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UNIQUENESS OF UNIFORM NORM AND C^*-NORM IN $L^p(G,\omega)$

Ibrahim Akbarbaglu — Majid Heydarpour — Saeid Maghsoudi

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ABSTRACT. Let G be a locally compact abelian group with a fixed Haar measure and ω be a weight on G. For $1 < p < \infty$, we study uniqueness of uniform and C^*-norm properties of the invariant weighted algebra $L^p(G,\omega)$.

1. Introduction

A uniform norm on an algebra $\mathfrak{A}$ is a (not necessarily complete) norm $|\cdot|$ on $\mathfrak{A}$ satisfying the square property $|x^2| = |x|^2$ for all $x \in \mathfrak{A}$. In fact, in the presence of the square property, submultiplicativity is automatic. Notice that any two equivalent uniform norms are equal. Thus two uniform norms are either identical or different. If $|\cdot|$ is a uniform norm on $\mathfrak{A}$, then $\mathfrak{A}$ is semisimple and commutative; further, if $(\mathfrak{A}, \|\cdot\|)$ is a Banach algebra, then the spectral radius on $\mathfrak{A}$ is the greatest uniform norm on $\mathfrak{A}$. A Banach algebra $\mathfrak{A}$ has the unique uniform norm property ($UUNP$) if it admits exactly one uniform norm; in this case, the spectral radius on $\mathfrak{A}$ is the only uniform norm. Banach algebras with unique uniform norm have been introduced and extensively studied in [3], [4] and [5].

A C^*-norm on a $*$-algebra $\mathfrak{B}$ is an (not necessarily complete) algebra norm $|\cdot|$ on $\mathfrak{B}$ satisfying the C^*-property $|x^*x| = |x|^2$ for all $x \in \mathfrak{B}$. Like uniform norms, any two equivalent C^*-norms on $\mathfrak{B}$ are identical. A Banach $*$-algebra $\mathfrak{B}$

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has the unique C^*-norm property (UC^*NP) if it admits exactly one C^*-norm. This property was first studied by B. A. Barnes in [1]; for more recent studies see [5]. In an arbitrary $*$-semisimple, commutative Banach $*$-algebra, the $UUNP$ is stronger than the UC^*NP.

Let G be a locally compact (Hausdorff) abelian group with a fixed Haar measure. In this paper, for $1 < p < \infty$, we study $UUNP$ and UC^*NP for invariant algebra $L^p(G,\omega)$, where ω is a weight on G. In fact, under mild conditions on a weight ω, we prove that $UUNP$ and UC^*NP for the invariant algebra $L^p(G,\omega)$ are equivalent to the regularity of $L^p(G,\omega)$. Our results extend some interesting results presented in [2] for Beurling algebras.

2. Preliminaries

In this section we give definitions and prove some basic lemmas which are needed for the rest of the paper.

Let $\mathfrak{A}$ be a semisimple, commutative Banach algebra with the complete norm $\|\cdot\|$, $\Delta(\mathfrak{A})$ be the Gelfand space of $\mathfrak{A}$, and $\widehat{x}\colon \Delta(\mathfrak{A}) \to \mathbb{C}$, $\varphi \mapsto \varphi(x)$, be the Gelfand transform of an element x in $\mathfrak{A}$. For a closed subset E of $\Delta(\mathfrak{A})$, define

$$|x|_E = |\widehat{x}|_E := \sup\big\{|\widehat{x}(\varphi)| : \ \varphi \in E\big\}, \qquad (x \in \mathfrak{A}).$$

Then $|\cdot|_E$ is a uniform seminorm on $\mathfrak{A}$ and it is dominated by the spectral radius; the seminorm $|\cdot|_E$ is a uniform norm on $\mathfrak{A}$ if and only if E is a set of uniqueness for $\mathfrak{A}$, that is, for any $x \in \mathfrak{A}$, if $\varphi(x) = 0$ for all $\varphi \in E$, then $x = 0$. On the other hand, let $|\cdot|$ be a uniform norm on $\mathfrak{A}$. Set

$$E := \big\{\varphi \in \Delta(\mathfrak{A}) : \ \varphi \ \text{is} \ |\cdot|\text{-continuous}\big\}.$$

Then E is the largest closed subset of $\Delta(\mathfrak{A})$ such that $|x| = |x|_E$ for all $x \in \mathfrak{A}$.

