. If for each $x \in S$, $x \neq 0_S$ there are $x(n, k) \in S$, $k = 1, 2, \dots, n-1$, such that $0_S < x(n, 1) < \dots < x(n, k-1) < x$, $x \ominus x(n, k-1) = x(n, k-1) \ominus x(n, k-2) = \dots = x(n, 2) \ominus x(n, 1) = x(n, 1)$, then S is said to be *divisible by n* . If S is divisible by n for all natural numbers $n > 1$, then S is said to be *divisible*.

Example 3.2. Consider the set S of all rational numbers of the form $k/2^n$, where k, n are natural numbers and $0 \leq k \leq 2^n$, carrying the natural D -poset structure. It is easy to see that the D -poset S is linearly ordered and fails to be divisible by 3. On the other hand, S is *totally non-atomic* (does not contain any atom).

LEMMA 3.3. Let S be a linearly ordered σ -complete D -poset. Then S is a bold algebra.

Proof. It follows from the general theory of D -posets (cf. [6: Theorem 55, Theorem 58]) that a linearly ordered D -poset is an MV -algebra. Further, it is known that each σ -complete MV -algebra is a bold algebra (cf. [7]). $\square$

LEMMA 3.4. Let $\mathcal{X} \subseteq I^X$ be a linearly ordered (reduced) bold algebra. Let $u, v \in \mathcal{X}$.

- (i) If $v < u$, then for each $x \in X$ we have $v(x) < u(x)$.
- (ii) If $v(x) = u(x)$ for any $x \in X$, then $v = u$.

Proof.

(i). Otherwise, for $w = u - v$ and some $x \in X$ we would have $w(x) = v(x) - u(x) = 0$ and for some $y \in X$ we would have $w(y) > 0$. Consequently, for some positive natural number k we would have $kw(x) = 0$, $kw(y) \geq 1$. Define

$2w = w \oplus w$ and inductively $(n+1)w = nw \oplus w$. Since $kw \in \mathcal{X}$, the two distinct elements kw and $1_X - kw$ would be incomparable, a contradiction.

(ii). Clearly, (ii) follows from (i). $\square$

LEMMA 3.5. *Let $\mathcal{X} \subseteq I^X$ be a linearly ordered, σ -complete and totally non-atomic (reduced) bold algebra.*

(i) *For each $x \in X$, $\{u(x); u \in \mathcal{X}\} = [0, 1]$.*

(ii) $\text{card}(X) = 1$.

Proof. (i) follows from the previous lemma and the assumption that $\mathcal{X}$ is σ -complete and totally non-atomic and it implies that $\mathcal{X}$ contains all constant functions q_X , where q is a rational number, $0 \leq q \leq 1$. Clearly, this is possible only in case when $\text{card}(X) = 1$. $\square$

COROLLARY 3.6. *Let S be a linearly ordered σ -complete D -poset. If S is totally non-atomic, then S and I are isomorphic.*

In [5] it is proved that if S is a linearly ordered, σ -complete D -poset which is totally non-atomic, then there is a state on S . Since S and I are isomorphic and the isomorphism is a state, our previous lemma generalizes the main result in [5].

THEOREM 3.7. *Let S be a linearly ordered σ -complete D -poset. Then the following are equivalent:*

- (i) *S and I are isomorphic;*
- (ii) *S is totally non-atomic;*
- (iii) *S is divisible;*
- (iv) *If T is a linearly ordered, σ -complete and divisible D -poset and S is a sub- D -poset of T , then $S = T$.*

Proof. Clearly, (i) implies (ii) and (iii).

(i) $\implies$ (iv). Let T be a linearly ordered, σ -complete and divisible D -poset and let S be a sub- D -poset of T . As proved above, T is a bold algebra, $T \subseteq I^X$. Since T is divisible, it contains all constant functions q_X , where q is a rational number, $0 \leq q \leq 1$. Necessarily, $T = I = S$.

(ii) $\implies$ (i). See Corollary 3.6.

(iii) $\implies$ (i). According to Lemma 3.3, S is a (reduced) bold algebra. Since it is divisible, it contains all constant functions q_X , where q is a rational number, $0 \leq q \leq 1$. But S is linearly ordered and σ -complete, hence S and I are isomorphic.

