

EXTENDING SEMILATTICES TO FRAMES USING SITES AND COVERAGES

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Dedicated to Professor Ján Jakubík on the occasion of his 90th birthday

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ABSTRACT. Each meet semilattice S is well known to be freely extended to a frame by its down-sets $\mathfrak{D}S$. In this article we present, first, the complete range of frame extensions generated by S ; it turns out to be a sub-coframe of the coframe C of sublocales of $\mathfrak{D}S$, indeed, an interval in C , with $\mathfrak{D}S$ as the top and the extension of S respecting all the exact joins in S as the bottom. Then, the Heyting and Boolean case is discussed; there, the bottom extension is shown to coincide with the Dedekind-MacNeille completion. The technique used is a technique of sites, generalizing that used in [JOHNSTONE, P. T.: *Stone Spaces*. Cambridge Stud. Adv. Math. 3, Cambridge University Press, Cambridge, 1982].

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Introduction

As models of topology, frames play a central role in mathematics. As complete lattices in which binary meets commute with arbitrary joins, they can be understood as the superposition of the join structure upon the meet semilattice structure. But these two structures are more deeply intertwined. The down-set functor $\mathfrak{D}$ constructs the free frame over the meet semilattice, and, in fact, the category of frames is the Eilenberg-Moore category of the functor $\mathfrak{D}$ naturally

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extended to a monad, as observed in [4] and elsewhere. And one of the frame extensions of a meet semilattice is the maximal essential embedding in its injective hull ([5]).

Now in the natural embedding of a meet semilattice S into the frame $\mathfrak{D}S$, the joins are created freely. But it often happens that S already has some joins which we would like to preserve in an embedding into a frame. For some of them, here termed the exact joins, this turns out to be possible, but for others it is not, for reasons we explain.

We may wish to preserve all the exact joins. But we are also interested in more modest extensions, those respecting only some of the joins. To analyze them we employ the concept of site and coverage as introduced by Johnstone in his classic tome [6]. (See page 57, and then page 77 of this text for thorough attributions and a detailed discussion of the history of the ideas involved.)

Concentrating on a given set of already existing exact joins leads to special sites, the $\bigvee$ -sites (see 1.2.1(a)). But they are not the only sites of interest in our context; in particular the well-known Dedekind-MacNeille completion (which of course does not have to produce a frame) leads to a site of another character, the Dedekind-MacNeille site (see 1.2.1(b)) denoted by ϕ_{DM} , which calls for comparison with the previously mentioned ones.

After necessary preliminaries we construct in Section 2 a frame completion of a site as a left adjoint to the natural forgetfull functor $\mathbf{Frm} \rightarrow \mathbf{Site}$. This technique, together with the injective hull theorem from [5] allows us to prove (in Section 3) that the frame extensions of a meet-semilattice S constitute an interval in the co-frame of sublocales of $\mathfrak{D}S$ with bottom the frame completion of the finest $\bigvee$ -site. Section 4 is devoted to the Heyting and Boolean case. We prove that the bottom frame completion coincides with the Dedekind-MacNeille completion (so that in particular, ϕ_{DM} is in this case generated by a $\bigvee$ -site). For Boolean algebras we can show what happens by an explicit analysis of the (pseudo)complements.

1. Coverages and sites

1.1. In this article a meet semilattice is always a bounded meet semilattice, i.e., a meet semilattice with 0 and 1, and meet semilattice morphisms will be assumed to preserve 0 and 1. The resulting category will be denoted by $\mathbf{SLat}$.

Recall that a frame is a complete lattice L satisfying the frame distributivity axiom

$$\left(\bigvee A\right) \wedge b = \bigvee_{a \in A} (a \wedge b)$$

for all $A \subseteq L$ and $b \in L$. Frame morphisms preserve all joins, including the empty join 0, and all finite meets, including the empty meet 1. The resulting category will be denoted by **Frm**. Dismissing the join-structure of frames results in the forgetfull functor

$$\mathfrak{V}: \mathbf{Frm} \rightarrow \mathbf{SLat}.$$

1.1.1. For a semilattice S consider the poset

$$\mathfrak{D}S = \{U \mid \emptyset \neq U = \downarrow U, U \subseteq S\}$$

where $\downarrow U = \{x \mid x \leq u \in U\}$, ordered by inclusion (it is obviously a frame), and a semilattice homomorphism

$$\alpha_S = (a \mapsto \downarrow a = \downarrow \{a\}): S \rightarrow \mathfrak{D}S.$$

1.2. A *coverage* of a meet semilattice S is a relation

$$\sqsubseteq \subseteq S \times \mathfrak{D}S$$

(read $a \sqsubseteq A$ as “ a is covered by A ”) such that

- $a \sqsubseteq U$ for all $a \in U$ (“ U covers all of its elements”), and
- $b \leq a \sqsubseteq U$ implies $b \sqsubseteq U$ (“if U covers a then it covers all the elements smaller than a ”).

1.2.1. Examples.

(a) **$\bigvee$ -coverages.** Let $\mathcal{A} \subseteq \mathfrak{D}S$ be a subset such that for all $A \in \mathcal{A}$ there exists a supremum $\bigvee A$ in S . With $\mathcal{A}$ we associate a coverage $\sqsubseteq_{\mathcal{A}}$ by setting

$$a \sqsubseteq_{\mathcal{A}} A \iff a \leq \bigvee A.$$

Such coverages will be called *$\bigvee$ -coverages* and will be the main concern of this article: we will be interested in special complete extensions of a semilattice, and a $\bigvee$ -coverage $a \sqsubseteq_{\mathcal{A}} A$ will point out some of the existing suprema to be preserved.

(b) **The Dedekind-MacNeille coverage.** Set

$$b \sqsubseteq A \iff b \leq x \text{ for all the upper bounds } x \text{ of } A$$

$$(\text{that is, for all the } x \in \text{ub } A = \bigcap_{a \in A} \uparrow a).$$

This coverage is closely connected with the Dedekind-MacNeille completion $\text{DM } S$ (see 4.1 below). In general, $\text{DM } S$ is of course not a frame: it respects all joins which hold in S , including those that do not distribute over meets, and even if S is a distributive lattice, $\text{DM } S$ need not itself be one. But in some cases it is, and this will be our concern in Section 4.

