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CONCENTRATION COMPACTNESS OF A LANDAU-LIFSCHITZ TYPE ENERGY WITH RADIAL STRUCTURE

Bei Wang* — Yuze Cai** — Yutian Lei***

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ABSTRACT. This paper is concerned with the asymptotic behavior of a p-Landau-Lifschitz type functional with radial structure as parameter goes to zero. We study the concentration compactness properties in the case of $p \geq n$ where n is the dimension. By analyzing the functional globally, we show that the singularity of p-Landau-Lifschitz energy concentrates on the origin.

1. Introduction

Let $G \subset \mathbf{R}^n$ $(n \geq 2)$ be a bounded and simply connected domain with smooth boundary ∂G. Denote $S^{n-1} = \{x \in \mathbf{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_n^2 = 1, x_{n+1} = 0\}$ and $S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$. We sometimes write the vector-valued function $u = (u_1, u_2, \ldots, u_n, u_{n+1}) = (u', u_{n+1})$. Let $g = (g', 0)$ be a smooth map from ∂G into S^{n-1} satisfying $\deg(g', \partial G) = d > 0$. Consider the energy functional

$$E_\varepsilon(u, G) = \frac{1}{p} \int_G |\nabla u|^p \mathrm{d}x + \frac{1}{2\varepsilon^p} \int_G u_{n+1}^2 \mathrm{d}x,$$

where the small parameter $\varepsilon > 0$ and $p \geq n$. By the direct method in the calculus of variations, the minimizer u_ε exists in the space $W' = \{u \in W^{1,p}(G, S^n) :$

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$u|_{\partial G} = g\}$. Clearly, the minimizer is a weak solution of the system

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u|\nabla u|^p + \frac{1}{\varepsilon^p}\left[u(u_{n+1})^2 - u_{n+1}e_{n+1}\right] \qquad \text{in} \quad G, \tag{1.1}$$

where $e_{n+1} = (0, 0, \dots, 0, 1)$. Maybe (1.1) has no classical solution since it is degenerate. Therefore, following Uhlenbeck's idea, we consider the regularized functional

$$E_\varepsilon^\tau(u, G) = \frac{1}{p}\int_G (|\nabla u|^2 + \tau)^{p/2} \mathrm{d}x + \frac{1}{2\varepsilon^p}\int_G u_{n+1}^2 \mathrm{d}x.$$

Similar to the derivation of [8: (4.2) and (4.8)], when $p \geq n$, the regularized minimizer u_ε^τ of $E_\varepsilon^\tau(u, G)$ satisfies

$$\lim_{\tau\to 0} u_\varepsilon^\tau = u_\varepsilon, \quad \text{in } W^{1,p}(G), \tag{1.2}$$

$$\lim_{\tau\to 0} u_\varepsilon^\tau = u_\varepsilon, \quad \text{in } C_{loc}^{1,\beta}(G \setminus \{a_j\}_{j=1}^N), \quad \text{for some } \beta \in (0, 1/2). \tag{1.3}$$

where u_ε is a minimizer of $E_\varepsilon(u, G)$, and $a_j (j = 1, 2, \dots, N)$ are singularities of a p-harmonic map with boundary value g'. We call u_ε a *regularized minimizer* of $E_\varepsilon(u, G)$.

In the 2-dimensional case, the functional $E_\varepsilon(u, G)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planner ferromagnets and antiferromagnets (cf. [6], [11] and [12]). The asymptotic behavior of minimizers of $E_\varepsilon(u, G)$ has been considered in [4]. One of the main results is the following proposition.