Let $\mathfrak{B}$ be a $*$-semisimple, commutative Banach $*$-algebra. A complex homomorphism $\varphi \in \Delta(\mathfrak{B})$ is called self-adjoint if $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in \mathfrak{B}$. Let $\widetilde{\Delta}(\mathfrak{B}) := \big\{\varphi \in \Delta(\mathfrak{B}) : \ \varphi \ \text{is self-adjoint}\big\}$. Then it is easy to see that $\widetilde{\Delta}(\mathfrak{B})$ is closed in $\Delta(\mathfrak{B})$. If E is a closed subset of $\widetilde{\Delta}(\mathfrak{B})$ and if E is a set of uniqueness for $\mathfrak{B}$, then $|\cdot|_E$ is a C^*-norm on $\mathfrak{B}$. Conversely, if $|\cdot|$ is a C^*-norm on $\mathfrak{B}$, then there exists a largest closed subset E of $\widetilde{\Delta}(\mathfrak{B})$ such that $|x| = |x|_E$ for all $x \in \mathfrak{B}$. Now it is clear that every C^*-norm is a uniform norm.

Throughout the paper, let G be a locally compact abelian (LCA) group with a fixed Haar measure λ. Let ω be a *weight* function on G; that is, a strictly positive Borel measurable function on G. For $1 < p < \infty$, denote by $L^p(G,\omega)$ the set of all complex-valued Borel measurable functions f on G such that $f\omega \in L^p(G)$; the usual Lebesgue space as defined in [10]. Then $L^p(G,\omega)$ is a Banach space with the norm $\|\cdot\|_{p,\omega}$ defined by

$$\|f\|_{p,\omega} = \left(\int\limits_G |f\omega|^p \,\mathrm{d}\lambda \right)^{1/p} \qquad (f \in L^p(G,\omega)).$$

The dual space of $L^p(G,\omega)$ is the Banach space $L^q(G,1/\omega)$ under the duality

$$\langle f,g\rangle = \int\limits_G f(x)g(x)\,\mathrm{d}\lambda(x) \qquad (f \in L^p(G,\omega),\quad g \in L^q(G,1/\omega)),$$

where q is the exponential conjugate to p. For measurable functions f and g on G, the convolution multiplication

$$f * g(x) = \int\limits_G f(y)g(y^{-1}x)\,\mathrm{d}\lambda(y)$$

is defined at each point $x \in G$ for which this makes sense. For $p=1$, it is well known that $L^p(G,\omega)$ is an algebra if and only if ω is weakly *submultiplicative*; that is, there exists a constant $c \geq 1$ such that

$$\omega(xy) \leq c\,\omega(x)\omega(y)$$

for all $x,y \in G$; for a study of some aspects of these algebras see, for example, [7] and [9]. For $1 < p < \infty$, a sufficient condition for $L^p(G,\omega)$ be an algebra is

$$\omega^{-q} * \omega^{-q} \leq \omega^{-q}$$

locally almost everywhere, where q is exponential conjugate to p. But this condition is not necessary. In fact, Kuznetsova in [12] has constructed a weight ω on any free group G of infinite cardinality such that $L^p(G,\omega)$ is a Banach algebra for $1 < p \leq 2$, while ω does not satisfy the above mentioned condition.

Weights ω and ν are called equivalent if, for some constants c_1, c_2,

$$0 < c_1 \leq \frac{\omega}{\nu} \leq c_2 < \infty$$

holds locally almost everywhere. In this case, the spaces $L^p(G,\omega)$ and $L^p(G,\nu)$ are topologically isomorphic for any $p \geq 1$.

The space $L^p(G,\omega)$ is called (left translation) *invariant* if $L_y f \in L^p(G,\omega)$ provided that $f \in L^p(G,\omega)$, where $L_y f(x) = f(y^{-1}x)$ for $x,y \in G$. We say $L^p(G,\omega)$ is an *invariant algebra* when $L^p(G,\omega)$ is left translation invariant and closed under the convolution multiplication.