(c) **The trivial coverage of a frame.** In a frame L consider the trivial coverage

$$a \sqsubseteq A \iff a \leq \bigvee A$$

(thus, it is the $\bigvee$ -coverage associated with the system of *all* the U in $\mathfrak{D}L$).

1.3. An alternative formulation of coverage. With the coverage $\sqsubseteq$ on S associate a mapping

$$\phi = \phi_{\sqsubseteq} \equiv (U \mapsto \{a \mid a \sqsubseteq U\}) : \mathfrak{D}S \rightarrow \mathfrak{D}S.$$

The following are simple:

1.3.1. OBSERVATIONS.

- (1) $U \subseteq \phi(U)$,
- (2) Let $\sqsubseteq_i$, $i = 1, 2$, be coverages on S , and let $\phi_i : \mathfrak{D}S \rightarrow \mathfrak{D}S$ be the associated mappings. Then $\sqsubseteq_1 \subseteq \sqsubseteq_2$ iff $\phi_1 \leq \phi_2$.

On the other hand, a mapping $\phi : \mathfrak{D}S \rightarrow \mathfrak{D}S$ satisfying (1) above gives rise to a cover

$$a \sqsubseteq_{\phi} A \iff a \in \phi(A)$$

and the formulas $\sqsubseteq \mapsto \phi_{\sqsubseteq}$, $\phi \mapsto \sqsubseteq_{\phi}$ constitute a mutually inverse order-preserving correspondence. Consequently, we will in the sequel as a rule represent coverages as mappings $\phi : \mathfrak{D}S \rightarrow \mathfrak{D}S$ satisfying $U \subseteq \phi(U)$.

1.3.2. Note. Recall the examples 1.2.1.

- (a) The $\bigvee$ -coverage associated with $\mathcal{A}$ is represented as

$$\phi_{\mathcal{A}}(A) = \begin{cases} \downarrow \bigvee A & \text{if } A \in \mathcal{A} \\ A & \text{otherwise.} \end{cases}$$

- (b) The *Dedekind-Mac Neille coverage* is

$$\phi_{\text{DM}}(U) = \text{lbub}(U)$$

where $\text{lb}(A)$ is the set of all lower bounds of A , that is, $\bigcap_{a \in A} \downarrow a$. Note that unlike $\phi_{\mathcal{A}}$, ϕ_{DM} is always monotone.

(c) The trivial coverage of a frame, which is, of course the $\bigvee$ -coverage associated with the whole of $\mathfrak{D}L$, will be denoted by

$$\sigma_L.$$

1.4. Sites. A *site* $(S, \sqsubseteq)$ is a meet semilattice S endowed with a coverage.

If $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$ are sites then a *site homomorphism* $f : (S_1, \sqsubseteq_1) \rightarrow (S_2, \sqsubseteq_2)$ is a homomorphism $S_1 \rightarrow S_2$ such that

$$a \sqsubseteq_1 U \implies f(a) \sqsubseteq_2 \downarrow f[U]$$

The resulting category will be denoted

Site.

A $\bigvee$ -site is a site $(S, \sqsubseteq)$ with a $\bigvee$ -coverage $\sqsubseteq$.

1.4.1. We will mostly work with sites in the representation (S, ϕ) as in 1.3. Then the definition of a site homomorphism $f: (S_1, \phi_1) \rightarrow (S_2, \phi_2)$ transforms to

$$a \in \phi_1(U) \implies f(a) \in \phi_2(\downarrow f[U]) = \phi_2(\mathfrak{D}f(U)).$$

In other words,

$$\mathfrak{D}f(\phi_1(U)) \subseteq \phi_2(\mathfrak{D}f(U)).$$

1.4.2. The functor (embedding) $\mathfrak{U}: \mathbf{Frm} \rightarrow \mathbf{Site}$. Recall the σ_L from 1.3.2. We immediately see that

the site homomorphisms $f: (L, \sigma_L) \rightarrow (M, \sigma_M)$ are precisely the frame homomorphisms $f: L \rightarrow M$.

Hence if we set $\mathfrak{U}L = (L, \sigma_L)$ and $\mathfrak{U}f = f$ we obtain a full embedding

$\mathfrak{U}: \mathbf{Frm} \rightarrow \mathbf{Site}$.

1.5. Exact $\bigvee$ -coverages and sites. Extending a semilattice to a frame respecting a given system of existing joins is possible only if those joins distribute over all finite meets. For dealing with such extensions we will need the following concept.

Recall from [1] the operations

$$c \downarrow d \equiv \{x \mid x \wedge d \leq c\} \quad \text{and} \quad c \uparrow d \equiv \{x \mid x \vee c \geq d\}.$$

We say that b is the *exact join* of a subset A if b is an upper bound of A such that $b \in c \downarrow d$ for all $c < d$ such that $A \subseteq c \downarrow d$.

Recall the basic facts about exact joins.

- (1) An exact join is a join. Therefore no subset can have two different exact joins.
- (2) b is the exact join of A iff for each x the join of $\{a \wedge x \mid a \in A\}$ exists and is equal to $b \wedge x$.
- (3) Any join in a Heyting algebra is exact. For suppose $b = \bigvee A$ in a Heyting algebra S , and suppose that $A \subseteq c \downarrow d$ for elements $c < d$. Then $A \leq d \rightarrow c$ implies $b \leq d \rightarrow c$, hence $b \wedge d \leq c$, i.e., $b \in c \downarrow d$.

We will most often use (2) as our working definition of an exact join.

Now recall 2.2.1(a). A set $\mathcal{A} \subseteq \mathfrak{D}S$ will be called *exact* if all the $A \in \mathcal{A}$ have an exact join. Accordingly, we will also speak of the associated coverage $\sqsubseteq_{\mathcal{A}}$ (resp. site $(S, \sqsubseteq_{\mathcal{A}})$) as an *exact coverage* resp. *exact site*.

