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BOUNDEDNESS OF THE EXTREMAL SOLUTION FOR SOME p-LAPLACIAN PROBLEMS

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ABSTRACT. We investigate the regularity of extremal solutions to some p-Laplacian Dirichlet problems with strong nonlinearities. Under adequate assumptions we prove the smoothness of the extremal solutions for some classes of nonlinearities. Our results suggest that the extremal solution's boundedness for some range of dimensions depends on the nonlinearity f.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p > 1$. We consider the quasi-linear elliptic problem

$$\begin{cases} -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the p-Laplace operator, λ is a positive parameter and f satisfies the following assumptions:

(2) $f\colon \mathbb{R} \to \mathbb{R}^+$ *is an nondecreasing* C^2 *function*, $f(0) > 0$, *such that* $f(t)^{\frac{1}{p-1}}$ *is superlinear at infinity, that is,* $\lim\limits_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty$ *and* $f(t)^{\frac{1}{p-1}}$ *is convex in* $[0, +\infty)$.

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Throughout the paper, we say that u is a *solution* of (1) if $u \in W_0^{1,p}(\Omega)$, $f(u) \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \,\mathrm{d}x = \lambda \int_\Omega f(u)\varphi \,\mathrm{d}x \qquad \text{for all} \quad \varphi \in C_0^1(\Omega). \tag{3}$$

These solutions, which may be unbounded, are usually called *weak energy solutions*. For short, we will refer to them simply as solutions. In addition, by the strong maximum principle (see [16,20]) one has $u > 0$ in Ω.

On the other hand, we say that $u \in W_0^{1,p}(\Omega)$ is a *regular solution* of (1) if u is a solution and $f(u) \in L^\infty(\Omega)$. By well known regularity results for degenerate elliptic equations, we can have that every regular solution belongs to $C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0,1]$ (see [6,13,19]).

Problem (1) has been studied in some papers. In [9,10], for $f(u) = \mathrm{e}^u$ Gareia-Azoreno, Peral and Puel obtain the existence of the family of minimal regular solutions (λ, u_λ) for $\lambda \in (0, \lambda^*)$. Here minimal means smaller than any other possible solution of the problem. Consider the increasing limit

$$u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda. \tag{4}$$

In contrast with the case $p = 2$ involving the Laplacian (which has been extensively studied, see [2,3,5,7,12,14,15,21,22] and so on), it is not always clear that the limit u^* is a weak solution of (1) for $\lambda = \lambda^*$. When one can establish that u^* is a weak solution (this may depend on the assumptions on N, p, Ω and f), it is called the *extremal solution*. The authors show in [9,10] that u^* is a weak energy solution independently of N and that u^* is a bounded solution in addition

$$N < p + \frac{4p}{p-1}. \tag{5}$$

Moreover, if $N \geq p + \frac{4p}{p-1}$ and the domain Ω is the unit ball of $\mathbb{R}^N$ then u^* is unbounded.

Under assumption (2), Cabré and Sanchón in [4] prove the existence of the family of minimal regular solutions for $\lambda \in (0, \lambda^*)$ and the nonexistence of regular solutions for $\lambda > \lambda^*$ of (1). Moreover, it is also proved that for every $\lambda \in (0, \lambda^*)$ the minimal solution u_λ is semi-stable in the sense that the second variation of the energy functional associated with (1) is nonnegative definite. Consequently, every minimal solution u_λ satisfies

$$\lambda \int_\Omega f'(u_\lambda)\psi^2(u_\lambda) \,\mathrm{d}x \leq (p-1) \int_\Omega \psi'^2(u_\lambda) |\nabla u_\lambda|^p \,\mathrm{d}x \tag{6}$$

for all $\psi \in C^1([0,\infty))$ such that $\psi(0) = 0$. In addition, [4] establishes a range of dimensions where problem (1) with $\lambda = \lambda^*$ admits a bounded extremal solution for some nonlinearities with power growth at infinity. Independently, Ferrero studies problem (1) for $f(u) = (1+u)^m$ with $m > p-1$ and obtains similar results in [8].

In [17] it is proved, for $p \geq 2$, that u^* is a solution of (1) with $\lambda = \lambda^*$ if $N < p(1+p')$, where $p' = \frac{p}{p-1}$, and $u^* \in L^\infty(\Omega)$ if $N < p+p'$. But it is still an open problem to prove the boundedness (or not) of the extremal solution when $p(1+p') \leq N < p+4p'$ even for $p = 2$. Later on, in [18] the author shows some extended results.