The following proposition shows that our conditions for weights do not affect the generality as far as algebraic and topological properties are concerned; for a proof, see [13: Lemma 2] and [15: Theorem 3].

Proposition 2.1. *Let G be a LCA group, ω be a weight on G, $1 < p < \infty$ and q be the exponential conjugate to p. Then the following statements hold.*

(i) *If $L^p(G,\omega)$ is an invariant algebra and $\omega^{-q} \in L^1(G)$, then ω is equivalent to a submultiplicative weight ν with $\nu \geq 1$.*

(ii) *Each algebra $L^p(G,\omega)$ is isometrically isomorphic to an algebra $L^p(G,\nu)$ for some weight ν on G with $\nu^{-q} \in L^1(G)$.*

By an ω-bounded generalized character on G we mean a nonzero complex-valued continuous function ρ on G such that

$$|\rho(x)| \leq \omega(x) \quad (x \in G) \qquad \text{and} \qquad \rho(xy) = \rho(x)\rho(y) \quad (x, y \in G).$$

Let $\widehat{G}(\omega)$ denote the set of all ω-bounded generalized characters on G equipped with the topology of uniform convergence on compact subsets of G. If ω is weakly submultiplicative, then the map

$$T\colon \widehat{G}(\omega) \cap L^q(G, 1/\omega) \to \Delta(L^p(G,\omega))$$

defined as $T(\rho) = \varphi_\rho$, where

$$\varphi_\rho(f) = \int_G f(x)\rho(x)\,\mathrm{d}\lambda(x) \qquad (f \in L^p(G,\omega))$$

and $1/p + 1/q = 1$, is a homeomorphism. So the Gelfand topology and the topology of uniform convergence on compact subsets of G coincide in this case; the proof is analogous to the corresponding result for $L^1(G,\omega)$; see, for example, [11].

For a complex-valued function f on G define $f^*(x) = \overline{f(x^{-1})}$. A weight function ω on G is called symmetric if $\omega(x) = \omega(x^{-1})$ for all $x \in G$. The following result has been proved in [15].

Proposition 2.2. *Let G be a LCA group, ω be a weight on G, and $1 < p < \infty$. If $L^p(G,\omega)$ is a Banach algebra, then it is semisimple and its spectrum contains a homeomorphic image of the dual group $\widehat{G}$. Moreover, if ω is a symmetric weight, then $L^p(G,\omega)$ with $*$ as an involution is a semisimple Banach $*$-algebra.*

We always have $\theta\rho \in \Delta(L^p(G,\omega))$ for all $\theta \in \widehat{G}$ and $\rho \in \Delta(L^p(G,\omega))$. We note that for each $\rho \in \Delta(L^p(G,\omega))$, the map defined by

$$|f|_\rho = \sup\big\{|\varphi_{\theta\rho}(f)| : \ \theta \in \widehat{G}\big\}$$

for $f \in L^p(G,\omega)$ is a uniform norm on $L^p(G,\omega)$.

We need the following two lemmas in the sequel.

Lemma 2.1. *Let G be a LCA group and $E \neq F$ be closed subsets of $\widehat{G}$. If $|\cdot|_E$ and $|\cdot|_F$ are uniform norms, then they are distinct.*

Proof. Let $\theta \in E \setminus F$. By the regularity of $L^1(G)$, there exists a nonzero function $f \in L^1(G)$ such that $\widehat{f}(F) = \{0\}$ and $\widehat{f}(\theta) = 1$. Pick $g \in C_c(G)$ such that $\|f-g\|_1 < 1/3$. Then $g \in L^p(G,\omega)$, and

$$|\widehat{g}(\rho)| = |\widehat{g}(\rho) - \widehat{f}(\rho)| \leq \ \|\widehat{f} - \widehat{g}\|_\infty \leq \|f-g\|_1 < 1/3 \qquad (\rho \in F)$$

Moreover,

$$|\widehat{g}(\theta)| \geq |\widehat{f}(\theta)| - |\widehat{f}(\theta) - \widehat{g}(\theta)| \geq 1 - \|f-g\|_1 > 2/3$$