2. The frame of ideals of a site

2.1.1. Recall 1.1.1. If, for a semilattice homomorphism $f: S \rightarrow T$, we set $\mathfrak{D}f \equiv (U \mapsto \downarrow f[U]) : \mathfrak{D}S \rightarrow \mathfrak{D}T$, we obtain a functor

$$\mathfrak{D}: \mathbf{SLat} \rightarrow \mathbf{Frm},$$

and $\alpha = (\alpha_S)_S$ is a transformation $\text{Id} \rightarrow \mathfrak{D}\mathfrak{D}$.

It is well known that for every frame L and every semilattice homomorphism f there is exactly one frame morphism g such that the diagram

$$\begin{array}{ccc} \mathfrak{D}S & \xrightarrow{g} & L \\ \alpha_L \uparrow & \nearrow f & \\ S & & \end{array}$$

commutes. It is given by the formula $g(U) = \bigvee f[U]$.

This makes $\mathfrak{D}$ a left adjoint to the forgetful functor $\mathfrak{V}: \mathbf{Frm} \rightarrow \mathbf{SLat}$ (the free functor); α is one of the adjunction units and the other one, $v: \mathfrak{D}\mathfrak{V} \rightarrow \text{Id}$, is defined by $v_L(U) = \bigvee U$.

2.2. The ideal frame $\mathfrak{I}(S, \phi)$. First recall that, unlike in most algebras, in frames it is very easy to construct congruences, or, rather, quotients. (For details, see, e.g., [8] and [9] or [2].) Given a relation $R \subseteq L \times L$ on a frame L , define an element $s \in L$ to be *R-saturated*, or simply *saturated*, if

$$\forall a, b, c \in L \quad [aRb \implies (a \wedge c \leq s \iff b \wedge c \leq s)].$$

The saturated elements are closed under arbitrary meets in L , and we can define

$$\nu(x) \equiv \bigwedge \{s \mid s \text{ saturated and } x \leq s\}$$

The monotone mapping

$$\nu = \nu_R: L \rightarrow L \tag{*}$$

thus associated with the relation R is a *nucleus* in the standard sense that

- (1) $x \leq \nu(x)$,
- (2) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$, and
- (3) $\nu\nu(x) = \nu(x)$.

We set

$$L/R \equiv \nu[L] = \{x \mid \nu(x) = x\}.$$

By abuse of notation it is customary to use the same symbol ν for the restriction

$$\nu = \nu_R: L \rightarrow L/R. \tag{**}$$

Then we obtain a *frame homomorphism* such that:

- aRb implies $\nu(a) = \nu(b)$, and
- for every frame morphism $h: L \rightarrow M$ such that $aRb \implies h(a) = h(b)$ there is precisely one frame morphism $\bar{h}: L/R \rightarrow M$ such that $\bar{h} \circ \nu = h$; moreover, we have $\bar{h}(x) = h(x)$ for all $x \in L/R$.

Caution. The mappings ν in the senses (*) and (**) have the same formula, but the change in the range is essential: the latter is a frame homomorphisms, the former not. But introduction of a more precise notation does not seem worthwhile. The reader will certainly easily recognize the range in question.

2.2.1. Now let (S, ϕ) be a site. Consider the relation on $\mathfrak{D}S$

$$R_\phi \equiv \{(U, \phi(U)) \mid U \in \mathfrak{D}S\}$$

and set

$$\mathfrak{I}(S, \phi) \equiv \mathfrak{D}S/R_\phi.$$

This frame is made up of the R_ϕ -saturated down-sets V ; one readily sees that they are characterized by the condition

$$\forall U \in \mathfrak{D}S \quad \forall c \in S \quad [U \cap \downarrow c \subseteq V \implies \phi(U) \cap \downarrow c \subseteq V],$$

and if ϕ is such that $\phi(U) \cap \downarrow c \subseteq \phi(U \cap \downarrow c)$ then this condition reduces to

$$\forall U \in \mathfrak{D}S \quad [U \subseteq V \implies \phi(U) \subseteq V].$$

Such saturated down-sets will be referred to as *ideals*, and we will speak of $\mathfrak{I}(S, \phi)$ as the *frame of ideals of the site* (S, ϕ) . The associated nucleus will be denoted by $\phi^*(U)$, so that U is an ideal iff $\phi^*(U) = U$.

Note that if $\phi = \text{id} \equiv (U \mapsto U)$ then $\mathfrak{I}(S, \phi) = \mathfrak{D}S$.

2.2.2. PROPOSITION. ϕ^* is the smallest nucleus on $\mathfrak{D}S$ such that $\phi \leq \phi^*$.

Proof. We have $\phi(U) \leq \phi^*(\phi(U)) \leq \phi^*(\phi^*(U)) = \phi^*(U)$. On the other hand, if ν is a nucleus and $\phi(U) \leq \nu(U)$ then $\phi(U) \leq \nu(\phi(U)) \leq \nu\nu(U) = \nu(U)$, and ϕ^* is the least nucleus equalizing the relation R_ϕ . $\square$

2.3. The ideal functor. For a site (S, ϕ) define

$$\delta_{(S, \phi)} \equiv \phi^* \cdot \alpha_S = (a \mapsto \phi^*(\downarrow a)) : (S, \phi) \rightarrow \mathfrak{U}\mathfrak{I}(S, \phi),$$

where α_S is as in 2.1.