Proposition 1.1. ([4: Theorem 4.2]) *Assume $p = n = 2$, u_ε is a minimizer of $E_\varepsilon(u, G)$, and $a_1, a_2, \dots, a_d$ are singularities of a harmonic map with boundary value g'. Then as $\varepsilon \to 0$, there exists a subsequence ε_k such that*

$$\lim_{\varepsilon_k\to 0} \frac{u_{\varepsilon_k 3}^2}{2\varepsilon_k^2} = \frac{\pi}{2}\sum_{j=1}^d \delta_{a_j}, \qquad \textit{in the weak * topology of } C(\overline{G}), \tag{R1}$$

$$\lim_{\varepsilon_k\to 0} \frac{|\nabla u_{\varepsilon_k}|^2}{|\log\varepsilon_k|} = 2\pi\sum_{j=1}^d \delta_{a_j}, \qquad \textit{in the weak * topology of } C(\overline{G}), \tag{R2}$$

where δ_{a_j} is the Dirac mass at a_j.

If the term $\frac{u_{n+1}^2}{\varepsilon^p}$ is replaced by $\frac{(1-|u|^2)^2}{2\varepsilon^p}$ and S^n is replaced by $\mathbf{R}^n$, the functional is the Ginzburg-Landau functional. Many researches studied the case $p = n = 2$ (cf. [2] and the references therein). In particular, in [2: Theorems VII.2 and VII.3] derived the similar results to (R1) and (R2). Paper [10] generalized those results to the case of $p > n = 2$. Other work on $p \neq n$ can be seen in [1] and [16]. Nineteen open problems were introduced in [2]. When $p = n > 2$, the asymptotic behavior of minimizers was in [2: Problem 17], which was studied in [3], [5] and [14]. Their work shows that the study of minimizers of n-Ginzburg-Landau functional is connected tightly with the corresponding properties of the n-harmonic map. Afterwards, paper [9] obtained (R1) and (R2) when $p = n > 2$.

In this paper, we will study the Landau-Lifschitz type functional $E_\varepsilon(u, G)$ with radial structure. Let $G = B = \{x \in \mathbf{R}^n : \ |x| < 1\}$. The minimizer u_ε of $E_\varepsilon(u, B)$ in the space

$$W = \left\{ u(x) = \left(\frac{x}{|x|} \sin f(r), \cos f(r) \right) \in W^{1,p}(B, S^n) : \ f(1) = \frac{\pi}{2} \right\},$$

is named *radial minimizer*, where $r = |x|$. Similar to Remark in [7: p. 68], $f_\varepsilon(r)$ also belongs to $\{f \in C[0,1] : \ f(0) = 0\}$.

When $p > n = 2$, Paper [7] studied the asymptotics of the radial minimizer. Those results can be extended to the higher dimensions without any obstacle. First, in [7: Proposition 2.1] implies that there exists $C > 0$ which is independent of ε, such that

$$\varepsilon^{p-n} \int_B |\nabla u_\varepsilon|^p \mathrm{d}x \le C, \tag{1.4}$$

$$\frac{1}{\varepsilon^n} \int_B |u_{\varepsilon n+1}|^2 \mathrm{d}x \le C. \tag{1.5}$$

Next, similar to [7: Proposition 3.2], for any compact subset K of $B \setminus \{0\}$, there exists a constant $C > 0$ which is independent of ε, such that

$$E_\varepsilon(u_\varepsilon, K) \le C. \tag{1.6}$$

In addition, [7: Theorem 1.1] shows that, for any $\gamma \in (0,1)$, there exists a constant $h = h(\gamma) > 0$ which is independent of ε, such that

$$|u'_\varepsilon(x)|^2 \ge 1 - \gamma^2, \quad \text{for } x \in B \setminus B(0, h\varepsilon). \tag{1.7}$$

In this paper, we establish the properties (R1) and (R2) for $p \ge n > 2$.