In this paper, we would study some model classes of the nonlinearity of problem (1), which make the extremal solution of this problem possess the regularity property.

First, we assume that $f(u)$ satisfies the condition:

(7) *there exists positive constants a and b, $a \geq b$, such that* $b \leq \frac{f'(u)}{f(u)} \leq a$ *holds for any* $u \geq 0$,

then our first main result can be stated as follows.

Theorem 1.1. *If f satisfies conditions* (2) *and* (7), *u^* is the function defined in* (4). *Then u^* is a regular extremal solution of* (1) *for any $N < N^* := p\big(1 + \frac{4b}{a(p-1)}\big)$. In particular, $u^* \in L^\infty(\Omega)$.*

In the above theorem, if we take $a = b$, then $N^* = p\big(1 + \frac{4}{p-1}\big)$. For $p = 2$, Theorem 1.1 has been obtained in [7]. Now, we point out that there are some examples for f satisfying conditions (2) and (7). For example, $f(u) = \mathrm{e}^u + \varepsilon g(u)$ with a small parameter $\varepsilon > 0$ and a function $g(u)$ satisfying $g(u) > 0$, $0 \leq g'(u) \leq C\mathrm{e}^u$, and $g''(u) \geq 0$ for any $u > 0$. Another example is $f(u) = \mathrm{e}^u \sum\limits_{i=0}^{n} a_i u^i$ if the coefficients a_i satisfy some constraints such that $f(u)$ satisfies (2). So our assumption (7) and the condition (21) in [18] are not covered each other.

Next, we assume that $f(u)$ satisfies the following condition:

$$\liminf_{t\to+\infty} \frac{f(t)f''(t)}{f'^2(t)} = \gamma > 0, \qquad \limsup_{t\to+\infty} \frac{f(t)f''(t)}{f'^2(t)} = \Gamma, \tag{8}$$

then our second result is the following.

Theorem 1.2. *Assume $f(u)$ satisfies* (2) *and* (8), *u^* is the function defined by* (4), *then the following assertions hold:*

(i) *Let $p \geq 2$, in addition, $\gamma - \frac{p-2}{p-1}\Gamma \geq \frac{p-2}{(p-1)^2}$, then u^* is a solution of* (1) *and $u^* \in L^\infty(\Omega)$ for any $N < N_1^*(p)$, where*

$$N_1^*(p) = p\Big\{p - 1 - (p-2)\Gamma + \frac{2}{p-1}\Big(1 + \sqrt{2 - p + (p-1)[(p-1)\gamma - (p-2)\Gamma]}\Big)\Big\};$$

(ii) *If f is a convex function and $1 < p \leq 2$, then u^* is a solution of* (1) *and $u^* \in L^\infty(\Omega)$ for any $N < N_2^*(p)$, where*

$$N_2^* = p\Big\{p - 1 - (p-2)\gamma + \frac{2}{p-1}\Big(1 + \sqrt{2 - p + (p-1)\gamma}\Big)\Big\}.$$

We remark that if $1 < p \leq 2$ and $\gamma \geq 1$, then we have that $N_2^* \geq H(p)$, where $H(p)$ is given in [18]. For $p \geq 2$, if $\gamma = \Gamma$, then $N_1^*(p) = p\{p - 1 - (p-2)\gamma + \frac{2}{p-1}(1 + \sqrt{1 + (p-1)(\gamma - 1)})\}$. In this case, we assume in addition that $\gamma \leq 1$, then $N_1^* \geq N(p)$, here $N(p)$ is given in [18]. Hence, our result extends partially the main results in [18]. On the other hand, for $p = 2$, we find that $N_2^* = 6 + 4\sqrt{\gamma}$, this has been given in [22]. And for $f = e^u$ we have $\gamma = \Gamma = 1$ and $N_1^* = N_2^* = p(1 + \frac{4}{p-1})$, which is well known result. Hence, our results extend partially the main results of above literatures. Our main idea is to find some suitable test functions for obtaining appropriate estimations in virtue of (6), as in [5, 21, 22] and so on.

The paper is organized as follows. In section 2, we give a known regularity result for solutions of (1). In section 3 and 4, we prove Theorem 1.1 and 1.2.

2. Preliminaries

We consider the problem

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where $g \in L^q(\Omega)$ for some $q \geq 1$. Then the following result can be found in [11] or [1].