2.3.1. LEMMA. $\delta_{(S, \phi)}$ is a site homomorphism.

Proof. $\delta_{(S, \phi)}$ is a semilattice homomorphism since α_S and ϕ^* are. If $a \in \phi(U)$ then $\downarrow a \subseteq \phi(U)$ and

$$\phi^*(\downarrow a) \subseteq \phi^*(\phi(U)) = \phi^*(U) = \phi^*\left(\bigvee_{a \in U} \downarrow a\right) = \bigvee_{a \in U} \phi^*(\downarrow a) = \bigvee \delta[U]$$

□

2.3.2. PROPOSITION. *Let (S, ϕ) be a site and let L be a frame. Let $f: (S, \phi) \rightarrow \mathfrak{U}L$ be a site homomorphism. Then there is precisely one frame homomorphism $h: \mathfrak{I}(S, \phi) \rightarrow L$ such that the following diagram (with $\mathfrak{U}$ from 1.4.2) commutes.*

$$\begin{array}{ccc} (S, \phi) & \xrightarrow{\delta_{(S, \phi)}} & \mathfrak{U}\mathfrak{I}(S, \phi) \\ & \searrow f & \downarrow \mathfrak{U}h \\ & & \mathfrak{U}L \end{array}$$

Proof. Since the elements $\alpha(a) = \downarrow a$, $a \in S$, generate $\mathfrak{D}S$, and since ϕ^* is onto, there is at most one such h . Now consider, first, the unique frame homomorphism g given in 2.1.1 such that $g \cdot \alpha_S = f$ and recall that $g(U) = \bigvee f[U]$. Since f is a site homomorphism we have $f(a) \in f[U]$ whenever $a \in \phi(U)$, hence

$$\bigvee f[\phi(U)] \leq \bigvee f[U] \leq \bigvee f[\phi(U)],$$

with the result that $g[\phi[U]] = g[U]$. Thus g respects the relation R_ϕ , from which it follows that there is a frame homomorphism $h: \mathfrak{I}(S, \phi) \rightarrow L$ such that $h \cdot \phi^* = g$. We have $\mathfrak{U}h \cdot \delta_{(S, \phi)} = h \cdot \phi^* \cdot \alpha_S = g \cdot \alpha_S = f$. □

2.3.3. For a site homomorphism $f: (S, \phi) \rightarrow (T, \psi)$ define $\mathfrak{I}f: \mathfrak{I}(S, \phi) \rightarrow \mathfrak{I}(T, \psi)$ as the unique $h: \mathfrak{I}(S, \phi) \rightarrow (T, \psi)$ such that $\mathfrak{U}h \cdot \delta_{(S, \phi)} = \delta_{(T, \psi)} \cdot f$; thus we obtain a functor

$$\mathfrak{I}: \mathbf{Site} \rightarrow \mathbf{Frm}.$$

We have:

THEOREM. $\mathfrak{I}: \mathbf{Site} \rightarrow \mathbf{Frm}$ is left adjoint to $\mathfrak{U}: \mathbf{Frm} \rightarrow \mathbf{Site}$. For $f: (S, \phi) \rightarrow (T, \psi)$ we have

$$\mathfrak{I}f(U) = \bigvee_{a \in U} \psi^*(\downarrow f(a)).$$

One of the adjunction units is $\delta = (\delta_{(S, \phi)})_{(S, \phi)}: Id \rightarrow \mathfrak{U}\mathfrak{I}$, and the other, $\varepsilon = (\varepsilon_L)_L: \mathfrak{I}\mathfrak{U} \rightarrow Id$, consists of the isomorphisms given by $\varepsilon_L(\downarrow a) = \downarrow a$.

Proof. It is a standard fact that the situation outlined in Proposition 2.3.2 creates an adjunction as claimed, and the formula for $\mathfrak{I}f$ is obtained from the fact that this is what the g associated with $\delta_{(T, \psi)} \cdot f$ yields. Furthermore, the other adjunction unit morphisms ε_L are obtained as the unique $h: \mathfrak{I}\mathfrak{U}L \rightarrow L$ such

that $\mathfrak{U}h \cdot \delta_{\mathfrak{U}L} = \text{id}$, and this, using the g from 2.1.1 again, makes $\varepsilon_L(U) = \bigvee U$. Finally, an ideal in $\mathfrak{U}L = (L, \sigma_L)$ is a down-set U such that $U = \downarrow \bigvee U$, hence

$$\varepsilon_L = (\downarrow a \mapsto a): \mathfrak{J}\mathfrak{U}(L) = \{\downarrow a \mid a \in L\} \rightarrow L.$$

□

2.4. Faithful sites. A site (S, ϕ) is said to be *faithful* if $\phi^*(\downarrow a) = \downarrow a$ for all $a \in S$. Later on, we will also need a weaker property: (S, ϕ) is said to be *dense* if just $\phi^*(\downarrow 0) = \downarrow 0$. So (S, ϕ) is dense if $\downarrow 0 = \{0\}$ is saturated, that is, if

$$\forall U \in \mathfrak{D}S \quad \forall c \in S \quad [U \cap \downarrow c = \{0\} \implies \phi(U) \cap \downarrow c = \{0\}]$$

2.4.1. PROPOSITION. $\delta_{(S, \phi)}$ is one-one iff (S, ϕ) is faithful.

Proof. Suppose $\phi(\downarrow a) \not\subseteq \downarrow a$. Choose $b \in \phi(\downarrow a) \setminus \downarrow a$. Then $b \not\leq a$ and hence $c \equiv b \wedge a < b$ while $\phi^*(\downarrow c) = \phi^*(\downarrow a) \cap \phi^*(\downarrow b) = \phi^*(\downarrow b)$. □

2.4.2. By 1.4.2 we can view **Frm** as a full subcategory of **Site**. Theorem 2.3.3 then can be interpreted as that it is a reflective subcategory with the reflection $\delta = (\delta_{(S, \phi)})_{(S, \phi)}$.

Restricted to the subcategory of faithful sites, this reflection is a monoreflection.

2.4.3. Notes.

1. Since the nucleus ϕ^* is often not quite transparent, the definition of faithfulness above may sound somewhat ad hoc, tailored to obtain proposition 2.4.1. In the next section we will see, however, that in the case of $\bigvee$ -sites (and this is the case we are particularly interested in) it has a very natural and transparent equivalent.

2. The expression “dense site” for the weaker property has a standard geometric meaning. The property amounts to the topological density of $\mathfrak{J}(S, \phi)$ as a sublocale of $\mathfrak{D}S$ — see 3.4 below.

3. Frame extensions of meet semilattices

3.1. A *frame extension of a semilattice* S is a (meet semilattice) homomorphism $\tau: S \rightarrow L$ with L a frame such that

- τ is one-to-one, and
- L is join-generated by $\tau[S]$.