Theorem 1.2. *Assume $p > n > 2$, u_ε is a regularized radial minimizer. Then as $\varepsilon \to 0$, there exists a subsequence ε_k such that*

$$\frac{1}{2\varepsilon_k^n} |u_{\varepsilon_k n+1}|^2 \to L\delta_o, \qquad \textit{weakly star in } C(\overline{B}). \tag{1.8}$$

$$\varepsilon_k^{p-n} |\nabla u_{\varepsilon_k}|^p \to \frac{np}{p-n} L\delta_o, \qquad \textit{weakly star in } C(\overline{B}). \tag{1.9}$$

Here δ_o is the Dirac mass at the origin, and the positive constant L satisfies

$$\begin{aligned}&\frac{(n-1)^{p/2}|\partial B|}{np}\sup_{\gamma\in(0,1)}(1-\gamma^2)^{p/2}h^{n-p}(\gamma)\\&\leq L\leq(1-\frac{n}{p})\min_W E_1(u,B)+\frac{(n-1)^{p/2}}{p^2}|\partial B|,\end{aligned}\tag{1.10}$$

where $h(\gamma)$ is a positive constant in (1.7).

Theorem 1.3. *Assume $p=n>2$, u_ε is a regularized radial minimizer. Then as $\varepsilon\to 0$,*

$$\frac{|\nabla u_\varepsilon|^n}{|\log\varepsilon|}\to(n-1)^{\frac{n}{2}}|\partial B|\delta_o,\qquad \textit{weakly star in } C(\overline{B}).\tag{1.11}$$

$$\frac{|u_{\varepsilon n+1}|^2}{2\varepsilon^n}\to\frac{(n-1)^{\frac{n}{2}}|\partial B|}{n^2}\delta_o,\qquad \textit{weakly star in } C(\overline{B}).\tag{1.12}$$

Remark 1. We conjecture that L in the limits of (1.8) and (1.9) should be independent of the subsequence. If it is true, then (1.8) and (1.9) hold not only for ε_k, but also for the whole u_ε. However, this conjecture seems difficult to prove. How to deal with the absence of the conformal transformation of $\int|\nabla u|^p\mathrm{d}x$ when $p\neq n$ is also an open problem.

Remark 2. Theorem 1.2 shows that the energy $\varepsilon^{p-n}E_\varepsilon(u_\varepsilon,B)$ concentrates on the origin. In addition, if $p=n$ in (1.10), then $L=\frac{(n-1)^{n/2}}{n^2}|\partial B|$, (1.8) becomes (1.12). Moreover, if $p=n=2$ in (1.10), then $L=\frac{\pi}{2}$, and (1.8), (1.11) and (1.12) are identical with (R1) and (R2).

Remark 3. When $p<n$, the energy $E_\varepsilon(u_\varepsilon,B)\leq E_\varepsilon(u_*,B)=C$, where $u_*=(\frac{x}{|x|},0)$. The concentration compactness of this p-Landau-Lifschitz energy like (R1) and (R2) does not happen.

2. Proof of Theorem 1.2

In this section, we assume $p>n>2$, and u_ε is the regularized radial minimizer.

Proof of (1.8) and (1.9). In view of (1.4) and (1.5), there exist two Radon measures ω_1 and ω_2, such that as $\varepsilon\to 0$,

$$\varepsilon_k^{p-n}|\nabla u_{\varepsilon_k}|^p\to\omega_1,\qquad \text{weakly star in } C(\overline{B}),\tag{2.1}$$

$$\frac{1}{2\varepsilon_k^n}u^2_{\varepsilon_k n+1}\to\omega_2,\qquad \text{weakly star in } C(\overline{B}),\tag{2.2}$$

for some subsequence ε_k of ε. Sometimes we also denote u_{ε_k} by u_ε if it is not confusing. Furthermore, (1.6) implies that as $\varepsilon \to 0$,

$$\varepsilon^{p-n} \int_K |\nabla u_\varepsilon|^p \mathrm{d}x \to 0,$$
$$\frac{1}{2\varepsilon^n} \int_K u_{\varepsilon n+1}^2 \mathrm{d}x \to 0,$$

where K is an arbitrary compact subset of $B \setminus \{0\}$. These results lead to $\operatorname{supp}(\omega_i) = \{0\}$ for $i = 1, 2$. Then we can find constants L and L' such that

$$\omega_1 = L'\delta_o, \qquad \omega_2 = L\delta_o. \tag{2.3}$$

Next, we shall point out the relation between L and L'.

