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ON OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF FOURTH ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, oscillatory and asymptotic properties of solutions of nonlinear fourth order neutral dynamic equations of the form

$$\big(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2}\big)^{\Delta^2}+q(t)G\big(y(\alpha_2(t))\big)-h(t)H\big(y(\alpha_3(t))\big)=0 \quad \text{(H)}$$

and

$$\big(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2}\big)^{\Delta^2}+q(t)G\big(y(\alpha_2(t))\big)-h(t)H\big(y(\alpha_3(t))\big)=f(t), \quad \text{(NH)}$$

are studied on a time scale $\mathbb{T}$ under the assumption that $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty$ and for various ranges of $p(t)$. In addition, sufficient conditions are obtained for the existence of bounded positive solutions of the equation (NH) by using Krasnosel'skii's fixed point theorem.

1. Introduction

In this paper we study the oscillatory and asymptotic properties of solutions of the nonlinear fourth order neutral dynamic equations

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\alpha_2(t)))-h(t)H(y(\alpha_3(t)))=0 \quad \text{(H)}$$

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and

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2}+q(t)G(y(\alpha_2(t)))-h(t)H(y(\alpha_3(t)))=f(t), \tag{NH}$$

for $t\in[t_0,\infty]_{\mathbb{T}}$, $t_0\geqslant 0$, where $\mathbb{T}$ is a time scale (i.e., a closed subset of the real numbers) such that $\sup\mathbb{T}=\infty$ and $t_0\in\mathbb{T}$. This formulation is quite general in that it includes as special cases the well known fourth order Emden-Fowler type ordinary differential equation

$$y^{(4)}+q(t)|y|^{\gamma}\operatorname{sgn}y=0$$

and its discrete analog, i.e., the difference equation

$$\Delta^4 y_n+q_n|y_n|^{\gamma}\operatorname{sgn}y_n=0.$$

Variations on these equations such as those with time delays, forcing terms, or equations involving neutral terms have been widely studied in the literature and these are included in the forms of (H) and (NH) as well. What is also important to point out here is that using the framework of time scales, a single result may apply to differential equations and difference equations at the same time. (This is demonstrated in Section 4 which contains some examples of the main results in this paper.) At the same time, the results would apply to other time scales such as the quantum time scale $\mathbb{T}=q^{\mathbb{N}_0}=\{t:\ t=q^k,\ k\in\mathbb{N}_0\}$ with $q>1$, $\mathbb{T}=\mathbb{N}_0^2=\{t^2:\ t\in\mathbb{N}_0\}$, $\mathbb{T}=\{\sqrt{n}:\ n\in\mathbb{N}_0\}$, etc. The study of dynamic equations on time scales originated with the seminal work of Stefan Hilger [4] to unify the study of continuous and discrete mathematics and has proved to be particularly useful in the study of differential and difference equations. This approach has been popularized with the publication of the monographs by Bohner and Peterson [1, 2] to which we refer the reader for additional background and common notation.

We consider equations (H) and (NH) under the assumption that

$$\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty \tag{H_0}$$

as opposed to the more restrictive condition

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\Delta t=\infty$$

that is often used. We will present results for various ranges of $p(t)$. Sufficient conditions are also obtained for the existence of bounded positive solutions of (NH) by using Krasnosel'kii's fixed point theorem.

Thandapani and Arockiasamy [8] considered the fourth order non-linear neutral difference equation

$$\Delta^2(r_n\Delta^2(y_n+p_ny_{n-k}))+f(n,y_{\sigma(n)})=0,\qquad n\in N(n_0), \tag{1.1}$$

where $N(n_0) = \{n_0, n_0+1, n_0+2, \dots\}$, $f\colon N(n_0)\times\mathbb{R}\to\mathbb{R}$ is a continuous function with $uf(n,u) > 0$ for all $u \neq 0$, $\{r_n\}$ and $\{p_n\}$ are positive real sequences, $\{\sigma_n\}$ is an increasing sequence of integers, and k is a non negative integer. Under the assumption that $0 \leqslant p_n < p < 1$, they obtained necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1) with various asymptotic properties as well as necessary and sufficient conditions for all solutions of (1.1) to oscillate. Clearly, if we consider $f(n, y_{\sigma(n)}) = q(n)G(y(n-k))$, then the work in [8] is a particular case of [7] for the corresponding ranges of p_n.

In [6], the authors studied the oscillatory and asymptotic behavior of solutions of the fourth order nonlinear neutral dynamic equations

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\alpha_2(t))) = 0 \tag{1.2}$$

and

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2} + q(t)G(y(\alpha_2(t))) = f(t), \tag{1.3}$$

under the assumptions that $q(t) > 0$ and $\int\limits_{t_0}^{\infty} \frac{t}{r(t)}\,\mathrm{d}t < \infty$, and for various ranges of $p(t)$. Their work showed that if $q(t) < 0$, then it would be possible to obtain analogous results for the oscillation and asymptotic behavior of solutions of (1.2) and (1.3). The problem remains open as to what happens if $q(t)$ is allowed to change signs. Note that if $q(t) = q^+(t) - q^-(t)$, where $q^+(t) = \max\{0, q(t)\}$ and $q^-(t) = \max\{0, -q(t)\}$, then (1.2) and (1.3) can be viewed as

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\alpha_2(t))) - q^-(t)G(y(\alpha_2(t))) = 0 \tag{1.4}$$

and

$$(r(t)(y(t)+p(t)y(\alpha_1(t)))^{\Delta^2})^{\Delta^2} + q^+(t)G(y(\alpha_2(t))) - q^-(t)G(y(\alpha_2(t))) = f(t), \tag{1.5}$$

respectively.

Clearly, (1.4)–(1.5) are particular cases of (H)–(NH). Here we generalize the results of [6, 7] to fourth order dynamic equations on time scales. To the best of our knowledge there are no papers to date on fourth order nonlinear dynamic equations with positive and negative coefficients. The results obtained in this paper are new and generalize the earlier work in [6–8].

In equations (H) and (NH), we assume that $r \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $p, f \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $q, h \in C_{rd}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, $G, H \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$ and $uH(u) > 0$ for $u \neq 0$, G is nondecreasing, H is bounded, $\alpha_1, \alpha_2, \alpha_3 \in C_{rd}(\mathbb{T},\mathbb{T})$ are strictly increasing functions such that

$$\lim_{t\to\infty}\alpha_1(t) = \lim_{t\to\infty}\alpha_2(t) = \lim_{t\to\infty}\alpha_3(t) = \infty, \qquad \text{and} \qquad \alpha_1(t),\ \alpha_2(t),\ \alpha_3(t) \leqslant t.$$

We also denote the inverse of α_1 by $\alpha_1^{-1} \in C_{rd}(\mathbb{T},\mathbb{T})$. We define the time scale interval by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty)\cap\mathbb{T}$.