(iv) $\implies$ (i). Let T be a linearly ordered, σ -complete and divisible D -poset and let S be a sub- D -poset of T . Since (iii) implies (i), the assertion follows. This completes the proof. $\square$

COROLLARY 3.8. *Let S be a linearly ordered, σ -complete D -poset. Then S is totally non-atomic iff it is divisible.*

The next example shows an important difference between the two properties.

Example 3.9. Let $\mathbf{B}_I$ be the σ -field of all Borel measurable subsets of $[0, 1]$ and let $\mathcal{M}(\mathbf{B}_I)$ be the corresponding generated Łukasiewicz tribe. Denote $\mathcal{X} = \{u \in \mathcal{M}(\mathbf{B}_I); u(0), u(1) \in \{0, 1\}\}$ the corresponding bold algebra. Clearly, $\mathcal{X}$ is a Łukasiewicz tribe (it is sequentially closed in I^I), hence σ -complete, $\mathbf{B}_I \subsetneq \mathcal{X} \subsetneq \mathcal{M}(\mathbf{B}_I)$, $\mathcal{X}$ is totally non-atomic and fails to be divisible. Notice that $\mathcal{M}(\mathbf{B}_I)$ is divisible.

To sum up, the Boolean cogenerator $\{0, 1\}$ is *the minimal linearly ordered σ -complete (atomic) D -poset* and the cogenerator $[0, 1]$ of fuzzy random events is *the maximal linearly ordered, σ -complete and divisible (totally non-atomic) D -poset*.

From the viewpoint of CPT and FPT, important classes of D -posets are *linearly ordered, σ -complete, atomic and totally non-atomic* at a basic level, *bold algebras* (lattice ordered D -posets), *divisible and sequentially closed bold algebras* (generated Łukasiewicz tribes) at a more advanced level.

Previous results lead to several distinguished categories. The category D of D -posets and D -homomorphisms serves as the reference category and properties related to generalized random events lead to the following hierarchy of subcategories of D .

The category ID of D -posets of fuzzy sets and sequentially continuous D -homomorphisms captures the idea of objects cogenerated by $I = [0, 1]$ (the domain of probability measures) carrying the initial (weak) structure.

Its full subcategory BID consists of the D -posets of fuzzy sets the order in which is a lattice (with respect to the coordinatewise order) and hence exactly bold algebras. Note that both σ -fields of sets (classical random events) and measurable functions into I (fuzzy random events) are lattices.

Its full subcategory $CBID$ consists of Łukasiewicz tribes, i.e., bold algebras which are sequentially closed in I^X . Each σ -field of sets is a Łukasiewicz tribe. Sequential closedness is a must both in CPT and FPT - stochastics is about limits.

Let $\mathcal{X} \subseteq I^X$ be a Łukasiewicz tribe. Then (cf. [27: 8.1.4. Theorem]) $\{u \in \mathcal{X}; u \in \{0, 1\}^X\}$ is the set of all indicator functions of a σ -algebra $\mathbf{A}_{\mathcal{X}}$ of subsets of X and each $u \in \mathcal{X}$ is a measurable function, i.e. $\mathbf{A}_{\mathcal{X}} \subseteq \mathcal{X} \subseteq \mathcal{M}(\mathbf{A}_{\mathcal{X}})$. Further, if $\mathcal{X}$ contains all constant functions ranging in I , then $\mathcal{X} = \mathcal{M}(\mathbf{A}_{\mathcal{X}})$ and $\mathcal{X}$ is said to be a *generated* Łukasiewicz tribe.

Finally, denote $CGBID$ the full subcategory consisting of the generated Łukasiewicz tribes. Recall that the objects of $CGBID$ are exactly divisible Łukasiewicz tribes.

4. *CGBID* is an epireflective in *BID*

It is known that the extension of a probability measure p from a field of sets $\mathbf{A}$ over the generated σ -field $\sigma(\mathbf{A})$ and further to the probability integral $\tilde{p}(f) = \int f \, dp$, $f \in \mathcal{M}(\mathbf{A})$ has a categorical background ([9], [10], [11]). In [18] the following problem has been stated.

PROBLEM. Is *CGBID* an epireflective subcategory of *BID*?