Lemma 2.1. *Assume that $g \in L^q(\Omega)$, for some $q \geq 1$, and that u is a solution of* (9)*. The following assertions hold:*

(i) *If $q > \frac{N}{p}$, then $u \in L^\infty(\Omega)$. Moreover,*

$$\|u\|_\infty \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, q and $|\Omega|$.

(ii) *If $q = \frac{N}{p}$, then $u \in L^r(\Omega)$ for $1 \leq r < +\infty$ and*

$$\|u\|_r \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, r and $|\Omega|$.

(iii) *If* $1 \le q < \frac{N}{p}$, *then* $|u|^r \in L^1(\Omega)$ *for* $0 < r < r_1$, *where* $r_1 := \frac{(p-1)Nq}{N-pq}$. *Moreover,*

$$\||u|^r\|_1^{1/r} \le C\|g\|_q^{\frac{1}{p-1}},$$

where C *is a constant depending only on* N, p, q, r *and* $|\Omega|$.

3. Proof of Theorem 1.1

In this section, we will use (3) and (6) to prove Theorem 1.1. Theorem 1.1 is a consequence of the following proposition.

Proposition 3.1. *Assume that* $f(u)$ *satisfies the condition* (2) *and for some* $a, b, q, s > 0$ *with*

$$bq > a(p-1)s^2 \tag{10}$$

such that

$$b \le \frac{f'(u)}{f^{1+q-2s}} \le a. \tag{11}$$

Then the function u^* *given by* (4) *belongs to* $L^\infty(\Omega)$ *and is a regular extremal solution to* (1) *for* $N < p(1+q)$.

Proof. Let u_λ, $0 < \lambda < \lambda^*$ be the sequence of minimal (regular) solutions of the problem (1) which converges to the function u^*. Now taking

$$\varphi(t) = f^q(t) - f^q(0) := f^q(t) - A \quad \text{and} \quad \psi(t) = f^s(t) - f^s(0) := f^s(t) - B,$$

from (3) we have

$$q\int_\Omega |\nabla u_\lambda|^p f^{q-1}(u_\lambda) f'(u_\lambda)\,\mathrm{d}x = \lambda \int_\Omega f(u_\lambda)\big[f^q(u_\lambda) - A\big]\,\mathrm{d}x. \tag{12}$$

On the other hand, it follows from (6) and (11) that

$$\begin{aligned}\lambda \int_\Omega f'(u_\lambda)\big[f^s(u_\lambda) - B\big]^2\,\mathrm{d}x &\le s^2(p-1)\int_\Omega |\nabla u_\lambda|^p f^{2s-2}(u_\lambda) f'^2(u_\lambda)\,\mathrm{d}x \\ &\le a(p-1)s^2 \int_\Omega |\nabla u_\lambda|^p f^{q-1}(u_\lambda) f'(u_\lambda)\,\mathrm{d}x.\end{aligned} \tag{13}$$

Therefore, taking account of (11), (12) and (13) we deduce that

$$\begin{aligned}\lambda b\int_\Omega f^{1+q-2s}(f^{2s}-2Bf^s)\,\mathrm{d}x &\le \lambda b\int_\Omega f^{1+q-2s}(f^s-B)^2\,\mathrm{d}x\\ &\le \lambda\int_\Omega f'(f^s-B)^2\,\mathrm{d}x\\ &\le a(p-1)s^2\int_\Omega |\nabla u_\lambda|^p f^{q-1}f'\,\mathrm{d}x\\ &= \frac{a(p-1)s^2\lambda}{q}\int_\Omega f(f^q-A)\,\mathrm{d}x,\end{aligned}$$

which implies

$$\left(b-\frac{a(p-1)s^2}{q}\right)\int_\Omega f^{1+q}\,\mathrm{d}x \le 2bB\int_\Omega f^{1+q-s}\,\mathrm{d}x.$$

Thus, applying the Hölder inequality we get

$$\int_\Omega f^{1+q}\,\mathrm{d}x \le C\int_\Omega f^{1+q-s} \le C\biggl(\int_\Omega f^{1+q}\biggr)^{\frac{1+q-s}{1+q}},$$

that is,

$$\biggl(\int_\Omega f^{1+q}\biggr)^{\frac{s}{1+q}} \le C, \qquad C>0. \tag{14}$$