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}} \{\alpha_1(t), \alpha_2(t), \alpha_3(t)\}$. By a *solution* of (H) (or (NH)) we mean a function $y \in C_{rd}([t_{-1},\infty)_{\mathbb{T}},\mathbb{R})$ with $y(t)+p(t)y(\alpha_1(t)) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)(y(t) + p(t)y(\alpha_1(t)))^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and such that (H) (or (NH)) is satisfied identically on $[t_0,\infty)_{\mathbb{T}}$. A solution is called *oscillatory* if it is neither eventually positive nor eventually negative, and it is called *nonoscillatory* otherwise. In this paper we do not discuss eventually identically vanishing solutions. An equation will be called oscillatory if all its solutions are oscillatory.

2. Homogeneous oscillations

In this section, we obtain sufficient conditions for the oscillation of solutions of equation (H). We will need the following lemmas in the sequel.

Lemma 2.1. ([6: Lemma 3.1]) *Let* (H_0) *hold and u be a twice rd-continuously differentiable function on $[t_0,\infty]_{\mathbb{T}}$ such that $r(t)u^{\Delta^2}(t) \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(r(t)u^{\Delta^2}(t))^{\Delta^2} \le 0$ for large $t \in [t_0,\infty)_{\mathbb{T}}$. If $u(t) > 0$ eventually, then one of the following cases* (a) *or* (b) *holds for large t, and if $u(t) < 0$ eventually, then one of the cases* (b), (c), (d), *or* (e) *holds for large t, where*

(a) $u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) > 0 \quad$ *and* $\quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,$

(b) $u^{\Delta}(t) > 0, \quad u^{\Delta^2}(t) < 0 \quad$ *and* $\quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,$

(c) $u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \quad$ *and* $\quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0,$

(d) $u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) < 0 \quad$ *and* $\quad (r(t)u^{\Delta^2}(t))^{\Delta} < 0,$

(e) $u^{\Delta}(t) < 0, \quad u^{\Delta^2}(t) > 0 \quad$ *and* $\quad (r(t)u^{\Delta^2}(t))^{\Delta} > 0.$

Lemma 2.2. ([6: Lemma 3.2]) *Let the conditions of* Lemma 2.1 *hold. If $u(t) > 0$ eventually, then $u(t) > R_T(t)(r(t)u^{\Delta^2}(t))^{\Delta}$ for $t \geqslant T \geqslant t_0$, where*

$$R_T(t) = \int_T^{\rho(t)} \frac{(t-\sigma(s))(s-T)}{r(s)} \Delta s.$$

Remark 1. Notice that $R_T(t)$ is an increasing function.

Lemma 2.3. ([6: Lemma 3.3]) *Let $F, H, P\colon [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ satisfy*

$$F(t) = H(t) + P(t)H(\alpha(t)) \qquad \text{for} \quad t \in [\hat{t},\infty)_{\mathbb{T}},$$

where $\hat{t} \in [t_0,\infty)_{\mathbb{T}}$, $\alpha(t) \le t$, $\alpha(t) \to \infty$ as $t \to \infty$ and $\alpha(t) \geqslant t_0$ for all $t \in [\hat{t},\infty)_{\mathbb{T}}$. Assume that there exist P_1, $P_2 \in \mathbb{R}$ such that $P(t)$ is in one of the following ranges:

(1) $-\infty < P(t) \leqslant 0,$

(2) $0 \leqslant P(t) \leqslant P_1 < 1,$

(3) $1 < P_2 \leqslant P(t) < \infty.$

If $H(t) > 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $\liminf_{t\to\infty} H(t) = 0$, *and* $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ *exists, then* $L = 0$.

Discussions of the oscillatory behavior of solutions of differential equations and difference equations for various ranges of values of $p(t)$ can be found in [3] and [9], respectively.

Lemma 2.4 (**Krasnosel'skii's fixed point theorem**). ([5: Lemma 3]) *Let* S *be a bounded, convex and closed subset of the Banach space* X. *Suppose there exists two operators* $A, B\colon S \to X$ *such that*

(i) $Ax + By \in S$ *for all* x, $y \in S$,

(ii) A *is a contraction mapping,*

(iii) B *is completely continuous.*

Then $A + B$ *has a fixed point in* S, *that is,* $Ax + Bx = x$ *for some* $x \in S$.

The results in our paper will make use of the following conditions on the functions in equations (H) and (NH):

(H_1) $\int\limits_{t_0}^{\infty} \frac{\sigma(s)}{r(s)} \int\limits_{s}^{\infty} \sigma(t)h(t)\Delta t\Delta s < \infty$;

(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geqslant \lambda G(u+v)$ for $u > 0$ and $v > 0$;

(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;

(H_4) $\int\limits_{0}^{\pm c} \frac{\mathrm{d}u}{G(u)} < \infty$ for all $c > 0$;

(H_5) $\int\limits_{\hat{t}}^{\infty} Q(t)\Delta t = \infty$, $Q(t) = \min\{q(t), q(\alpha_1(t))\}$ for $t \geqslant \hat{t}$, $\hat{t} \in [t_0, \infty)_{\mathbb{T}}$ such that $\alpha_1(t) \geqslant t_0$ for all $t \in [\hat{t}, \infty)_{\mathbb{T}}$.

Also, when the function Q above is employed we will need the that there exists $\tilde{t} \in [t_0, \infty)_{\mathbb{T}}$ such that $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t) \geqslant t_0$ and

$$(\alpha_1 \circ \alpha_2)(t) = (\alpha_2 \circ \alpha_1)(t) \tag{Λ}$$

for $t \in [\tilde{t}, \infty)_{\mathbb{T}}$.