Thus $|g|_F \leq 1/3 < |g|_E$. This completes the proof. □

In the sequel, for a weight ω on G and any $x \in G$ we set

$$\mathfrak{L}(x) = \sum_{n\in\mathbb{Z}} \frac{\ln \omega(x^n)}{1+n^2}.$$

Lemma 2.2. *Let G be a LCA group and $\omega \geq 1$ be a submultiplicative weight on G. Then $\widehat{G} = \widehat{G}(\omega)$ if $\mathfrak{L}(x)$ is finite for all $x \in G$.*

Proof. Suppose $\mathfrak{L}(x)$ is finite for all $x \in G$. It is clear that $\widehat{G} \subseteq \widehat{G}(\omega)$. For proving the reverse inclusion, suppose that there exists $\theta \in \widehat{G}(\omega) \setminus \widehat{G}$. Then there is $x \in G$ such that $|\theta(x)| \neq 1$. We may assume $|\theta(x)| > 1$, because $\theta(x)\theta(x^{-1}) = 1$. We have

$$\lim_{n\to\infty} \frac{\ln \omega(x^n)}{n} \geq \lim_{n\to\infty} \frac{|\theta(x^n)|}{n} > \ln|\theta(x)| \neq 0,$$

which contradicts our assumption on finiteness of $\mathfrak{L}(x)$. Thus, $\widehat{G}(\omega) \subseteq \widehat{G}$, and hence $\widehat{G} = \widehat{G}(\omega)$. □

Let us remark that the converse of Lemma 2.2 does not hold; for a counterexample see [1: Example 2.8].

3. Main results

We start by the following proposition which is a direct consequence of [5: Theorem 2.2] and Proposition 2.2

Proposition 3.1. *Let G be a LCA group, ω be a weight on G, and $1 < p < \infty$. Then the following statements hold.*

(i) *Let $L^p(G,\omega)$ be an invariant algebra. Then $L^p(G,\omega)$ admits either exactly one uniform norm or infinitely many uniform norms.*

(ii) *If ω is symmetric, then the invariant algebra $L^p(G,\omega)$ has either exactly one C^*-norm or infinitely many C^*-norms.*

We need the following lemma for proving our main theorem. The proof is similar to [3: Proposition 4.4], and we omit it.

Lemma 3.1. *Let G be a LCA group, ω be a weight on G, and $1 < p < \infty$. Let $L^p(G,\omega)$ be an invariant algebra. If $L^p(G,\omega)$ has the $UUNP$, then for any $\theta \in \Delta(L^p(G,\omega))$, the map $T_\theta \colon \widehat{G} \to \Delta(L^p(G,\omega))$, $T_\theta(\rho) = \rho\theta$ is a homeomorphism.*

Let us recall that a Banach algebra $\mathfrak{A}$ is called regular if given any closed subset E of $\Delta(\mathfrak{A})$ and $\varphi \in \Delta(\mathfrak{A}) \setminus E$, there exists $x \in \mathfrak{A}$ such that $\varphi(x) \neq 0$ and $\psi(x) = 0$ for all $\psi \in E$.

The next result is our main theorem.

Theorem 3.2. *Let G be a LCA group, ω be a weight on G, and $1 < p < \infty$. Let $L^p(G,\omega)$ be an invariant algebra. Then the following are equivalent.*

(a) *$L^p(G,\omega)$ has a minimum uniform norm.*

(b) *$L^p(G,\omega)$ has $UUNP$.*

(c) *$L^p(G,\omega)$ is regular.*

Proof. Since every regular semisimple commutative Banach algebra has a minimum uniform norm, we only have to show the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

(a) $\Longrightarrow$ (b). Assume that $L^p(G,\omega)$ has a minimum uniform norm $|\cdot|$. Let $(\tilde{L}^p(G,\omega), |\cdot|)$ be the completion of $L^p(G,\omega)$ with respect to $|\cdot|$. Since $L^p(G,\omega)$ is dense in $\tilde{L}^p(G,\omega)$ and elements of $\Delta(\tilde{L}^p(G,\omega))$ are continuous, the restriction map is a bijection from $\Delta(\tilde{L}^p(G,\omega))$ onto the set of all elements of $\Delta(L^p(G,\omega))$ which are continuous with respect to the norm $|\cdot|$ on $\Delta(L^p(G,\omega))$. This is nothing but the set

$$E = \big\{\theta \in \Delta(L^p(G,\omega)) : \ |\varphi_\theta(f)| \le |f| \ \text{ for all } \ f \in L^p(G,\omega)\big\}.$$