This shows that $f(u_\lambda)$ is uniformly bounded in $L^{1+q}(\Omega)$. By the Lemma 2.1 we obtain that u_λ is uniformly bounded in $L^\infty(\Omega)$ for $N<p(1+q)$. Therefore $u^*=\lim\limits_{\lambda\uparrow\lambda^*}u_\lambda$ belongs to $L^\infty(\Omega)$ and is a regular extremal solution of (1) for $\lambda=\lambda^*$. □

Proof of Theorem 1.1. Taking an arbitrary $q>0$ such that $q<\frac{4b}{a(p-1)}$ and setting $s=\frac{q}{2}$, by the Proposition 3.1 we can obtain easily the desired result for $N<p\bigl(1+\frac{4b}{a(p-1)}\bigr)$. □

4. Proof of Theorem 1.2

In this section, we begin with the following lemma.

Lemma 4.1. *Let f be a positive, nondecreasing, convex C^2-function, and u_λ be a solution of* (1). *Assume that*

$$\|f^{(\beta-1)(p-1)+1}(u_\lambda)(f'(u_\lambda))^{p-1}\|_{L^1(\Omega)} \le C_1$$

for some $\beta \ge 1$, then for $N < p[1+(p-1)\beta]$ there holds

$$\|u_\lambda\|_{L^\infty(\Omega)} \le C$$

for some positive constant $C > 0$.

Proof. Since $f^\beta(u_\lambda)$ is convex for $\beta \ge 1$, a direct computation shows that

$$\begin{aligned} -\Delta_p f^\beta(u_\lambda) = & -\beta^{p-1}\Big\{|\nabla u_\lambda|^p\Big[(\beta-1)(p-1)f^{p\beta-p-\beta}(f')^p \\ & \qquad + (p-1)f^{(\beta-1)(p-1)}(f')^{p-2}f''\Big]\Big\} \\ & - \beta^{p-1}f^{(\beta-1)(p-1)}(f')^{p-1}\operatorname{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda) \\ \le & \ \lambda\beta^{p-1}f^{(\beta-1)(p-1)+1}(f')^{p-1}. \end{aligned} \tag{15}$$

Let $v = v_\lambda$ be the solution of

$$\begin{cases} -\Delta_p v = \lambda\beta^{p-1}f^{(\beta-1)(p-1)+1}(u_\lambda)(f')^{p-1}(u_\lambda) & \text{in } \Omega, \\ v = f^\beta(0) & \text{on } \partial\Omega, \end{cases} \tag{16}$$

from (15) we then have $-\Delta_p f^\beta(u_\lambda) \le -\Delta_p v$ in Ω and $f^\beta(u_\lambda) = f^\beta(0) = v$ on the boundary. So the weak comparison principle yields that $f^\beta(u_\lambda) \le v$. From the boundedness of $f^{(\beta-1)(p-1)+1}(u_\lambda)(f')^{p-1}(u_\lambda)$ in $L^1(\Omega)$ and the Lemma 2.1, we conclude that v is bounded uniformly in $L^r(\Omega)$, where $r = \infty$ for $N < p$, $r \in [1,\infty)$ for $N = p$ and $r \in (0, \frac{(p-1)N}{N-p})$ for $N > p$. Therefore $f(u_\lambda)$ is bounded uniformly in $L^{\beta r}(\Omega)$. Using the Lemma 2.1 again, we find that $u_\lambda \in L^\infty(\Omega)$ for any $N < p + p(p-1)\beta$. □

We now apply Lemma 4.1 and the inequality (6) to prove Theorem 1.2.

Proof of Theorem 1.2. First, we choose some suitable functions φ and ψ such that $\varphi(0) = \psi(0) = 0$ and $\varphi'^2(t) = \frac{1}{p-1}\psi'(t)$, then by taking $\psi(u_\lambda)$ as a test function in (6) and using Eq. (1), we get

$$\begin{aligned}\lambda\int\limits_\Omega f'(u_\lambda)\varphi^2(u_\lambda)\,\mathrm{d}x &\le \int\limits_\Omega \psi'(u_\lambda)|\nabla u_\lambda|^p\,\mathrm{d}x\\ &= -\int\limits_\Omega \operatorname{div}(|\nabla u_\lambda|^{p-2}\nabla u_\lambda)\psi(u_\lambda)\,\mathrm{d}x\\ &= \lambda\int\limits_\Omega f(u_\lambda)\psi(u_\lambda)\,\mathrm{d}x.\end{aligned}\tag{17}$$

Next, without loss of generality, we assume that $f(0) = 1$. Let

$$\varphi(t) = (f')^{\frac{p-2}{2}}(t)(f^\alpha(t) - 1),$$

where $\alpha > 0$ is a constant to be determined later.