It is not difficult to see that the radial minimizer u_ε^τ solves the system

$$\begin{aligned} - \operatorname{div}[v^{(p-2)/2}\nabla u] = u|\nabla u|^2 v^{(p-2)/2} & \\ + \frac{1}{\varepsilon^p}(u u_{n+1}^2 - u_{n+1}e_{n+1}) & \qquad \text{in } B, \end{aligned} \tag{2.4}$$

where $v = |\nabla u|^2 + \tau$. Multiplying (2.4) by $x \cdot \nabla u$ and integrating by parts, we can obtain the Pohozaev type identity

$$\begin{aligned} - \int_{\partial B_R(0)} |x| v^{\frac{p-2}{2}} |\partial_\nu u|^2 \mathrm{d}s + \int_{B_R(0)} v^{\frac{p-2}{2}} |\nabla u|^2 \mathrm{d}x & \\ - \frac{n}{p} \int_{B_R(0)} v^{\frac{p}{2}} \mathrm{d}x + \frac{1}{p} \int_{\partial B_R(0)} |x| v^{\frac{p}{2}} \mathrm{d}s & \\ = - \frac{1}{2\varepsilon^p} \int_{\partial B_R(0)} |x| u_{n+1}^2 \mathrm{d}s + \frac{n}{2\varepsilon^p} \int_{B_R(0)} u_{n+1}^2 \mathrm{d}x & \end{aligned} \tag{2.5}$$

for any $R \in (0, 1]$. Hereafter, we denote f_ε by f. By (1.6) and the mean value theorem, there exists $\sigma \in (1/4, 1/2)$ such that

$$r^{n-1}[(f_r)^2 + r^{-2}(n-1)\sin^2 f]^{p/2}\Big|_{r=\sigma} + \frac{r^{n-1}}{\varepsilon^p}\cos^2 f\Big|_{r=\sigma} \le C. \tag{2.6}$$

Then, we take $R=\sigma$ in (2.5) and multiply it by ε^{p-n}. Letting $\tau\to 0$ and using (1.2) and (1.3), we have

$$
\begin{aligned}
&-\varepsilon^{p-n}\sigma^n[(f_r)^2+\frac{n-1}{r^2}\sin^2 f]^{p/2}\Big|_\sigma \\
&\qquad\qquad +\left(1-\frac{n}{p}\right)\varepsilon^{p-n}\int_0^\sigma[(f_r)^2+\frac{n-1}{r^2}\sin^2 f]^{p/2}r^{n-1}\mathrm{d}r \\
&\qquad\qquad +\frac{\sigma^n}{p}\varepsilon^{p-n}[(f_r)^2+\frac{n-1}{r^2}\sin^2 f]^{p/2}\Big|_\sigma \\
&=-\frac{\sigma^n}{2\varepsilon^n}\cos^2 f\Big|_\sigma+\frac{n}{2\varepsilon^n}\int_0^\sigma\cos^2 f r^{n-1}\mathrm{d}r.
\end{aligned}
$$

Using (2.6), we get

$$
\left(1-\frac{n}{p}\right)\varepsilon^{p-n}\int_{B_\sigma(0)}|\nabla u_\varepsilon|^p\mathrm{d}x-\frac{n}{2\varepsilon^n}\int_{B_\sigma(0)}u_{\varepsilon n+1}^2\mathrm{d}x\to 0 \tag{2.7}
$$

as $\varepsilon\to 0$. Combining this result with (2.1)–(2.3), we obtain

$$
L'=\frac{np}{p-n}L.
$$

Thus, (1.8) and (1.9) are proved. □

Proof of (1.10).

Step 1. Upper bound.