The extensions are (pre)ordered by declaring for $\tau_i: S \rightarrow L_i$

$\tau_2 \leq \tau_1$ if there is a homomorphism $\rho: L_1 \rightarrow L_2$ such that $\rho \cdot \tau_1 = \tau_2$.

τ_1 and τ_2 are said to be *equivalent* if $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$, in which case the connecting ρ are obviously isomorphisms.

Note that the natural embedding $\alpha_S: S \rightarrow \mathfrak{D}S$ is a frame extension of S . It will be shown to be the largest one in the preorder $\leq$. More generally, $\delta_{(S, \phi)}$ are extensions if ϕ is faithful.

3.2. PROPOSITION. *A $\bigvee$ -site (S, ϕ_A) is faithful iff all the suprema of the form $\bigvee A$, $A \in \mathcal{A}$, distribute over all the elements of S . That is, $\delta_{(S, \phi_A)}$ is a frame extension of the semilattice S iff $\mathcal{A}$ is exact.*

Proof. If ϕ is exact we have by Remark (2) in 1.5 and 2.2.1 that $\phi(U) \cap \downarrow c = \phi(U \cap \downarrow c)$, and hence we can use the saturation condition in reduced form. We have $\downarrow \bigvee U \subseteq \downarrow a$ iff $U \subseteq \downarrow a$; thus each $\downarrow a$ is saturated, and we have $\phi^*(\downarrow a) = \downarrow a$.

Now suppose $\bigvee A$ is not exact, say $A \subseteq c \downarrow d$ but $b \wedge d \not\subseteq c$ for some $c < d$. Then $A \cap \downarrow d \subseteq \downarrow c$, while $\phi^*(A \cap \downarrow d) = \phi^*(A) \cap \phi^*(\downarrow d) = \downarrow b \cap \downarrow d \not\subseteq \downarrow c$. We have shown that $\phi^*(\downarrow c) \neq \downarrow c$, i.e., the site is not faithful. $\square$

3.3. The subcategory of **Site** generated by V -sites (resp. exact V -sites) will be denoted

$$\bigvee \mathbf{Site} \quad \text{resp.} \quad \bigvee \mathbf{Site}_{\text{ex}}$$

Since the trivial sites associated with frames are exact $\bigvee$ -sites, the reflection from 2.4.2 restricts to one of $\bigvee \mathbf{Site}$ onto **Frm** and a monoreflection of $\bigvee \mathbf{Site}_{\text{ex}}$ onto **Frm**.

Note that the definition of a site homomorphism $f: (S, \phi_A) \rightarrow (T, \phi_B)$ translates in $\bigvee \mathbf{Site}$ to

$$\forall A \in \mathcal{A} \quad \left(f\left(\bigvee A\right) = \bigvee f[A] \right). \quad (3.3.*)$$

3.4. Sublocales. A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X , and a continuous map $f: Y \rightarrow X$ can be represented as the frame homomorphism

$$\Omega(f) \equiv (U \mapsto f^{-1}[U]) : \Omega(X) \rightarrow \Omega(Y).$$

This leads to a natural view of frames as generalized spaces. To deal with the contravariance, one works with the category **Loc** opposite to **Frm**, and obtains the extension, or generalization, of spaces in the form of the covariant functor

$$\Omega: \mathbf{Top} \rightarrow \mathbf{Loc},$$

which is in fact a full embedding for a very broad class of spaces. Instead of the formal “inverted arrows” from **Frm**, the morphisms of **Loc** are represented as

the right Galois adjoints $f: M \rightarrow L$ of the corresponding frame homomorphisms $h: L \rightarrow M$. These maps are usually termed *localic morphisms* or *localic maps*.

Subspaces, that are in **Frm** represented as onto frame homomorphisms $h: L \rightarrow M$, become in this representation subsets $A \subseteq L$ called *sublocales* characterized by being closed under all meets and such that for every $a \in A$ and $x \in L$ the Heyting $x \rightarrow a$ is in A . See [8] and [9] or [7].

Among many advantages of this approach is the natural order of the sublocales by simple set inclusion (which replaces the preorder $h_1 \preceq h_2$ of the onto frame homomorphisms defined by the existence of g such that $gh_2 = h_1$). The system of all sublocales, ordered by inclusion is a coframe.

For our purposes it is important to realize that the ideal frames $\mathfrak{I}(S, \phi)$ are sublocales of $\mathfrak{D}S$.

3.4.1. PROPOSITION. *Each frame extension $\tau: S \rightarrow L$ is equivalent to a $\gamma\alpha_s: S \rightarrow L'$ with L' a sublocale of $\mathfrak{D}S$, and γ the corresponding, uniquely defined, frame homomorphism.*

Proof. Consider the $g: \mathfrak{D}S \rightarrow L$ from 2.1.1 such that $\tau = g\alpha$ and the right Galois adjoint $g_*: L \rightarrow \mathfrak{D}S$ of g . Then set $L' = g_*[L]$. The mapping $h: L \rightarrow L'$ defined by $h(x) = g_*(x)$ is easy to see to be a frame isomorphism. Set $\gamma = h\tau$. $\square$

Consequently, frame extensions of S can be studied as (special) sublocales of $\mathfrak{D}S$. Obviously, $\mathfrak{D}S$ (more precisely, $\alpha_S: S \rightarrow \mathfrak{D}S$ is the largest one, and $\tau_1 \leq \tau_2$ as defined above is the same as $L_1 \subseteq L_2$).

3.4.2. PROPOSITION. *Let $\mathcal{A}$ be exact. Then $\delta_{(S, \phi_{\mathcal{A}})}: S \rightarrow \mathfrak{D}(S, \phi_{\mathcal{A}})$ is the least frame extension of S respecting the joins of all the $U \in \mathcal{A}$.*

Proof. By (3.3.*), $\delta_{(S, \phi_{\mathcal{A}})}: S \rightarrow \mathfrak{D}(S, \phi_{\mathcal{A}})$ respects the joins of all the $U \in \mathcal{A}$.