Theorem 2.1. *Assume that conditions* (H_0)–(H_5) *and* (Λ) *hold,* $\alpha_2(t) \le \alpha_1(t)$, *and* p_1, p_2, *and* p_3 *are positive real numbers. If*

(i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or*

(ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$

holds, then every solution of (H) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (H), say $y(t)$ is an eventually positive solution. (The proof in case $y(t) < 0$ eventually is similar and will be omitted.) There exist $t_1 \in [\tilde{t}, \infty)_{\mathbb{T}}$ such that $y(t)$, $y(\alpha_1(t))$, $y(\alpha_2(t))$, $y(\alpha_3(t))$ and $y(\alpha_2(\alpha_1(t))) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Define the functions

$$z(t) = y(t) + p(t)y(\alpha_1(t)), \tag{2.1}$$

$$k(t) = \int_t^\infty \frac{\sigma(s) - t}{r(s)} \int_s^\infty (\sigma(\theta) - s)h(\theta)H(y(\alpha_3(\theta)))\Delta\theta\Delta s. \tag{2.2}$$

Notice that condition (H_1) and the fact that H is a bounded function imply that $k(t)$ exists for all t. Now if

$$w(t) = z(t) - k(t) = y(t) + p(t)y(\alpha_1(t)) - k(t), \tag{2.3}$$

then a calculation shows

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\alpha_2(t))) \leqslant 0, \tag{2.4}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Clearly, $w(t)$, $w^\Delta(t)$, $(r(t)w^{\Delta^2}(t))$ and $(r(t)w^{\Delta^2}(t))^\Delta$ are monotonic functions on $[t_1, \infty)_{\mathbb{T}}$. In view of Lemma 2.1, we have two cases to consider, namely $w(t) > 0$ or $w(t) < 0$ for $t \geqslant t_2$ for some $t_2 > t_1$.

Suppose that $w(t) > 0$ for $t \geqslant t_2$; then there exists $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(\alpha_1(t))$, $w(\alpha_2(t)) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Using (H_2) and (H_3) gives

$$\begin{aligned}
0 &= (r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\alpha_2(t))) + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2} \\
&\quad + G(p)q(\alpha_1(t))G(y(\alpha_2(\alpha_1(t)))) \\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(y(\alpha_2(t)) \\
&\quad + py(\alpha_1(\alpha_2(t)))) \\
&\geqslant (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(z(\alpha_2(t)))
\end{aligned} \tag{2.5}$$

for $t \geqslant t_3$. From (2.2), it follows that $k(t) > 0$ and $k^\Delta(t) < 0$, so $w(\alpha_2(t)) > 0$ for $t \geqslant t_3$ implies $w(\alpha_2(t)) < z(\alpha_2(t))$ for $t \geqslant t_3$. Therefore, (2.5) yields

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(w(\alpha_2(t))) \leqslant 0 \tag{2.6}$$

for $t \geqslant t_3$. Choose $T' \in [T, \infty)_{\mathbb{T}}$ so that $\alpha_2(t) \geqslant T \geqslant t_3$ for all $t \in [T', \infty)_{\mathbb{T}}$. Applying (H_3) and Lemma 2.2, inequality (2.6) yields

$$\begin{aligned}
0 \geqslant\ & (r(t)w^{\Delta^2}(t))^{\Delta^2} + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2} \\
& + \lambda Q(t)G(R_T(\alpha_2(t)))G((r(\alpha_2(t))w^{\Delta^2}(\alpha_2(t)))^\Delta)
\end{aligned}$$

for $t \geqslant T'$. Hence,

$$\begin{aligned}
&\lambda Q(t)G(R_T(\alpha_2(t))) \\
&\quad \leqslant -[G((r(\alpha_2(t))w^{\Delta^2}(\alpha_2(t)))^\Delta)]^{-1}\{(r(t)w^{\Delta^2}(t))^{\Delta^2} \\
&\qquad + G(p)(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2}\}
\end{aligned}$$

$$\begin{aligned}\leqslant\ & -[G((r(t)w^{\Delta^2}(t))^{\Delta})]^{-1}(r(t)w^{\Delta^2}(t))^{\Delta^2}\\ & -G(p)G((r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta})]^{-1}(r(\alpha_1(t))w^{\Delta^2}(\alpha_1(t)))^{\Delta^2}.\end{aligned}$$

Since $\lim\limits_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, applying (H_4) to the above inequality gives

$$\int_{T'}^{\infty} Q(t)G(R_T(\alpha_2(t)))\Delta t < \infty,$$

which contradicts (H_5) since $R_T(t)$ is a monotone increasing function.

Next, we suppose that $w(t) < 0$ for $t \geqslant t_2 > t_1$. Then $z(t) - k(t) < 0$ implies $y(t) \leqslant z(t) = y(t) + p(t)y(\alpha_1(t)) < k(t)$. Thus, $y(t)$ is bounded. By Lemma 2.1, any one of the cases (b), (c), (d) or (e) holds. Consider case (b). Since $\lim\limits_{t\to\infty} k(t)$ exists, $\lim\limits_{t\to\infty} w(t)$ exists, and so $\lim\limits_{t\to\infty} z(t)$ exists. Furthermore, $\lim\limits_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists, and an integration of (2.4) implies

$$\int_{t_2}^{\infty} Q(t)G(y(\alpha_2(t)))\Delta t < \infty.$$

Hence, it is easy to verify that $\liminf\limits_{t\to\infty} y(t) = 0$ due to (H_5). It then follows from Lemma 2.3 that $\lim\limits_{t\to\infty} z(t) = 0$. Thus, $\lim\limits_{t\to\infty} y(t) = 0$ since $z(t) \geqslant y(t)$.

To see that cases (c) and (d) are not possible, first note that $w(t) < 0$, $y(t)$ is bounded, and $\lim\limits_{t\to\infty} k(t)$ exists and so $\lim\limits_{t\to\infty} w(t)$ exists. On the other hand, integrating $w^{\Delta^2}(t) < 0$ twice from t_2 to $t \geqslant t_2$, yields $\lim\limits_{t\to\infty} w(t) = -\infty$, which is a contradiction. In case (e), $r(t)w^{\Delta^2}(t)$ is nondecreasing on $[t_2,\infty)_{\mathbb{T}}$. Hence, for $t > t_3 \geqslant t_2$, $r(t)w^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3)$, and we have

$$tw^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3)\frac{t}{r(t)}.$$

Integrating the above inequality from t_3 to t, we obtain

$$tw^{\Delta}(t) - t_3w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(\sigma(s))\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s,$$

or

$$tw^{\Delta}(t) - t_3w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(s)\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s.$$

By (H_0),

$$w(t) \leqslant -t_3 w^{\Delta}(t_3) + w(t_3) - (r(t_3) w^{\Delta^2}(t_3)) \int_{t_3}^{t} \frac{s}{r(s)} \Delta s \to -\infty$$

as $t \to \infty$, which is a contradiction. This completes the proof of the theorem. □

The following corollary is immediate.

Corollary 2.1.1. *Under the conditions of* Theorem 2.1, *every unbounded solution of* (H) *oscillates.*

In our next theorem, we replace condition (H_4) by a different type of growth condition on the function G.