Similar to the proof of [7: Proposition 2.1], it is easy to derive

$$
\varepsilon^{p-n}E_\varepsilon(u_\varepsilon,B)\le\frac{(n-1)^{p/2}}{p(p-n)}|\partial B|+\min_W E_1(u,B)+C\varepsilon^{p-n}. \tag{2.8}
$$

Here $C>0$ is independent of ε. On the other hand, (1.8) and (1.9) lead to

$$
\lim_{\varepsilon\to 0}\left[\frac{\varepsilon^{p-n}}{p}|\nabla u_\varepsilon|^p+\frac{1}{2\varepsilon^n}u_{\varepsilon n+1}^2\right]=\frac{p}{p-n}L\delta_o,\quad\text{weakly star in } C(\bar{B}). \tag{2.9}
$$

This result, together with (2.8), implies the upper bound of L in (1.10).

Step 2. Lower bound.

From (1.7), we can deduce that, for any $\sigma>0$, there exists $C>0$ independent of ε, such that

$$
\begin{aligned}
\int_{h\varepsilon}^\sigma[(f_r)^2+r^{-2}(n-1)\sin^2 f]^{p/2}r^{n-1}\mathrm{d}r &\ge\int_{h\varepsilon}^\sigma r^{n-p-1}(n-1)^{p/2}\sin^p f\mathrm{d}r \\
&\ge\frac{(n-1)^{p/2}}{p-n}(1-\gamma^2)^{p/2}h^{n-p}(\gamma)\varepsilon^{n-p}-C(\sigma).
\end{aligned} \tag{2.10}
$$

Applying (2.7), we obtain that,

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^{p-n}E_\varepsilon(u_\varepsilon,B_\sigma(0)) = & \lim_{\varepsilon\to 0}\frac{|\partial B|\varepsilon^{p-n}}{p}\int_0^\sigma [(f_r)^2+r^{-2}(n-1)\sin^2 f]^{p/2}r^{n-1}\mathrm{d}r\\ &+\lim_{\varepsilon\to 0}\frac{|\partial B|}{2\varepsilon^2}\int_0^\sigma \cos^2 f r^{n-1}\mathrm{d}r\\ =&\frac{|\partial B|}{n}\lim_{\varepsilon\to 0}\varepsilon^{p-n}\int_0^\sigma [(f_r)^2+r^{-2}(n-1)\sin^2 f]^{p/2}r^{n-1}\mathrm{d}r.\end{aligned}$$

Inserting (2.10) into this result, we deduce that for any $\eta\in(0,1)$,

$$\lim_{\varepsilon\to 0}\varepsilon^{p-n}E_\varepsilon(u_\varepsilon,B_\sigma(0)) \geq \frac{(n-1)^{p/2}|\partial B|}{n(p-n)}(1-\gamma^2)^{p/2}h^{n-p}(\gamma).$$

Taking the supremum and writing

$$H := \sup_{\gamma\in(0,1)}(1-\gamma^2)^{p/2}h^{n-p}(\gamma),$$

we have

$$\lim_{\varepsilon\to 0}\varepsilon^{p-n}E_\varepsilon(u_\varepsilon,B_\sigma(0)) \geq \frac{(n-1)^{p/2}|\partial B|}{n(p-n)}H.$$

Combining this with (2.9), we obtain the lower bound of L in (1.10). □

3. Proof of Theorem 1.3

In this section, we assume $p=n>2$, and u_ε is the regularized radial minimizer. In view of [13: Theorem 2.7], (1.7) is still true.