Let us recall the notion of an *epireflection*. Let $\mathbf{B}$ be a full subcategory of a category $\mathbf{A}$. Let X be an object of $\mathbf{A}$. An object $r(X)$ of $\mathbf{B}$ is called a *reflection* of X in $\mathbf{B}$, or a $\mathbf{B}$ -reflection, if there exists a morphism $e_X: X \rightarrow r(X)$ such that for each morphism $f: X \rightarrow Y$, Y in $\mathbf{B}$, there exists a unique morphism $r(f): r(X) \rightarrow Y$ such that $r(f) \circ e_X = f$. The functor assigning to each object of $\mathbf{A}$ its reflection in $\mathbf{B}$, is called a reflector. If each object of $\mathbf{A}$ has a $\mathbf{B}$ -reflection, then $\mathbf{B}$ is said to be *reflective* in $\mathbf{A}$. A reflective subcategory is called *epireflective* if the canonical morphism $e_X: X \rightarrow r(X)$ is an epimorphism for every X ; in this case we speak of an *epireflector* and *epireflection*.

THEOREM 4.1. *CGBID* is an epireflective subcategory of *BID*.

Proof. Let $\mathcal{X} \subseteq I^X$ be a bold algebra and let $\sigma(\mathcal{X}) \subseteq I^X$ be the induced Łukasiewicz tribe. Denote $\mathbf{A}_{\sigma(\mathcal{X})}$ the set $\{u \in \sigma(\mathcal{X}); u \in \{0, 1\}^X\}$ of all indicator functions of $\sigma(\mathcal{X})$. Then $\mathcal{X} \subseteq \sigma(\mathcal{X})$ and $\mathbf{A}_{\sigma(\mathcal{X})} \subseteq \sigma(\mathcal{X}) \subseteq \mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$. We claim that $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ is the desired epireflection of $\mathcal{X}$. First, let $\mathbf{B} \subseteq \{0, 1\}^Y$ be a σ -field of subsets of Y and let h be a sequentially continuous D -homomorphism of $\mathcal{X}$ into the generated Łukasiewicz tribe $\mathcal{M}(\mathbf{B})$. We have to prove that h can be uniquely extended to a sequentially continuous D -homomorphism $\bar{h}$ of $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ into $\mathcal{M}(\mathbf{B})$. According to [11: Corollary 2.11], *CBID* is epireflective in *BID* and hence h can be uniquely extended to a sequentially continuous D -homomorphism h_σ of $\sigma(\mathcal{X})$ into $\mathcal{M}(\mathbf{B})$. Further, according to [14: Corollary 4.4], *CGBID* is epireflective in *CBID*, and hence h_σ can be uniquely extended over $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$. The resulting extension $\bar{h}$ has the desired properties. Secondly, we have to prove that the embedding $e_{\mathcal{X}}$ of $\mathcal{X}$ into $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ is an epimorphism. Suppose that two sequentially continuous D -homomorphisms f, g of $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ into a bold algebra $\mathcal{Y} \subseteq I^Y$ agree on $\mathcal{X}$. We have to prove that $f = g$. Since $\mathcal{Y}$ is a sub- D -poset of the induced Łukasiewicz tribe $\sigma(\mathcal{Y})$ and $\sigma(\mathcal{Y})$ is the sub- D -poset of the corresponding generated Łukasiewicz tribe $\mathcal{M}(\mathbf{B}_{\sigma(\mathcal{Y})})$, f and g can be considered as sequentially continuous D -homomorphisms of $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$ into $\mathcal{M}(\mathbf{B}_{\sigma(\mathcal{Y})})$. The restrictions $f \upharpoonright \mathcal{X}$, $g \upharpoonright \mathcal{X}$ of f and g , respectively, agree on $\mathcal{X}$ and can be considered as a sequentially continuous D -homomorphism h of $\mathcal{X}$ into $\mathcal{M}(\mathbf{B}_{\sigma(\mathcal{Y})})$. In view of the first part of the proof, h has a unique extension $\bar{h}$ over $\mathcal{M}(\mathbf{A}_{\sigma(\mathcal{X})})$, and hence $f = \bar{h} = g$. This completes the proof. $\square$

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