Theorem 2.2. *Assume that conditions* (H_0)–(H_3), (H_5), *and* (Λ) *hold,* $\alpha_2(t) \leq \alpha_1(t)$, *and*

(H_6) $G(x_1)/x_1^{\gamma} \geqslant G(x_2)/x_2^{\gamma}$ *for* $x_1 \geqslant x_2 > 0$ *and some* $\gamma \geqslant 1$.

If (i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or*

(ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$

holds, then every solution of (H) *is either oscillatory or converges to zero as* $t \to \infty$.

P r o o f. Proceeding as in the proof of Theorem 2.1, in case $w(t) > 0$ we can again obtain inequality (2.6) for $t \geqslant t_3$. In view of (2.4) and Lemma 2.1, $w(t)$ is increasing, and since $\alpha_2(t)$ is increasing, there exists $k > 0$ and $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $w(\alpha_2(t)) > k$ for all $t \geqslant t_4$. Using (H_6) and Lemma 2.2, we obtain

$$\begin{aligned}
G(w(\alpha_2(t))) &= \left(\frac{G(w(\alpha_2(t))}{w^{\gamma}(\alpha_2(t))}\right) w^{\gamma}(\alpha_2(t)) \\
&\geqslant \left(\frac{G(k)}{k^{\gamma}}\right) w^{\gamma}(\alpha_2(t)) \\
&> \left(\frac{G(k)}{k^{\gamma}}\right) R_T^{\gamma}(\alpha_2(t))((r(\alpha_2(t)) w^{\Delta^2}(\alpha_2(t)))^{\Delta})^{\gamma}
\end{aligned}$$

for $t > t_4$. Thus, (2.6) yields,

$$\begin{aligned}
\lambda Q(t) &\left(\frac{G(k)}{k^{\gamma}}\right) R_T^{\gamma}(\alpha_2(t))((r(\alpha_2(t)) w^{\Delta^2}(\alpha_2(t)))^{\Delta})^{\gamma} \\
&< \lambda Q(t) G(w(\alpha_2(t))) \\
&\leqslant \lambda Q(t) G(w(\alpha_2(t))) \\
&\leqslant -(r(t) w^{\Delta^2}(t))^{\Delta^2} - G(p)(r(\alpha_1(t)) w^{\Delta^2}(\alpha_1(t)))^{\Delta^2}
\end{aligned}$$

for $t > t_4$. Since $\lim_{t\to\infty}(r(t)w^{\Delta^2}(t))^{\Delta}$ exists and $R_T(t)$ is nondecreasing, proceeding as in the proof of Theorem 2.1 we obtain

$$\int_{t_4}^{\infty} R_T^{\gamma}(\alpha_2(t))Q(t)\Delta t < \infty,$$

which contradicts (H_5). The proof in case $w(t) > 0$ is the same as in Theorem 2.1. □

We again have a corollary.

Corollary 2.2.1. *Under the conditions of* Theorem 2.2, *every unbounded solution of* (H) *oscillates.*

In our next theorem we are able to replace conditions (H_3) and (H_4) in Theorem 2.1 with conditions (H_7) and (H_8) below.

Theorem 2.3. *Assume that conditions* (H_0)–(H_2), (H_5), *and* (Λ) *hold,* $\alpha_2(t) \le \alpha_1(t)$, *and*

(H_7) $G(u)G(v) \geqslant G(uv)$ *for* $u, v > 0$,

(H_8) $G(-u) = -G(u)$ *for* $u \in \mathbb{R}$.

If (i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or*

(ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$

holds, then every solution of (H) *is either oscillatory or converges to zero as* $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 2.1, in case $w(t) > 0$ we again have (2.6) for $t \geqslant t_3$. Since $w(\alpha_2(t))$ is nondecreasing, there exist $k > 0$ and $t_4 > t_3$ such that $w(\alpha_2(t)) > k$ for $t \geqslant t_4$, so $z(\alpha_2(t)) \geqslant w(\alpha_2(t)) \geqslant k$ for $t \geqslant t_4$. Consequently, (2.5) yields

$$\lambda G(k)\int_{t_4}^{\infty} Q(t)\Delta t < \infty$$

contradicting (H_5). The remainder of the proof is similar to the proof of Theorem 2.1. □

We again have a corollary for the unbounded solutions.

Corollary 2.3.1. *Under the conditions of* Theorem 2.3, *every unbounded solution of* (H) *oscillates.*

Remark 2. Notice that in Theorem 2.1 and Corollary 2.1.1, G is sublinear, whereas in Theorem 2.2 and Corollary 2.2.1, G is superlinear. But in Theorem 2.3 and Corollary 2.3.1, G could be linear, sublinear, or superlinear.

Next, we consider the case where $p(t)$ is negative. Here p_4, p_5, and p_6 are negative real numbers.

Theorem 2.4. *Let $-1 < p_4 \leqslant p(t) \leqslant 0$ and conditions* (H_0), (H_1), (H_3), (H_4), *and*

(H_9) $\int\limits_{t_0}^{\infty} q(t)\Delta t = \infty$

hold. Then every solution of (H) *is either oscillatory or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (H), say $y(t) > 0$ eventually. From (2.1)–(2.3) we obtain (2.4) for $t \geqslant t_1$. By Lemma 2.1, $w(t)$ is monotonic. If $w(t) > 0$ for $t \geqslant t_2 > t_1$, then either case (a) or case (b) of Lemma 2.1 holds. Consequently, $w(t) \geqslant R_T(t)(r(t)w^{\Delta^2}(t))^{\Delta}$ for $t \geqslant T > t_1$ by Lemma 2.2. Moreover, $w(t) \leqslant y(t)$ since $p(t) \leqslant 0$. Choose $t_3 \in [T,\infty)_{\mathbb{T}}$ such that $\alpha_2(t) \geqslant T$ for all $t \in [t_3,\infty)_{\mathbb{T}}$. Then, $y(\alpha_2(t)) \geqslant R_T(\alpha_2(t))(r(\alpha_2(t))w^{\Delta^2}(\alpha_2(t)))^{\Delta}$ for $t \geqslant t_3$ and (2.4) becomes

$$\int\limits_{t_3}^{\infty} q(t)G(R_T(\alpha_2(t)))\Delta t < \infty,$$

which contradicts (H_9) since G, R_T, and α_2 are increasing functions. Hence, $w(t) < 0$ for $t \geqslant t_2$, and so one of the cases (b), (c), (d), or (e) of Lemma 2.1 holds.