Proposition 3.1. *Assume h is the constant in* (1.7). *Then for any $\sigma>0$, we can find $C>0$ which is independent of ε, such that*

$$\int_{B_{h\varepsilon}(0)\cup(B\setminus B_\sigma(0))}|\nabla u_\varepsilon|^n\mathrm{d}x\leq C; \tag{3.1}$$

$$\int_{B_\sigma(0)\setminus B_{h\varepsilon}(0)}|\nabla f|^n\mathrm{d}x\leq C; \tag{3.2}$$

$$\frac{1}{2\varepsilon^n}\int_B u_{\varepsilon n+1}^2\mathrm{d}x\leq C. \tag{3.3}$$

P r o o f. Clearly,

$$|\nabla u_\varepsilon|^n \ge |\nabla f|^n + \sin^n f |\nabla \frac{x}{|x|}|^n. \tag{3.4}$$

According to [13: Lemmas 2.2 and 3.1], we know

$$E_\varepsilon(u_\varepsilon, B) \le \frac{1}{n}(n-1)^{n/2}|\partial B||\log\varepsilon| + C; \tag{3.5}$$

$$\int_{B_\sigma(0)\setminus B_{h\varepsilon}(0)} |\nabla \frac{x}{|x|}|^n \mathrm{d}x \ge (n-1)^{n/2}|\partial B||\log\varepsilon| - C. \tag{3.6}$$

Eq. (3.5) subtracts (3.6). Then by applying (3.4) we obtain

$$\frac{1}{n}\int_{B_{h\varepsilon}(0)\cup(B\setminus B_\sigma(0))} |\nabla u_\varepsilon|^n \mathrm{d}x + \frac{1}{n}\int_{B_\sigma(0)\setminus B_{h\varepsilon}(0)} |\nabla f|^n \mathrm{d}x + \frac{1}{2\varepsilon^n}\int_B u_{\varepsilon n+1}^2 \mathrm{d}x$$
$$\le C + \frac{1}{n}\int_{B_\sigma(0)\setminus B_{h\varepsilon}} (1-\sin^n f)\left|\nabla \frac{x}{|x|}\right|^n \mathrm{d}x.$$

Using in [13: Lemma 3.2], we can deduce (3.1)–(3.3). Proposition 3.1 is proved. $\square$

P r o o f o f (1.11). By virtue of (3.5), there exists a Radon measure μ_1 such that

$$\lim_{\varepsilon_k\to 0} \frac{|\nabla u_{\varepsilon_k}|^n}{|\log\varepsilon_k|} = \mu_1 \qquad \text{in the weak * topology of } C(\overline{B}).$$

Here ε_k is some subsequence of ε.

In view of (3.1), for arbitrary compact subset K of $\overline{B}\setminus\{0\}$, there holds

$$\lim_{\varepsilon\to 0} \frac{|\nabla u_\varepsilon|^n}{|\log\varepsilon|} = 0 \qquad \text{in} \quad L^1(K).$$

Therefore, we can find $m_1 \ge 0$ such that

$$\mu_1 = m_1\delta_o. \tag{3.7}$$

By virtue of (3.5),

$$m_1 \le (n-1)^{n/2}|\partial B|. \tag{3.8}$$

On the other hand, (3.6) implies

$$m_1 \ge (n-1)^{n/2}|\partial B|.$$

Combining this with (3.8) yields

$$m_1 = (n-1)^{n/2}|\partial B|.$$

Substituting this result into (3.7), and noting the limit $\mu_1 = m_1\delta_o$ is unique, we get (1.11). $\square$

Proof of (1.12).

Step 1.

By virtue of (3.3), there exists a Radon measure μ_2, such that

$$\lim_{\varepsilon_k\to 0}\frac{u^2_{\varepsilon_k n+1}}{2\varepsilon_k^n}=\mu_2, \qquad \text{in the weak * topology of } C(\overline{B}). \tag{3.9}$$

Similar to the derivation of [8: (4.2), (4.6), (4.8) and Theorem 1.3 in], we can also obtain (1.2) and (1.3), and as $\varepsilon_k\to 0$,

$$\sup_K\frac{u^2_{\varepsilon_k n+1}}{2\varepsilon_k^n}\le C\varepsilon_k^n\to 0; \tag{3.10}$$

$$\lim_{\varepsilon_k\to 0}u_{\varepsilon_k}=\left(\frac{x}{|x|},0\right) \qquad \text{in } C^{1,\beta}(K) \text{ for some } \beta\in(0,1/2), \tag{3.11}$$

where K is an arbitrary compact subset of $B\setminus\{0\}$. Eqs. (3.9) and (3.10) lead to

$$\mu_2=m_2\delta_o, \qquad 0\le m_2\in\mathbf{R}.$$

Step 2.