Now for an extension $\tau: S \rightarrow L$ consider the $g: \mathfrak{D}S \rightarrow L$ from 2.1, defined by $g(U) = \bigvee \tau[U]$. Recall 2.2. We have $g(\downarrow \bigvee U) = \bigvee \tau[\downarrow \bigvee U] = \bigvee \downarrow \tau(\bigvee U) = \tau(\bigvee U) = \bigvee \tau[U] = g(U)$ and hence there is a ρ such that $\rho\phi^* = g$. $\square$

3.4.3. The ideals $U \in \mathfrak{I}(S, \phi_{\mathcal{A}})$. Recall 2.2.1 and 1.3.2. The characterization of an ideal (that is, saturated U) in the case of a $\bigvee$ -site $(S, \phi_{\mathcal{A}})$ transforms to

$$\forall A \in \mathcal{A} \quad \forall c \in S \quad \left(\{a \wedge c \mid a \in A\} \subseteq U \implies \bigvee A \wedge c \in U \right).$$

3.5. Two extremal sites associated with a meet semilattice. In a meet semilattice S consider the system $\mathcal{A}_S$ of all subsets $A \subseteq S$ with joins in S , and the system $\mathcal{A}_S^e$ of all the subsets $A \in \mathcal{A}_S$ whose joins are exact. Denote by

ϕ_S and ϕ_S^e the associated maps $\mathfrak{D}S \rightarrow \mathfrak{D}S$. Thus we have $\vee$ -sites (S, ϕ_S) and (S, ϕ_S^e) uniquely determined by S . Defining

$$\mathbf{V}\mathbf{Site}_{\text{all}} \quad (\mathbf{V}\mathbf{Site}_{\text{allex}})$$

as the categories of $\vee$ -sites with homomorphisms preserving all the (exact) existing joins, we see that they are full subcategories of $\mathbf{V}\mathbf{Site}$, the former generated by the (S, ϕ_S) 's and the latter by the (S, ϕ_S^e) 's.

Now taking into account that the σ_L of a frame L (see 1.3.2) is equal to both ϕ_L and ϕ_L^e , we have $\mathbf{Frm}$ represented as a full subcategory of both $\mathbf{V}\mathbf{Site}_{\text{all}}$ and $\mathbf{V}\mathbf{Site}_{\text{allex}}$, and we have the reflections

$$\begin{aligned} \delta_S &\equiv \delta_{(S, \phi_S)}: S \rightarrow \mathfrak{I}(S) = \mathfrak{I}(S, \phi_S), \quad \text{and} \\ \delta_S^e &\equiv \delta_{(S, \phi_S^e)}: S \rightarrow \mathfrak{I}^e(S) = \mathfrak{I}(S, \phi_S^e). \end{aligned}$$

The latter is a monoreflection (and hence also an epireflection), and it has a very special status, a status brought to light by a deep and beautiful result of Bruns and Lakser [5].

Recall that, in any reasonable category, an essential extension is a monomorphism $\delta: S \rightarrow T$ with the feature that any morphisms $\tau: T \rightarrow R$ such that $\tau \cdot \delta$ is monic is itself monic. An object I is injective if any morphism $\tau: S \rightarrow I$ lifts over any monic $\rho: S \rightarrow T$, i.e., there is some $\delta: T \rightarrow I$ for which $\delta \cdot \rho = \tau$. And if it exists, an essential monomorphism $\delta: S \rightarrow I$ into an injective object I is often referred to as the embedding of S into its injective envelope.

3.6. THEOREM. (Bruns and Lakser [5]) *The map $\delta_S^e: S \rightarrow \mathfrak{U}\mathfrak{I}^e(S)$ is the embedding of the semilattice S into its injective envelope.*

Proof. Bruns and Lakser proved this result for the category of unbounded meet semilattices. The same proof works, mutatis mutandis, for $\mathbf{SLat}$. $\square$

3.7. THEOREM. *All the frame extensions of a meet semilattice S constitute the interval $[\mathfrak{I}^e(S), \mathfrak{D}S]$ in the coframe of sublocales of $\mathfrak{D}S$.*

Proof. Consider the following diagram.

$$\begin{array}{ccccc} & & L & & \\ & \nearrow \hat{\tau} & \uparrow \tau & \nwarrow \hat{\delta} & \\ \mathfrak{D}S & \xleftarrow{\alpha} & S & \xrightarrow{\delta_S^e} & \mathfrak{I}^e(S) \end{array}$$

Given a frame extension τ , with L join-generated by $\tau[S]$, let $\hat{\tau}$ be the extension of τ described in 2.1.1. Note that $\hat{\tau}$ is onto because S generates L . Then let $\hat{\delta}$

be the extension of δ_S^e guaranteed by the injective property of $\mathfrak{I}^e(S)$. Again, $\widehat{\delta}$ is onto because S generates $\mathfrak{I}^e(S)$. $\square$

It is worth noting that Theorem 3.7 can be deduced from Proposition 3.4.2, thereby avoiding use of the Bruns-Lakser result.

3.8. It is reasonable to ask at this point how one might recognize when a given frame extension generated by a meet semilattice is minimum in the order imposed.

PROPOSITION. *The following are equivalent for a frame extension $S \rightarrow L$ generated by a meet semilattice S .*

- (1) $S \rightarrow L$ is least among such extensions, i.e., any other is an initial factor of $S \rightarrow L$.
- (2) $S \rightarrow L$ is isomorphic to δ_S^e over S .
- (3) $S \rightarrow L$ is an essential monomorphism in **SLat**.
- (4) Any frame homomorphism which identifies two distinct points of L must identify two distinct points of S .
- (5) For all $a < b$ in L there exist $c < d$ in S such that $b \vee c \geq d$ and $a \wedge d \leq c$.