We claim that $y(t)$ is bounded. If this is not the case, then there is an increasing sequence $\{\tau_n\}_{n=1}^{\infty} \subset [t_2,\infty)_{\mathbb{T}}$ such that $\tau_n \to \infty$, $y(\tau_n) \to \infty$ as $n \to \infty$, and $y(\tau_n) = \max\bigl\{y(t) : \ t_2 \leqslant t \leqslant \tau_n\bigr\}$. We may choose τ_1 large enough so that $\alpha_1(\tau_1) \geqslant t_2$. Hence,

$$0 \geqslant w(\tau_n) \geqslant y(\tau_n) + p(\tau_n)y(\alpha_1(\tau_n)) - k(\tau_n) \geqslant (1+p_4)y(\tau_n) - k(\tau_n).$$

Since $k(\tau_n)$ is bounded and $1 + p_4 > 0$, we have $w(\tau_n) > 0$ for large n, which is a contradiction. Thus, our claim holds.

The proofs that cases (c), (d), and (e) cannot hold are similar to the corresponding cases in the proof of Theorem 2.1. If (b) holds, then as in proof of Theorem 2.1 we obtain $\liminf\limits_{t\to\infty} y(t) = 0$. Hence, $\lim\limits_{t\to\infty} z(t) = 0$ by Lemma 2.3. Consequently,

$$\begin{aligned}
0 = \limsup_{t\to\infty} z(t) &\geqslant \limsup_{t\to\infty}(y(t) + p_4 y(\alpha_1(t))) \\
&\geqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_4 y(\alpha_1(t))) \\
&= \limsup_{t\to\infty} y(t) + p_4 \limsup_{t\to\infty} y(\alpha_1(t)) \\
&= (1+p_4)\limsup_{t\to\infty} y(t).
\end{aligned}$$

Since $(1+p_4) > 0$, $\limsup_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$. This completes the proof of the theorem. $\square$

Corollary 2.4.1. *Under the conditions of* Theorem 2.4, *every unbounded solution of* (H) *oscillates.*

Remark 3. Conditions (H_7) and (H_8) do not imply that condition (H_3) holds. In fact, if

$$G(u) = (a + b|u|^{\lambda})|u|^{\mu} \operatorname{sgn} u \qquad \text{where} \quad \lambda \geqslant 0, \;\; \mu > 0, \;\; a \geqslant 1, \;\; b \geqslant 1,$$

then (H_7) and (H_8) hold but (H_3) does not.

Remark 4. The prototype function for G satisfying (H_2), (H_7), and (H_8) is

$$G(u) = (a + |u|^{\lambda})|u|^{\mu} \operatorname{sgn} u, \qquad \text{where} \quad \lambda \geqslant 0, \;\; \mu > 0, \;\; \lambda + \mu \geqslant 1, \;\; a \geqslant 1.$$

Theorem 2.5. *Assume that conditions* (H_0), (H_1), (H_3), (H_4), *and* (H_9) *hold. If* $-\infty < p_5 \leqslant p(t) \leqslant p_6 < -1$, *then every bounded solution of* (H) *either oscillates or tends to zero as* $t \to \infty$.

P r o o f. Let $y(t)$ be a bounded nonoscillatory solution of (H), say $y(t)$ is eventually positive. With (2.1), (2.2), and (2.3) as above, we obtain (2.4) for $t \geqslant t_1$. Hence, from Lemma 2.1, $w(t)$ is monotonic. If $w(t) > 0$ for $t \geqslant t_2 > t_1$, then one of the cases (a) or (b) of Lemma 2.1 holds. Consequently, $w(t) \geqslant R_T(t)(r(t)w^{\Delta^2}(t))^{\Delta}$ for $t \geqslant T > t_2$ by Lemma 2.2. Moreover, $w(t) \leqslant y(t)$. Choose $t_3 \in [T,\infty)_{\mathbb{T}}$ such that $\alpha_2(t) \geqslant T$ for all $t \in [t_3,\infty)_{\mathbb{T}}$. Then, $y(\alpha_2(t)) \geqslant R_T(\alpha_2(t))(r(\alpha_2(t))w^{\Delta^2}(\alpha_2(t)))^{\Delta}$ for $t \geqslant t_3$, and (2.4) becomes

$$\int_{t_3}^{\infty} q(t)G(R_T(\alpha_2(t)))\Delta t < \infty,$$

contradicting (H_9) since G, R_T, and α_2 are increasing. Hence, $w(t) < 0$ for $t \geqslant t_2$, so one of the cases (b), (c), (d), or (e) of Lemma 2.1 holds.

In case (b), since $w(t) < 0$, $w^{\Delta}(t) > 0$, and $\lim_{t\to\infty} k(t)$ exists, we have $\lim_{t\to\infty} z(t)$ exists. Furthermore, $\lim_{t\to\infty} (r(t)w^{\Delta^2}(t))^{\Delta}$ exists. Integrating (2.4) from t_3 to t, we obtain

$$\int_{t_3}^{\infty} q(t)G(y(\alpha_2(t)))\Delta t < \infty,$$

which implies that $\liminf_{t\to\infty} y(t) = 0 = \liminf_{t\to\infty} y(\alpha_2(t))$ due to (H_5). Hence, $\lim_{t\to\infty} z(t) = 0$ by Lemma 2.3. Therefore,

$$
\begin{aligned}
0 = \liminf_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(\alpha_1(t)))\\
&\leqslant \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(t)y(\alpha_1(t)))\\
&= \limsup_{t\to\infty} y(t) + p_6 \limsup_{t\to\infty} y(\alpha_1(t))\\
&= (1+p_6)\limsup_{t\to\infty} y(t).
\end{aligned}
$$

Since $(1+p_6) < 0$, we have $\limsup\limits_{t\to\infty} y(t) \le 0$, so $\lim\limits_{t\to\infty} y(t) = 0$.

Cases (c) and (d) are not possible since $w(t) < 0$ for $t \ge t_2$, $y(t)$ is bounded, and $\lim\limits_{t\to\infty} k(t)$ exists.