Let $R=\sigma$ and $p=n$ in (2.5), then

$$\begin{aligned}\frac{1}{n}\int\limits_{\partial B_\sigma(0)}v^{n/2}|x|\mathrm{d}s-\int\limits_{\partial B_\sigma(0)}v^{(n-2)/2}|\partial_\nu u|^2|x|\mathrm{d}s-\tau\int\limits_{B_\sigma(0)}v^{(n-2)/2}\mathrm{d}x\\
=-\frac{1}{2\varepsilon^n}\int\limits_{\partial B_\sigma(0)}|x|u^2_{n+1}\mathrm{d}s+\frac{n}{2\varepsilon^n}\int\limits_{B_\sigma(0)}u^2_{n+1}\mathrm{d}x.\end{aligned}$$

Here $v=|\nabla u|^2+\tau$. Letting $\tau\to 0$ and using (1.2) and (1.3), we have

$$\begin{aligned}\frac{1}{n}\int\limits_{\partial B_\sigma(0)}|\nabla u_\varepsilon|^n|x|\mathrm{d}s-\int\limits_{\partial B_\sigma(0)}|\nabla u_\varepsilon|^{n-2}|\partial_\nu u_\varepsilon|^2|x|\mathrm{d}s\\
=-\frac{1}{2\varepsilon^n}\int\limits_{\partial B_\sigma(0)}|x|u^2_{n+1}\mathrm{d}s+\frac{n}{2\varepsilon^n}\int\limits_{B_\sigma(0)}u^2_{n+1}\mathrm{d}x.\end{aligned} \tag{3.12}$$

Clearly, (3.10) shows that

$$\frac{1}{2\varepsilon_k^n}\int\limits_{\partial B_\sigma(0)}|x|u^2_{n+1}\mathrm{d}s\to 0 \tag{3.13}$$

when $\varepsilon_k \to 0$. Next, from (3.11) we can deduce that as $\varepsilon_k \to 0$,

$$\begin{aligned} &f \to \frac{\pi}{2}, && \text{for} \quad r = \sigma; \\ &|\partial_\nu u|^2 = |f_r|^2 \to 0, && \text{for} \quad r = \sigma; \\ &\int_{\partial B_\sigma(0)} |\nabla u_{\varepsilon_k}|^{n-2} |x| \mathrm{d}s \le C. \end{aligned}$$

Thus, as $\varepsilon_k \to 0$,

$$\int_{\partial B_\sigma(0)} |\nabla u_{\varepsilon_k}|^{n-2} |\partial_\nu u_{\varepsilon_k}|^2 |x| \, \mathrm{d}s \to 0. \tag{3.14}$$

At last, there holds

$$|\nabla u|^n \ge \sin^n f |\nabla \left|\frac{x}{|x|}\right|^n = \left|\nabla \frac{x}{|x|}\right|^n - (1 - \sin^n f) \left|\nabla \frac{x}{|x|}\right|^n.$$

On the other hand, (3.10) implies that as $\varepsilon_k \to 0$,

$$\int_{\partial B_\sigma(0)} (1 - \sin^n f) \left|\nabla \frac{x}{|x|}\right|^n \mathrm{d}s \to 0.$$

Therefore,

$$\begin{aligned} \lim_{\varepsilon_k \to 0} \frac{1}{n} \int_{\partial B_\sigma(0)} |\nabla u_{\varepsilon_k}|^n |x| \mathrm{d}s &\ge \frac{1}{n} \int_{\partial B_\sigma(0)} \left|\nabla \frac{x}{|x|}\right|^n |x| \, \mathrm{d}s \\ &= (n-1)^{n/2} \frac{|\partial B|}{n}. \end{aligned} \tag{3.15}$$