Because $|\cdot|$ is a uniform norm on $\tilde{L}^p(G,\omega)$, for $f \in \tilde{L}^p(G,\omega)$ we have $|f^{2^n}| = |f|^{2^n}$, for all natural numbers n, and hence $|f| = \tilde{r}(f)$; the spectral radius of $\tilde{L}^p(G,\omega)$. For $f \in L^p(G,\omega)$, it follows that

$$|f| = \tilde{r}(f) = \sup\big\{|\varphi_\theta(f)| : \ \theta \in E\big\} = |f|_E.$$

So E is the set of uniqueness for $L^p(G,\omega)$. Note also that

$$\begin{aligned}\tilde{r}(f) &= \sup\big\{|\varphi(f)| : \ \varphi \in \Delta(\tilde{L}^p(G,\omega))\big\}\\ &\le \sup\big\{|\varphi_\theta(f)| : \ \theta \in \Delta(L^p(G,\omega))\big\}\\ &= r(f).\end{aligned}$$

Now, we only need to prove that $E = \Delta(L^p(G,\omega))$, from which it follows that $|\cdot|$ is the unique uniform norm.

For this end, let us note that $E = \widehat{G}E = \big\{\theta\rho : \ \theta \in \widehat{G} \text{ and } \rho \in E\big\}$. For proving this, we fix $\theta \in \widehat{G}$. We have $|f|_{\theta E} = |\theta f|_E$ for all $f \in L^p(G,\omega)$. Then $|\cdot|_{\theta E}$ is a uniform norm on $L^p(G,\omega)$. Since $|\cdot|_E$ is the minimum uniform norm on $L^p(G,\omega)$, we have

$$|f|_E \le |f|_{\theta E} \qquad (f \in L^p(G,\omega)).$$

This holds for each $\theta \in \widehat{G}$. So for any $\theta \in \widehat{G}$ and $f \in L^p(G,\omega)$, we have

$$|f|_{\overline{\theta}E} = |\overline{\theta} f|_E \le |f|_E.$$

Thus for each $\rho \in E$, the complex homomorphism $\varphi_{\overline{\theta}\rho}$ is $|\cdot|$-continuous. This means that $\overline{\theta}E \subseteq E$, and since $\theta \in \widehat{G}$ is arbitrary, $E = \widehat{G}E$.

Now, suppose, towards a contradiction, that $\rho_0 \in \Delta(L^p(G,\omega)) \setminus E$. Choose $\theta_0 \in E$ and let $\theta = |\theta_0|$ and $\rho = |\rho_0|$. Then $\theta \in E$ and $\rho \in \Delta(L^p(G,\omega)) \setminus E$. Pick $x \in G$ such that $\rho(x) < \theta(x)$. Let U be a relatively compact open neighbourhood of x in G and $\rho(s) < \theta(s)$ for each $s \in U$. Take $f = \chi_U \in L^p(G,\omega)$, the characteristic function of U. Then $|f|_\rho < |f|_\theta \le |f|_E$. Since $|\cdot|_E$ is the minimum uniform norm on $L^p(G,\omega)$, we have a contradiction. This completes the proof of (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (c). Let E be a proper closed subset of $\Delta(L^p(G,\omega))$ and let $\varrho \in \Delta(L^p(G,\omega)) \setminus E$. By Lemma 3.1, $F = \big\{\theta \in \widehat{G} : \ \theta\varrho \in E\big\}$ is a closed subset of $\widehat{G}$ which does not contain the trivial character 1_G. Choose a symmetric open neighbourhood U of 1_G in $\widehat{G}$ such that $F \cap U^2 = \varnothing$. Then $E \cap \varrho U^2 = \varnothing$. Because $L^p(G,\omega)$ is semisimple and has the $UUNP$ such that its Shilov boundary coincides with its spectrum, we conclude that $L^p(G,\omega)$ is weakly regular; see for example [4: p. 581] or [11: Corollary 4.6.7]. Thus there exists a nonzero function $g \in L^p(G,\omega)$ such that $\widehat{g}(\sigma) = 0$ for all $\sigma \in \widehat{G} \backslash \varrho U$. Since g is nonzero, $\widehat{g}(\varrho\theta_0) \ne 0$