If (e) holds, we have $r(t)w^{\Delta^2}(t)$ is nondecreasing on $[t_2,\infty)_{\mathbb{T}}$. Hence, for $t > t_3 \geqslant t_2$, $r(t)w^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3) > 0$, so

$$
tw^{\Delta^2}(t) \geqslant r(t_3)w^{\Delta^2}(t_3)\frac{t}{r(t)}.
$$

Integrating the above inequality from t_3 to t, we obtain

$$
tw^{\Delta}(t) - t_3 w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(\sigma(s))\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s,
$$

or

$$
tw^{\Delta}(t) - t_3 w^{\Delta}(t_3) - \int_{t_3}^{t} w^{\Delta}(s)\Delta s \geqslant r(t_3)w^{\Delta^2}(t_3)\int_{t_3}^{t}\frac{s}{r(s)}\Delta s.
$$

That is,

$$
w(t) \leqslant -t_3 w^{\Delta}(t_3) + w(t_3) - (r(t_3)w^{\Delta^2}(t_3))\int_{t_3}^{t}\frac{s}{r(s)}\Delta s \to -\infty
$$

as $t\to\infty$ by (H_0). This implies $\lim\limits_{t\to\infty} z(t) = -\infty$ contradicting the fact that $y(t)$ is bounded. This completes the proof of the theorem. □

3. Nonhomgeneous oscillations

This section is devoted to study the oscillatory and asymptotic behavior of solutions of the forced equation (NH) with suitable forcing functions. Our attention is restricted to forcing functions that eventually change signs. We will make use of the following hypotheses on $f(t)$.

(H_{10}) There exists $F \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty$, $rF^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^2})^{\Delta^2} = f$.

(H_{11}) There exists $F\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $\liminf_{t\to\infty} F(t)=-\infty$, $\limsup_{t\to\infty} F(t) = \infty$, $rF^{\Delta^2} \in C^2_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, and $(rF^{\Delta^2})^{\Delta^2} = f$.

Theorem 3.1. *Let either*

(i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or*

(ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$.

Assume that (H_0)–(H_2), (H_7), (H_8), (H_{11}), *and* (Λ) *hold. If*

(H_{12}) $$\limsup_{t\to\infty} \int_{t_0}^{t} Q(s)G(F(\alpha_2(s)))\Delta s = +\infty \quad \textit{and}$$
$$\liminf_{t\to\infty} \int_{t_0}^{t} Q(s)G(F(\alpha_2(s)))\Delta s = -\infty,$$

then equation (NH) *is oscillatory.*

P r o o f. Let $y(t)$ be a nonoscillatory solution of (NH), say, $y(t)$, $y(\alpha_1(t))$, $y(\alpha_2(t))$, $y(\alpha_3(t))$, and $y(\alpha_2(\alpha_1(t)))$ are all positive for $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t_1 \in [\tilde{t},\infty)_{\mathbb{T}}$. Defining $z(t)$, $k(t)$, and $w(t)$ as in (2.1), (2.2), and (2.3) respectively, equation (NH) becomes

$$(r(t)w^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\alpha_2(t))) = f(t). \tag{3.1}$$

Let

$$v(t) = w(t) - F(t) = z(t) - k(t) - F(t). \tag{3.2}$$

Then, for $t \geqslant t_2$, equation (3.1) becomes

$$(r(t)v^{\Delta^2}(t))^{\Delta^2} = -q(t)G(y(\alpha_2(t))) \leqslant 0. \tag{3.3}$$

Thus, $v(t)$ is monotonic on $[t_1,\infty)_{\mathbb{T}}$.

Suppose $v(t) > 0$ for $t \geqslant t_2$ so that case (a) or (b) of Lemma 2.1 holds. Then $z(t) > k(t) + F(t) \geq F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that $\alpha_2(t) \geqslant t_2$ for all $t \in [t_3,\infty)_{\mathbb{T}}$. Then, $z(\alpha_2(t)) > F(\alpha_2(t))$ for $t \geqslant t_3$. Applying (H_2), (H_7), and (Λ) yields

$$\begin{aligned}
0 \;\geqslant\; & (r(t)v^{\Delta^2}(t))^{\Delta^2} + q(t)G(y(\alpha_2(t))) + G(p)[(r(\alpha_1(t))v^{\Delta^2}(\alpha_1(t)))^{\Delta^2} \\
& +q(\alpha_1(t))G(y(\alpha_2(\alpha_1(t))))] \\
\geqslant\; & (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha_1(t))v^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(y(\alpha_2(t)) \\
& +py(\alpha_1(\alpha_2(t))))] \\
\geqslant\; & (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha_1(t))v^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(z(\alpha_2(t)))] \\
\geqslant\; & (r(t)v^{\Delta^2}(t))^{\Delta^2} + G(p)[(r(\alpha_1(t))v^{\Delta^2}(\alpha_1(t)))^{\Delta^2} + \lambda Q(t)G(F(\alpha_2(t)))]
\end{aligned}$$

for $t \geqslant t_3$. Integrating the above inequality we obtain

$$\limsup_{t\to\infty} \int_{t_3}^{t} Q(s)G(F(\alpha_2(s)))\Delta s < \infty,$$

which contradicts (H_{12}).

Therefore, $v(t) < 0$ for $t \geqslant t_1$ and one of the cases (b), (c), (d), or (e) of Lemma 2.1 holds. In each of these cases $z(t) < k(t) + F(t)$. This implies $\liminf_{t\to\infty} z(t) < 0$ which is a contradiction. This completes the proof of the theorem. □

Theorem 3.2. *Let $-1 < p(t) \leqslant 0$. Suppose that (H_0), (H_1), (H_8) and (H_{11}) hold. If*

$$(H_{13}) \qquad \limsup_{t\to\infty} \int_{t_0}^{t} q(s)G(F(\alpha_2(s)))\Delta s = +\infty \quad \text{and}$$
$$\liminf_{t\to\infty} \int_{t_0}^{t} q(s)G(F(\alpha_2(s)))\Delta s = -\infty,$$

then every bounded solution of (NH) *oscillates.*

Proof. Proceeding as in the proof of Theorem 3.1, we obtain (3.3) for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus, $v(t)$ is monotonic, so $v(t) > 0$ or $v(t) < 0$ for large t. If $v(t) > 0$ for $t \geqslant t_2 > t_1$, then either case (a) or case (b) of Lemma 2.1 holds for $t \geqslant t_2$. Since $v(t)$ is monotonic, $z(t) > z(t) - k(t) > F(t)$ implies that $z(t) > F(t)$, so $y(t) > z(t) > F(t)$ for $t \geqslant t_2$. Choose $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $\alpha_2(t) \geqslant t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, for $t \geqslant t_3$, $y(\alpha_2(t)) > z(\alpha_2(t)) > F(\alpha_2(t))$. From (3.3), we have

$$q(t)G(F(\alpha_2(t))) \leqslant q(t)G(y(\alpha_2(t))) = -(r(t)v^{\Delta^2}(t))^{\Delta^2}$$

for $t \geqslant t_3$. An integration yields a contradiction to (H_{13}).