Proof. The equivalence of (1) and (2) is the content of Theorem 3.7 and the equivalence of (3) and (4) is a consequence of the fact that monomorphisms in **SLat** are one-one. The equivalence of (4) and (5) follows from the well known fact that, assuming $a < b$ and $x < y$, the finest frame congruence $\sim$ which identifies a with b is

$$x \sim y \iff y \vee a \geq b \text{ \& } x \wedge b \leq a.$$

(2) implies (3), for the Bruns-Lakser Theorem 3.6 informs us that δ_S^e is an essential extension in **SLat**. And (3) implies (2), for if (3) holds for the extension $S \rightarrow L$ then when Theorem 3.7 is applied to it the result is that $\widehat{\delta}$ is one-one and therefore an isomorphism. $\square$

4. The Dedekind-MacNeille Coverage, Boolean and Heyting Cases

4.1. Recall the Dedekind-MacNeille coverage from 1.2.1(b) and 1.3.2. It is well-known that the Dedekind-MacNeille completion of a poset $(X, \leq)$ can be constructed as

$$\text{DM } X \equiv \{\text{lb}(\text{ub } U) \mid U = \downarrow U\},$$

ordered by inclusion. In particular for a meet semilattice

$$\mathbf{DM} S = \{U \in \mathfrak{D}S \mid \phi_{\mathbf{DM}}(U) = U\}.$$

The starting poset X is embedded into $\mathbf{DM} X$ by sending x to $\downarrow x$. All existing joins and meets in X remain valid in $\mathbf{DM} X$, which is characterized by the fact that it is complete and that every point of $\mathbf{DM} X$ is both a join and a meet of elements of X .

In addition to this completion let us now introduce, for meet-semilattices S , the *Dedekind-MacNeille frame completion*

$$\mathbf{DMF}(S) = \mathfrak{I}(S, \phi_{\mathbf{DM}}).$$

This typically has to differ from $\mathbf{DM}(S)$, for instance because of the fact that $\mathbf{DM}(S)$ is not necessarily distributive, and it is in general not necessarily distributive even if S is a distributive lattice. But $\mathbf{DMF}(S)$ and $\mathbf{DM}(S)$ do coincide for Heyting algebras as we will show below. We will first prove the fact for the Boolean case and add a remark on Booleanization. There the formulas are fairly explicit. Then we will proceed with the general Heyting case where, however, we will have to use indirect reasoning.

4.2. The coverage $\phi_{\mathbf{DM}}$ is monotone and has the property that $\phi_{\mathbf{DM}}(\phi_{\mathbf{DM}}(U)) = \phi_{\mathbf{DM}}(U)$, which is reminiscent of a nucleus. This it cannot be, however, for it does not generally preserve meets, and hence we have only the trivial inequality $\phi_{\mathbf{DM}} \leq \phi_{\mathbf{DM}}^*$.

4.2.1. Example. For the Chinese lantern $S = \{0 < a, b, c < 1\}$ and $x = a, b, c$ we have $\phi_{\mathbf{DM}}(\downarrow x) = \downarrow x$ and $\phi_{\mathbf{DM}}^*(\downarrow x) = S$. The latter follows from $\phi_{\mathbf{DM}}(\downarrow \{x, y\}) = S$ for $x \neq y$ and, e.g., $\downarrow \{a, b\} \cap \downarrow \{a, c\} = \downarrow a$. This is of course a trivial example. We will present a more interesting one with an S “as distributive as possible” in 4.8 below.

4.3. Polars. The polar of $U \in \mathfrak{D}S$ is defined as

$$U^\perp \equiv \{a \mid (\forall x \in U)(a \wedge x = 0)\}.$$

4.3.1. LEMMA. $U^\perp$ is the pseudocomplement of U in $\mathfrak{D}S$. Consequently, $\nu(U) = U^{\perp\perp}$ is a nucleus.

Proof. $A = U \cap U^\perp = \{0\}$ since $u \in A$ implies $u \wedge u = 0$. If $V \cap U = \{0\}$ and $a \in V$ and $u \in U$ then $a \wedge u \in U \cap V$ and hence it is 0. $\square$

4.3.2. PROPOSITION. Let (S, ϕ) be dense (in particular, faithful, recall 2.4). Then $U^\perp$ is the pseudocomplement of U in $\mathfrak{I}(S, \phi)$.

Proof. It suffices to prove that $P \equiv U^\perp$ is an ideal. Suppose not, i.e., suppose that $\phi^*(P) \not\subseteq P$, say $c \in \phi^*(P) \setminus P$. Since $c \notin P = P^{\perp\perp}$, there is some $x \in P^\perp$ such that $d \equiv x \wedge c \neq 0$. This would yield

$$d \in \phi^*(P) \cap \downarrow d \subseteq \phi^*(P) \cap \phi^*(\downarrow d) = \phi^*(P \cap \downarrow d) = \phi^*(\downarrow 0) = \{0\}.$$

From this we conclude $d = 0$, contrary to assumption. $\square$

4.4. Pseudocomplemented meet-semilattices. The pseudocomplement of x , if it exists, will be denoted by x^* . Further we will set

$$*U = \{u^* \mid u \in U\}$$

4.4.1. PROPOSITION. *Let S be pseudocomplemented, Then (S, ϕ_{DM}) is dense.*

Proof. We have to prove that $\downarrow 0$ is saturated, that is (recall 2.2.1) that

$$\forall U \in \mathfrak{D}S \quad \forall c \in S \quad [U \cap \downarrow c = \{0\} \implies \phi(U) \cap \downarrow c = \{0\}]$$

Now if $U \cap \downarrow c = \{0\}$ then $U \subseteq \downarrow c^*$, that is, $c^* \in \text{ub } U$, and hence if $x \in \text{lb ub } U$ we have in particular $x \leq c^*$ and if it is, moreover, in $\downarrow c$ we have $x = 0$. $\square$

4.4.2. LEMMA. *In a pseudocomplemented meet-semilattice we have*

- (1) $U^\perp = \text{lb}(*U)$ and
- (2) $*\text{lb}(*U) \subseteq \text{ub } U$.

If S is Boolean then

$$*\text{lb}(*U) = \text{ub } U.$$

Proof.

- (1) $a \in U^\perp$ iff for all $u \in U$, $a \leq u^*$.
- (2) Let $x \leq u^*$ for all $u \in U$. Then $x^* \geq u^{**} \geq u$ for all $u \in U$.