for some $\theta_0 \in U$. Now, let $f = \bar{\theta_0} g$. Then $f \in L^p(G,\omega)$ and $\widehat{f}(\rho) = \widehat{g}(\rho\theta_0) = 0$ for all $\rho \in E$. This shows that $L^p(G,\omega)$ is regular. □

The following result has been proved in [13], we, for completeness, give a slightly simpler proof; see also [8].

Proposition 3.3. *Let G be a LCA group, ω be a weight on G, and $1 < p < \infty$. Let $L^p(G,\omega)$ be an invariant algebra. Then $L^p(G,\omega)$ is regular if and only if $\mathfrak{L}(x)$ is finite for all $x \in G$*

Proof. First note that regularity is preserved under algebra isomorphisms. This together with Proposition 2.1 allow us to assume that ω is submultiplicative, continuous, and $\omega \geq 1$. In particular $L^1(G,\omega)$ is an algebra.

If $\mathfrak{L}(x)$ is finite for all $x \in G$, then, by Domar Theorem, the algebra $L^1(G,\omega)$ separates points and closed sets in $\widehat{G}(\omega) = \widehat{G}$. Since, by our assumption, ω is submultiplicative, it can be easily verified that $L^p(G,\omega)$ is an $L^1(G,\omega)$-module, hence the Gelfand transform on $L^1(G,\omega) * L^p(G,\omega)$ coincide with the Fourier transform and this also implies that $L^p(G,\omega)$ separates points and closed sets in $\widehat{G}$. But, Proposition 2.1 yields that

$$\Delta(L^p(G,\omega)) = \widehat{G}(\omega) \cap L^q(G,\omega^{-1}) = \widehat{G},$$

and this establishes the regularity of $L^p(G,\omega)$.

Conversely, if the invariant algebra $L^p(G,\omega)$ separates points and closed sets in its spectrum, then this also holds in the dual group $\widehat{G} \subseteq \Delta(L^p(G,\omega))$. We claim that this also holds for $L^1(G,\omega)$. Take $\theta \in \widehat{G}$ and an open set U containing θ. We prove that there exists a function $f \in L^1(G,\omega)$ with $\hat{f}(\theta) = 1$ and $\operatorname{supp}(\hat{f}) \subseteq U$. Since $\widehat{G}$ is a locally compact group, there is a nonempty open subset $V \subseteq \widehat{G}$ such that $\theta V \subseteq \theta V^2 \subseteq U$. By the assumption, we can choose an element $f_1 \in L^p(G,\omega)$, $f_1 \neq 0$ with $\hat{f}_1(\theta) = 1$ and $\operatorname{supp}(\hat{f}_1) \subseteq \theta V$. An easy application of the inversion theorem shows that f_1 is continuous, and we may assume that $f_1(e) = 1$, where e is the identity element of G. By the regularity of $L^p(G,\omega)$, we can find a nonzero function $g \in C_b(G) \cap L^q(G)$ so that $\operatorname{supp}(\hat{g}) \subseteq V$. Next, choose $x \in G$ such that $g(x^{-1}) \neq 0$, and set $f_2 = L_x g$. Hence $f_2(e) \neq 0$. Finally, define $f = f_2/f_2(e)$. Then $f(e) = 1$ and $\operatorname{supp}(\hat{f}) = \operatorname{supp}(\hat{f}_2) \subseteq V$. Now, let $h = f f_1$, and note that h is continuous and nonvanishing, $\hat{h}(\theta) \neq 0$, and $\operatorname{supp}(\hat{h}) \subseteq U$. Moreover, $h\omega = f f_1 \omega \in L^q(G) L^p(G) \subseteq L^1(G)$ by Hölder's inequality. Now, the fact that $\mathfrak{L}(x)$ is finite for all $x \in G$, follows from Domar's criterion for the regularity of $L^1(G,\omega)$. □

Theorem 3.4. *Let G be a LCA group, $\omega \geq 1$ be a symmetric submultiplicative weight on G, and $1 < p < \infty$. Then $L^p(G,\omega)$ has UUNP if and only if it has UC*NP.*

Proof. By our assumptions on ω, $\widehat{G} \subseteq \Delta(L^p(G,\omega))$. Assume that $L^p(G,\omega)$ has $UUNP$. Because every C^*-norm on a commutative Banach $*$-algebra is a uniform norm it is clear that $L^p(G,\omega)$ has UC^*NP.