Now assume $v(t) < 0$ for $t \geqslant t_2$. Thus, $z(t) - k(t) < F(t)$, and condition (H_{11}) then implies that $\liminf_{t\to\infty} z(t) = -\infty$. This contradicts the fact that $y(t)$ is bounded and completes the proof of the theorem. □

Theorem 3.3. *Assume that (H_0)–(H_2), (H_7), (H_8), (H_{10}), (H_{12}), and (Λ) hold. If* (i) $0 \leqslant p(t) \leqslant p_1 < 1$ *or*

(ii) $1 < p_2 \leqslant p(t) \leqslant p_3 < \infty$

holds, then every unbounded solution of (NH) *oscillates.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (NH), say $y(t)$ is an eventually positive solution. Using (2.1), (2.2), (2.3), and (3.2), we obtain inequality (3.3). Thus, $v(t)$ is monotonic, so first assume $v(t) > 0$ for all $t \geqslant t \geqslant t_2$. Proceeding as in the proof of Theorem 3.1, we again obtain a contradiction.

Next, let $v(t) < 0$ for $t \geqslant t_2 > t_1$. From Lemma 2.1 it follows that one of the cases (b), (c), (d), or (e) holds. In case (b), $\lim_{t\to\infty} v(t)$ exists and hence $z(t) = v(t) + k(t) + F(t)$ implies $y(t) \leqslant v(t) + k(t) + F(t)$. That is, $y(t)$ is bounded, which is a contradiction.

For each of the cases (c), (d) and (e), $v(t)$ is nonincreasing on $[t_2, \infty)_{\mathbb{T}}$, so let $\lim_{t\to\infty} v(t) = l$ for some $l \in [-\infty, 0)$. If $l = -\infty$, then $y(t) \leqslant v(t) + k(t) + F(t)$, which in view of (H_{10}) implies that $y(t)$ eventually becomes negative. If $-\infty < l < 0$, then in cases (c) and (d), $v^{\Delta}(t)$ is decreasing. Successive integrations of $v^{\Delta^2}(t)$ again show that $\lim_{t\to\infty} v(t) = -\infty$. If case (e) holds, $y(t) \leqslant v(t) + k(t) + F(t) \leqslant k(t) + F(t)$, which contradicts the unboundedness of $y(t)$. This completes the proof of the theorem. $\square$

Our final theorem in this paper gives sufficient conditions for equation (NH) to have a bounded positive solution.

Theorem 3.4. *Assume that* $0 \leqslant p(t) \leqslant p_1 < 1$*, and* (H_1) *and* (H_{10}) *hold with*

$$\frac{-1}{8}(1 - p_1) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1 - p_1).$$

In addition, assume that G *and* H *are Lipschitzian on* $\mathbb{R}$ *with Lipschitz constants* G_1 *and* H_1*, respectively. If*

$$\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \int_{t}^{\infty} \sigma(s) q(s) \Delta s \Delta t < \infty,$$

then (NH) *admits a positive bounded solution.*

P r o o f. Choose $t_1 > t_0$ large enough so that

$$\int_{t_1}^{\infty} \frac{\sigma(t)}{r(t)} \int_{t}^{\infty} \sigma(s) h(s) \Delta s \Delta t < \min\left\{ \frac{1 - p_1}{4H(1)}, \frac{1 - p_1}{4G(1)} \right\}.$$

Let $X = BC_{rd}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_1, \infty)_{\mathbb{T}}$ with the supremum norm

$$\|x\| = \sup\bigl\{ |x(t)| : \ t \in [t_1, \infty)_{\mathbb{T}} \bigr\},$$

and let

$$S = \bigl\{ x \in X : \ \tfrac{1}{8}(1 - p) \leqslant x(t) \leqslant 1, \ \ t \in [t_1, \infty)_{\mathbb{T}} \bigr\}.$$

Then, S is a closed, bounded, and convex subset of X. Take $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t) \geqslant t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Define the mappings $A, B \colon S \to S$ by

$$Ax(t) = \begin{cases} Ax(t_2), & \text{for } \ t \in [t_1, t_2)_{\mathbb{T}}, \\ -p(t)x(\alpha_1(t)) + \frac{1+p}{2}, & \text{for } \ t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

and

$$Bx(t) = \begin{cases} Bx(t_2), & \text{for } t \in [t_1, t_2)_{\mathbb{T}}, \\ -\int\limits_t^\infty \frac{\sigma(s)-t}{r(s)} \int\limits_s^\infty (\sigma(u) - s)q(u)G(x(\alpha_2(u)))\Delta u)\Delta s & \\ \qquad\qquad +F(t) + k(t), & \text{for } t \in [t_2, \infty)_{\mathbb{T}}. \end{cases}$$

For $x \in S$, we have

$$\begin{aligned} k(t) &= \int\limits_t^\infty \frac{\sigma(s) - t}{r(s)} \int\limits_s^\infty (\sigma(u) - s)h(u)H(x(\alpha_3(u)))\Delta u\Delta s \\ &\leqslant H(1) \int\limits_t^\infty \frac{\sigma(s)}{r(s)} \int\limits_s^\infty \sigma(u)h(u)\Delta u\Delta s \\ &< \frac{1}{4}(1 - p_1). \end{aligned}$$

For all x, $y \in S$ and all $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$Ax(t) + By(t) \leqslant \frac{1 + p_1}{2} + \frac{1}{4}(1 - p_1) + \frac{1}{4}(1 - p_1) = 1$$

and

$$Ax(t) + By(t) \geqslant -p_1 + \frac{1 + p_1}{2} - \frac{1}{8}(1 - p_1) - \frac{1}{4}(1 - p_1) = \frac{1 - p_1}{8}.$$

Thus, $Ax + By \in S$.

To show that A is a contraction mapping on S, first notice that

$$\begin{aligned} \|Ax - Ay\| &= \left\| -p(t)x(\alpha_1(t)) + \frac{1 + p_1}{2} + p(t)y(\alpha_2(t)) - \frac{1 + p_1}{2} \right\| \\ &= \|-p(t)(x(\alpha_1(t)) - y(\alpha_1(t)))\| \\ &\leqslant p_1\|x(\alpha_1(t)) - y(\alpha_1(t))\| \\ &= p_1\|x(t) - y(t)\|. \end{aligned}$$

Since $p_1 < 1$, A is a contraction mapping.