Now let S be Boolean and $x \geq u$ for all $u \in U$. Then $x^* \leq u^*$ for all $u \in U$ and hence $x^* \in \text{lb}(*U)$. As $x = x^{**}$, $x \in *\text{lb}(*U)$. $\square$

4.4.3. Denote by $\mathfrak{B}(L)$ the standard Booleanization $\{x \in L \mid x = x^{**}\}$ of a frame L . We have

THEOREM. *For a Boolean algebra B ,*

$$\text{DM}(B) = \text{DMF}(B) = \mathfrak{B}(\mathfrak{D}(B)).$$

Proof. By 4.4.2 we have $(U^\perp)^\perp = \text{lb}(*U^\perp) = \text{lb}(*(\text{lb}(*U))) = \text{lb ub}(U) = \phi_{\text{DM}}(U)$. Thus, ϕ_{DM} coincides with the Booleanization nucleus $\beta = (U \mapsto (U^\perp)^\perp)$. $\square$

4.5. The previous theorem concerned Boolean algebras B with the Dedekind-MacNeille coverage. But it is also natural to consider them endowed with the maximal $\bigvee$ -coverage. (Recall that all suprema are exact in a Boolean algebra; see 1.5(3).) We have

THEOREM. *Let B be a Boolean algebra viewed as a $\bigvee$ -site with all existing joins. Then*

- (1) $\mathfrak{I}B$ is a Boolean algebra, and
- (2) $\mathfrak{I}B = \text{DM } B$.

PROOF. We will show that $U^{\perp\perp} = U$ for every $U \in \mathfrak{I}B$. Suppose not, i.e., suppose there exists some $b \in U^{\perp\perp} \setminus U$. This b , then, cannot be a supremum of $M = \{c \wedge b \mid c \in U\}$, whether $\bigvee U$ exists or not. Since b is an upper bound of M there has to be an upper bound a of M such that $a \not\leq b$. Thus we have, first, that $b \wedge a^* \neq 0$ and moreover that

$$\forall c \in U \quad [c \wedge (b \wedge a^*) = (c \wedge b) \wedge a^* \leq a \wedge a^* = 0],$$

with the result that $b \wedge a^* \in U^\perp$, yielding the contradiction $b \wedge (b \wedge a^*) = b \wedge a^* = 0$.

(2) Now the second statement follows from 4.4.3. $\square$

4.6. PROPOSITION. *Let P be an arbitrary polar in $\mathfrak{D}S$. Then for every exact $\mathcal{A} \subseteq \mathfrak{D}S$, $P \in \mathfrak{I}(S, \phi_{\mathcal{A}})$.*

PROOF. Recall 3.4.3. Let $P = U^\perp$. If $A \cup \downarrow c \subseteq U^\perp$ we have for each $a \in A$ and $u \in U$, $a \wedge c \wedge u = 0$. Hence $\bigvee A \wedge c \wedge u = \bigvee \{a \wedge c \wedge u \mid a \in A\} = 0$ and $\bigvee A \wedge c \in U^\perp$. $\square$

4.6.1. COROLLARY. *The Booleanization $\mathfrak{B}(\mathfrak{I}(S, \phi_{\mathcal{A}}))$ of any $\mathfrak{I}(S, \phi_{\mathcal{A}})$ with exact $\mathcal{A}$ coincides with $\mathfrak{B}(\mathfrak{D}S)$, that is, with $\mathfrak{I}(S, \phi_{\text{DM}})$.*

4.7. The Heyting case We will use the well-known but non-trivial fact that for a Heyting algebra S the Dedekind-MacNeille completion $\text{DM}(S)$ is Heyting and hence a frame.

4.7.1. PROPOSITION. *Let S be a Heyting algebra. Then the associated coverage ϕ_{DM} is a nucleus. As a consequence we see that it is faithful.*

PROOF. For the embedding into the frame

$$\gamma = (x \mapsto \downarrow x): S \rightarrow \text{DM}(S)$$

consider by 2.1.1 the homomorphism $g: \mathfrak{D}S \rightarrow \text{DM}(S)$ such that $g \cdot \alpha = \gamma$, that is, $g(\downarrow x) = \gamma(x) = \downarrow x$ for all $x \in S$. We have $g(U) = \bigvee \gamma[U]$ and hence

$$g(U) = \text{lub} \left(\bigcup \{ \downarrow x \mid x \in U \} \right) = \phi_{\text{DM}}(U).$$

The congruence associated with g is

$$E \equiv \{(U, V) \mid g(U) = g(V)\}.$$

and in turn the nucleus associated with E is

$$\nu_E(U) = \bigvee EU = g(U)$$

(if VEU then $V \leq g(V) = g(U)$). But then $\phi_{\text{DM}}(U) = g(U) = \nu_E(U)$, and hence $\phi_{\text{DM}} = \nu_E$ is a nucleus. $\square$

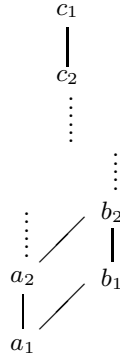
4.7.2. COROLLARY. *Let S be a Heyting algebra. Then*

$$\text{DMF}(S) = \mathfrak{I}(S, \phi_{\text{DM}}) = \text{DM}(S).$$

4.7.3. PROPOSITION. *Let S be a Heyting algebra. Then $\text{DM}(S) = \mathfrak{I}^e(S)$.*

Proof. The frame extension $\gamma: S \rightarrow \text{DM}(S)$ preserves all existing joins and hence $\gamma \leq \delta_S^e$. $\square$

4.8. Example. One cannot, of course, expect the coincidence of DM and $\mathfrak{I}$ to go much further. Consider the following meet-semilattice.



Let $L \equiv \{a_n, b_n, c_n \mid n \in \mathbb{N}\}$, partially ordered as follows: $a_n \leq a_{n+1}$, $b_n \leq b_{n+1}$, $a_n \leq b_n$, and $a_n, b_n \leq c_k$ for all $n, k \in \mathbb{N}$. By inspection, no infinite subsets have suprema, hence $\mathfrak{I}L = \mathfrak{D}L$ contains $U \equiv \{a_n\}$, while $\text{lb}(\text{ub } U) = \{a_n, b_n\}$. All the finite suprema distribute and hence all the existing suprema do.

Of course this example is not Heyting, and not even pseudocomplemented.

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