Letting $\varepsilon_k \to 0$ in (3.12), and inserting (3.13)–(3.15) into this result, we obtain

$$\lim_{\varepsilon_k \to 0} \frac{n}{2\varepsilon_k^n} \int_{B_\sigma} u_{\varepsilon_k n+1}^2 \mathrm{d}x \ge (n-1)^{n/2} \frac{|\partial B|}{n}.$$

This result, together with (3.9), implies

$$m_2 \ge (n-1)^{n/2} \frac{|\partial B|}{n^2}. \tag{3.16}$$

Step 3.
By Struwe's idea (cf. [15]), we claim that, if $\varepsilon = \varepsilon_k$ is a sequence converging to zero, then there exists an integer $k_0 > 0$ which is independent of ε, such that as $k > k_0$,

$$\frac{n}{2\varepsilon_k^n} \int_B u_{\varepsilon_k n+1}^2 \mathrm{d}x \le (n-1)^{n/2} \frac{|\partial B|}{n}. \tag{3.17}$$

In fact, if we denote $\nu(\varepsilon) = \inf\{E_\varepsilon(u,G): \ u \in W\}$, then the map $\varepsilon \to E_\varepsilon(u,G)$ is not increasing, and

$$-\frac{\partial}{\partial\varepsilon}E_\varepsilon(u,B) = \frac{n}{2\varepsilon^{n+1}}\int_B u_{n+1}^2 \mathrm{d}x.$$

Thus, for the minimizer $u = u_\varepsilon$ of $E_\varepsilon(u,G)$,

$$\begin{aligned}\frac{n}{2\varepsilon^{n+1}}\int_B u_{n+1}^2 \mathrm{d}x &= \lim_{\delta\to 0}\frac{E_\varepsilon(u,G) - E_{\varepsilon+\delta}(u,G)}{\delta}\\ &\le \lim_{\delta\to 0}\frac{\nu(\varepsilon)-\nu(\varepsilon+\delta)}{\delta} = -\nu'(\varepsilon),\end{aligned}$$

since $\nu(\varepsilon+\delta) \le E_{\varepsilon+\delta}(u,G) \le E_\varepsilon(u,G) = \nu(\varepsilon)$. Suppose (3.17) is not true, there must be a fixed $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$,

$$-\nu'(\varepsilon) > (n-1)^{n/2}|\partial B|(n\varepsilon)^{-1}.$$

Integrating from ε to ε_0, we have

$$\nu(\varepsilon) = -\int_\varepsilon^{\varepsilon_0}\nu'(\varepsilon)\mathrm{d}\varepsilon + \nu(\varepsilon_0) > \frac{1}{n}(n-1)^{n/2}|\partial B|\log\frac{\varepsilon_0}{\varepsilon} + \nu(\varepsilon_0).$$

It contradicts (3.5) as long as ε is sufficiently small. (3.17) is proved.

Combining (3.17) with (3.9), we get

$$m_2 \le (n-1)^{n/2}\frac{|\partial B|}{n^2}.$$

This result, together with (3.16), means

$$m_2 = (n-1)^{n/2}\frac{|\partial B|}{n^2}.$$

Substituting this into (3.9), and noting the limit $\mu_2 = m_2\delta_o$ is unique, we complete the proof of (1.12). □

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* *School of mathematics and information technology*
Jiangsu Second Normal University
Nanjing, Jiangsu, 210013
CHINA

** *Department of Basic Science*
Shazhou Professional Institute of Technology
Zhangjiagang, Jiangsu, 215600
CHINA

*** *Corresponding author:*
Jiangsu Key Laboratory for NSLSCS
School of Mathematics Sciences
Nanjing Normal University
Nanjing, Jiangsu, 210023
CHINA
E-mail: leiyutian@njnu.edu.cn