For the converse, suppose that $\Delta(L^p(G,\omega))$ does not coincide with $\widehat{G}$. In view of Lemma 2.2, there exists $x \in G$ with $\mathfrak{L}(x) = \infty$. Set

$$U = \{\theta \in \widehat{G} : \ \mathrm{Re}\,\theta(x) > 0\} = \{\theta \in \widehat{G} : \ |\theta(x) - 1| < \sqrt{2}\}.$$

Then $1_G \in U$, and so U is a non-empty, open subset of $\widehat{G}$. Since $\widehat{G}$ is locally compact, there is a non-empty open subset V of $\widehat{G}$ such that $V \subset \overline{V} \subset U$. Set $E = \widehat{G} \setminus V$. Then E is closed in $\widehat{G}$ and hence in $\Delta(L^p(G,\omega))$. Set $|f|_E = \sup\{|\widehat{f}(\theta)| : \ \theta \in E\}$ for all $f \in L^p(G,\omega)$. It is clear that $|\cdot|_E$ is a C^*-seminorm on $L^p(G,\omega)$. We show that it is in fact a C^*-norm on $L^p(G,\omega)$. Suppose, if possible, there exists a nonzero function $f \in L^p(G,\omega)$ such that $|f|_E = 0$. Then $\mathrm{supp}\,\hat{f} \subseteq \overline{V} \subseteq U$. An argument similar to the proof of [16: A.1.13] shows that $\mathfrak{L}(x)$ is finite; which is a contradiction. Hence $|\cdot|_E$ is a C^*-norm on $L^p(G,\omega)$. By Lemma 2.1, we have $|\cdot|_E \neq |\cdot|_{\widehat{G}}$. This means that $L^p(G,\omega)$ does not have UC^*NP, which is again a contradiction. This contradiction proves that $\triangle(L^p(G,\omega)) = \widehat{G}$. Hence $L^p(G,\omega)$ is Hermitian. So every uniform norm is a C^*-norm, and $L^p(G,\omega)$ has $UUNP$. □

We conclude this work with some examples.

Example 1.

(a) On every σ-compact LCA group G and for any $p > 1$ there is a weight ω on G such that $L^p(G,\omega)$ has $UUNP$. This is an immediate consequence of [13: Theorem 2.3] and Theorem 3.2.

(b) For $1 < p < +\infty$, let

$$\omega(x) = (a + \|x\|^{\alpha})^{\frac{s}{q}} b^{\frac{ln(1+\|x\|^{\beta})}{q}},$$

where $x \in \mathbb{R}^n$, q is the conjugate exponent to p, $s > n/\alpha$, $b > 1$, and $a, \alpha, \beta > 0$. Then $L^p(\mathbb{R}^n,\omega)$ is an invariant algebra. Indeed, observe first that

$$\omega(y) \geq \omega(x/2) \qquad \text{if} \quad 2\|y\| \geq \|x\| \tag{3.1}$$

and

$$\omega(x-y) \geq \omega(x/2) \qquad \text{if} \quad 2\|y\| < \|x\|. \tag{3.2}$$