Case (i): $p \ge 2$.
Observe first that the condition (2) implies f is convex on $[0, +\infty)$ in this case. For $\psi(t)$ we have the following estimate

$$\psi(t) \le \widetilde{C}(f')^{p-1}(f^{2\alpha} - 1)f^{-1} + C \tag{18}$$

for all $t > 0$, where $\widetilde{C} \in (0, 1)$ and C are constants.

In fact,

$$\begin{aligned}\psi(t) &= (p-1)\int\limits_0^t \varphi'^2(s)\,\mathrm{d}s\\ &= (p-1)\int\limits_0^t \Big[\frac{p-2}{2}(f')^{\frac{p}{2}-2}f''(f^\alpha - 1) + \alpha f^{\alpha-1}(f')^{\frac{p}{2}}\Big]^2\,\mathrm{d}s.\end{aligned}\tag{19}$$

Note that the condition (8) yields: given any $\varepsilon \in (0, \gamma)$ there exists $C \ge 0$ such that

$$\int\limits_0^t \frac{p-2}{2}(f')^{\frac{p}{2}-2}f''(f^\alpha - 1)\,\mathrm{d}s \le \int\limits_0^t \frac{p-2}{2}(f')^{\frac{p}{2}-2}(f^\alpha - 1)\frac{f'^2(\Gamma+\varepsilon)}{f}\,\mathrm{d}s + C, \tag{20}$$

so substituting (20) into (19) we obtain that

$$\begin{aligned}\psi(t) \le{}& (p-1)\left[\alpha + \frac{(p-2)(\Gamma+\varepsilon)}{2}\right]^2 \int_0^t f^{2\alpha-2} f'^p \,\mathrm{d}s \\ &+ \frac{(p-2)^2(\Gamma+\varepsilon)^2(p-1)}{4} \int_0^t f'^p f^{-2} \,\mathrm{d}s \\ &- (p-1)(p-2)(\Gamma+\varepsilon)\left[\alpha + \frac{(p-2)(\Gamma+\varepsilon)}{2}\right] \int_0^t f'^p f^{\alpha-2} \,\mathrm{d}s + C.\end{aligned} \tag{21}$$

Observing that

$$\begin{aligned}\int_0^t f^{2\alpha-2} f'^p \,\mathrm{d}s = {}& \frac{1}{2\alpha}(f'^{p-1}(f^{2\alpha}-1)f^{-1} \\ &+ \frac{1}{2\alpha}\int_0^t (f^{2\alpha}-1)f^{-2}[f' - (p-1)(f')^{p-2}f''f] \,\mathrm{d}s,\end{aligned} \tag{22}$$

as above by (8) we can find

$$\begin{aligned}&\int_0^t (f^{2\alpha}-1)f^{-2}[f'^p - (p-1)(f')^{p-2}f''f] \,\mathrm{d}s \\ &\qquad \le [1-(p-1)(\gamma-\varepsilon)] \int_0^t (f^{2\alpha}-1)f^{-2}f'^p \,\mathrm{d}s + C.\end{aligned} \tag{23}$$

Hence from (22) and (23) we obtain

$$\begin{aligned}\int_0^t f^{2\alpha-2} f'^p \,\mathrm{d}s \le{}& C^*\Big((f')^{p-1}(f^{2\alpha}-1)f^{-1} \\ &- [1-(p-1)(\gamma-\varepsilon)] \int_0^t f^{-2} f'^p \,\mathrm{d}s + C\Big),\end{aligned} \tag{24}$$

where

$$C^* = \frac{1}{2\alpha - 1 + (p-1)(\gamma-\varepsilon)}.$$

Substituting (24) into (21) and notice the fact

$$\frac{f^{-2}f'^p}{f^{2\alpha-2}f'^p} = \frac{1}{f^{2\alpha}} \to 0 \qquad \text{as } t \to \infty,$$

we can obtain

$$\psi(t) \le \widetilde{C}(f')^{p-1}(f^{2\alpha}-1)f^{-1} + C \qquad \text{for all } t > 0, \tag{25}$$

that is, (18) holds, where

$$\widetilde{C} = \frac{(p-1)[2\alpha + (p-2)(\Gamma+\varepsilon)]^2}{4[2\alpha - 1 + (p-1)(\gamma-\varepsilon)]} \in (0,1). \tag{26}$$