Hence

$$\begin{aligned}\omega^{-q} * \omega^{-q}(x) &= \int\limits_{\mathbb{R}^n} \omega^{-q}(y)\omega^{-q}(x-y)\,\mathrm{d}y \\ &= \int\limits_{\mathbb{R}^n \setminus A_x} \omega^{-q}(y)\omega^{-q}(x-y)\,\mathrm{d}y + \int\limits_{A_x} \omega^{-q}(y)\omega^{-q}(x-y)\,\mathrm{d}y \\ &\le \Bigg(\int\limits_{\mathbb{R}^n \setminus A_x} \omega^{-q}(y)\,\mathrm{d}y + \int\limits_{A_x} \omega^{-q}(x-y)\,\mathrm{d}y \Bigg)\omega^{-q}(x/2) \\ &\le \Bigg(\frac{2}{2^{\alpha s} b^{\beta ln 2}} \int\limits_{\mathbb{R}^n} \omega^{-q}(y)\,\mathrm{d}y \Bigg)\omega^{-q}(x),\end{aligned}$$

where for any $x \in \mathbb{R}^n$,

$$A_x = \big\{ y \in \mathbb{R}^n : \ 2\|y\| \ge \|x\| \big\}.$$

It is clear that $\mathfrak{L}(x)$ is finite for all $x \in \mathbb{R}^n$, and hence, by Theorem 3.2, $L^p(\mathbb{R}^n, \omega)$ has $UUNP$ for all $1 < p < \infty$.

(c) Let $1 < p < \infty$. For $x \in \mathbb{R}^n$ define Gevrey's weight by

$$\omega_\alpha(x) = \exp\left(\|x\|^\alpha / q\right),$$

where q is the exponential conjugate to p and $0 < \alpha < 1$. In [6] it has been shown that $L^p(\mathbb{R}^n, \omega_\alpha)$ is an invariant algebra for $1 < p < \infty$ and $0 < \alpha < 1$. It can be easily verified that $\mathfrak{L}(x)$ is finite for all $x \in \mathbb{R}^n$. So, Proposition 3.3 together with Theorem 3.2 show that $L^p(\mathbb{R}^n, \omega_\alpha)$ has $UUNP$ for all $1 < p < \infty$ and $0 < \alpha < 1$. Note that $L^p(\mathbb{R}^n, \omega_\alpha)$ is not an algebra for $\alpha = 1$.

(d) Let $1 < p < \infty$. For $x \in \mathbb{R}^n$ define

$$\omega_\alpha(x) = e^{\|x\|}\left(1 + \|x\|^2\right)^\alpha,$$

where $\alpha > \frac{n(p-1)}{2}$. Then $L^p(\mathbb{R}^n, \omega_\alpha)$ is a Banach algebra for which $UUNP$ fails. To show this, by a simple calculation, we can prove that

$$\omega_\alpha^{-q} * \omega_\alpha^{-q} \le \Bigg(2^{1+\frac{2\alpha}{p-1}} \int\limits_{\mathbb{R}^n} (1 + \|y\|^2)^{-\alpha q}\,\mathrm{d}y \Bigg)\omega_\alpha^{-q},$$

where q is the exponential conjugate to p. Hence the space $L^p(\mathbb{R}^n, \omega_\alpha)$ is an invariant Banach algebra for $1 < p < \infty$ and $\alpha > \frac{n(p-1)}{2}$. But, for all $x \in \mathbb{R}^n$,

$$\mathfrak{L}(x) = \sum_{n\in\mathbb{Z}} \frac{\ln \omega(nx)}{1+n^2} = \sum_{n\in\mathbb{Z}} \frac{n\|x\|}{1+n^2} + \sum_{n\in\mathbb{Z}} \frac{\ln\left(1 + \|x\|^2\right)^\alpha}{1+n^2} = +\infty.$$

Hence, it follows from Theorem 3.2 and Proposition 3.3 that $UUNP$ fails for $L^p(\mathbb{R}^n,\omega_\alpha)$. However, it is easy to see that $L^p(\mathbb{R}^n,\omega_{\alpha,\beta})$ has $UUNP$ for $1 < p < \infty$, $\alpha > \frac{n(p-1)}{2}$, and $0 < \beta < 1$, where

$$\omega_{\alpha,\beta}(x) = e^{\|x\|^\beta} \left(1 + \|x\|^2\right)^\alpha \qquad (x \in \mathbb{R}^n).$$

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Department of Mathematics
University of Zanjan
45195-313 Zanjan
IRAN

E-mail: ibrahim.akbarbaglu@znu.ac.ir
heydarpour@znu.ac.ir
s_maghsodi@znu.ac.ir