(26) implies

$$\alpha < \alpha^* := \frac{1}{2(p-1)}\Big[2 - (p-1)(p-2)(\Gamma+\varepsilon) + 2\sqrt{2-p+(p-1)\big[(p-1)(\gamma-\varepsilon) - (p-2)(\Gamma+\varepsilon)\big]}\,\Big].$$

Now going back to the inequality (17), we find that

$$\int_\Omega (f')^{p-1}(f^\alpha - 1)^2 \,\mathrm{d}x \le \int_\Omega f\Big[\tilde{C}(f')^{p-1}(f^{2\alpha}-1)f^{-1} + C\Big]\,\mathrm{d}x,$$

that is

$$(1-\tilde{C})\int_\Omega (f')^{p-1}f^{2\alpha}\,\mathrm{d}x \le 2\int_\Omega (f')^{p-1}f^\alpha\,\mathrm{d}x + C\int_\Omega f\,\mathrm{d}x,$$

which yields for any $\alpha \in (1, \alpha^*)$

$$\int_\Omega (f')^{p-1}f^{2\alpha}\,\mathrm{d}x \le C. \tag{27}$$

Since ε can be arbitrarily small, we conclude that for any

$$\alpha < \frac{1}{2(p-1)}\Big[2 - (p-1)(p-2)\Gamma + 2\sqrt{2-p+(p-1)\big[(p-1)\gamma - (p-2)\Gamma\big]}\,\Big],$$

there exists $C > 0$ such that (27) holds.

Case (ii): $1 < p \le 2$.
We now proceed as in the above case, the estimate (21) is replaced by

$$\begin{aligned}\psi(t) \le (p-1)C_\varepsilon^2 \int_0^t f^{2\alpha-2}f'^p\,\mathrm{d}s + (p-1)(C_\varepsilon - \alpha)^2\int_0^t f'^p f^{-2}\,\mathrm{d}s \\ - 2(p-1)C_\varepsilon(C_\varepsilon-\alpha)\int_0^t f'^p f^{\alpha-2}\,\mathrm{d}s + C,\end{aligned} \tag{28}$$

where

$$C_\varepsilon = \begin{cases} \frac{p-2}{2}(\gamma-\varepsilon)+\alpha & \text{if } \left|\frac{p-2}{2}(\gamma-\varepsilon)+\alpha\right| \geq \left|\frac{p-2}{2}(\Gamma+\varepsilon)+\alpha\right|, \\ \frac{p-2}{2}(\Gamma+\varepsilon)+\alpha & \text{if } \left|\frac{p-2}{2}(\gamma-\varepsilon)+\alpha\right| < \left|\frac{p-2}{2}(\Gamma+\varepsilon)+\alpha\right|. \end{cases}$$

Observing that (24) still holds, so we have also the estimate (25) with the constant

$$\widetilde{C} = \frac{(p-1)C_\varepsilon^2}{2\alpha-1+(p-1)(\gamma-\varepsilon)} \in (0,1). \tag{29}$$

(29) implies

$$\alpha < \frac{1}{2(p-1)}\left[2+(p-1)(2-p)(\gamma-\varepsilon)+2\sqrt{2-p+(p-1)(\gamma-\varepsilon)}\,\right].$$

Hence, the same procedure as in the case (i) gives estimate (27) for any

$$\alpha < \frac{1}{2(p-1)}\left[2+(p-1)(2-p)\gamma+2\sqrt{2-p+(p-1)\gamma}\,\right].$$

Therefore, it follows from Lemma 4.1 that if

$$N < \begin{cases} p\Big\{p-1-(p-2)\Gamma \\ \quad + \frac{2}{p-1}\Big(1+\sqrt{2-p+(p-1)[(p-1)\gamma-(p-2)\Gamma]}\,\Big)\Big\} & \text{for } p \geq 2; \\ p\Big\{p-1-(p-2)\gamma \\ \quad + \frac{2}{p-1}\Big(1+\sqrt{2-p+(p-1)\gamma}\,\Big)\Big\} & \text{for } 1 < p \leq 2, \end{cases}$$

then there holds $\|u_\lambda\|_{L^\infty(\Omega)} \leq C$. Therefore $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ belongs to $L^\infty(\Omega)$ and is a regular extremal solution of (1) for $\lambda = \lambda^*$. □

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