To show that B is completely continuous on S, we need to show that B is continuous and maps bounded sets into relatively compact sets. In order to show that B is continuous, let x, $x_k = x_k(t) \in S$ be such that $\|x_k - x\| = \sup\limits_{t \geqslant t_1}\{|x_k(t) - x(t)|\} \to 0$. Since S is closed, $x(t) \in S$. For $t \geqslant t_1$, we have

$$|(Bx_k) - (Bx)| = \left| F(t) + \int\limits_t^\infty \frac{\sigma(s) - t}{r(s)} \int\limits_s^\infty (\sigma(u) - s)h(u)H(x_k(\alpha_3(u)))\Delta u\Delta s \right.$$

$$
\begin{aligned}
&- \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)q(u)G(x_k(\alpha_2(u)))\Delta u \Delta s \\
&- F(t) - \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)h(u)H(x(\alpha_3(u)))\Delta u \Delta s \\
&+ \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)q(u)G(x(\alpha_2(u)))\Delta u \Delta s \Bigg| \\
= &\Bigg| \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)h(u)(H(x_k(\alpha_3(u))) \\
&- H(x(\alpha_3(u))))\Delta u \Delta s \\
&+ \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)q(u)(G(x(\alpha_2(u))) \\
&- G(x_k(\alpha_2(u))))\Delta u \Delta s \Bigg| \\
\leqslant &H_1 \|x_k - x\| \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty (\sigma(u)-s)h(u)\Delta u \Delta s \\
&+ G_1 \|x_k - x\| \int_t^\infty \frac{\sigma(s)-t}{r(s)} \int_s^\infty \sigma(u)q(u)\Delta u \Delta s \\
\leqslant &\frac{1}{2}(1-p)\|x - x_k\|.
\end{aligned}
$$

Since for all $t \geqslant t_1$, $\{x_k(t)\}$, converges uniformly to $x(t)$ as $k \to \infty$, we have $\lim_{k\to\infty} |(Bx_k)(t) - (Bx)(t)| = 0$ for $t \geqslant t_1$. Thus, B is continuous.

To show that BS is relatively compact, it suffices to show that the family of functions $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. The uniform boundedness is clear. To show that BS is equicontinuous, let $x \in S$ and $t'', t' \geqslant t_1$. Then

$$
\begin{aligned}
&|(Bx)(t'') - (Bx)(t')| \\
&= \Bigg| F(t'') + k(t'') - \int_{t''}^\infty \frac{\sigma(s)-t''}{r(s)} \int_s^\infty (\sigma(u)-s)q(u)G(x(\alpha_2(u)))\Delta u \Delta s
\end{aligned}
$$

$$- F(t') - k(t') + \int_{t'}^{\infty} \frac{\sigma(s) - t'}{r(s)} \int_{s}^{\infty} (\sigma(u) - s) q(u) G(x(\alpha_2(u))) \Delta u \Delta s \bigg|$$

$$= |F(t'') - F(t')| + |k(t'') - k(t')| + G(1) \int_{t'}^{t''} \frac{\sigma(s)}{r(s)} \int_{s}^{\infty} \sigma(u) q(u) \Delta u \Delta s$$

$$+ G(1)|t' - t''| \int_{t''}^{\infty} \frac{1}{r(s)} \int_{t''}^{\infty} \sigma(u) q(u) \Delta u \Delta s$$

so $|(Bx)(t'') - (Bx)(t')| \to 0$ as $t'' \to t'$. Therefore, $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[t_1, \infty)_{\mathbb{T}}$. Hence, BS is relatively compact. By Krasnosel'skii's fixed point theorem, there exists $x \in S$ such that $Ax + Bx = x$. Thus, the theorem is proved. □

Remark 5. Results similar to Theorem 3.4 can be proved for other ranges of $p(t)$.

4. Examples

In this section we present some examples to illustrate our main results in the paper.

Example 1. Let $\mathbb{T} = \mathbb{R}$ and consider the differential equation

$$\begin{aligned} &(y(t) + 81y(t/3))'''' + y^{1/3}(t/6) \\ &- \left(\mathrm{e}^{-t} + \mathrm{e}^{-\frac{35}{36}t} + \mathrm{e}^{-\frac{1}{3}t} + \mathrm{e}^{-\frac{11}{36}t} + \mathrm{e}^{-\frac{1}{18}t} + \mathrm{e}^{-\frac{1}{36}t}\right) \frac{y(t/36)}{1 + |y(t/36)|} = 0, \qquad t > 0. \end{aligned} \tag{4.1}$$

We have $r(t) = 1$, $p(t) = 81$, $q(t) = 1$, $h(t) = \mathrm{e}^{-t} + \mathrm{e}^{-\frac{35}{36}t} + \mathrm{e}^{-\frac{1}{3}t} + \mathrm{e}^{-\frac{11}{36}t} + \mathrm{e}^{-\frac{1}{18}t} + \mathrm{e}^{-\frac{1}{36}t}$, $G(u) = u^{1/3}$ and $H(u) = \frac{u}{1+|u|}$. Also, $\alpha_3(t) = t/36$ and $\alpha_2(t) = t/6 < t/3 = \alpha_1(t)$. It is easy to see that conditions (H_0) and (H_2)–(H_5) hold. For simplicity in showing that (H_1) holds, we only consider one term from $h(t)$. We have

$$\int_{t_0}^{\infty} \frac{\sigma(s)}{r(s)} \int_{s}^{\infty} \sigma(t) \mathrm{e}^{-t} \,\mathrm{d}t \,\mathrm{d}s = \int_{1}^{\infty} s \int_{s}^{\infty} t\mathrm{e}^{-t} \,\mathrm{d}t \,\mathrm{d}s = \int_{1}^{\infty} s(s\mathrm{e}^{-s} + \mathrm{e}^{-s}) \,\mathrm{d}s < \infty.$$

By Theorem 2.1, any solution of (4.1) either oscillates or tends to zero as $t \to \infty$. Here, $y(t) = \mathrm{e}^{-t}$ is a solution.

Example 2. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\Delta^2[n\Delta^2(y(n) + (1 + (-1)^n)y(n-4))] + 32(n + \mathrm{e}^{-n} + 1)y(n-5) \\ -64\mathrm{e}^{-n}\frac{y(n-1)}{1 + y^2(n-1)} = 0, \qquad n \geq 6, \qquad (4.2)$$

for $n \in [6, \infty)_{\mathbb{Z}}$. Here $r(n) = n$, $p(n) = (1 + (-1)^n)$, $q(t) = 32(n + \mathrm{e}^{-1} + 1)$, $h(t) = 64\mathrm{e}^{-n}$, $G(u) = u$ and $H(u) = \frac{u}{1+u^2}$. We also have $\alpha_3(n) = n - 1$ and $\alpha_2(t) = n - 5 < n - 4 = \alpha_1(n)$. Notice that condition (H_1) becomes

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\left(\frac{n+1}{n}\sum_{s=n}^{\infty}64(s+1)\mathrm{e}^{-s}\right) \\
&\quad \leq \sum_{n=n_0}^{\infty}\left(2\sum_{s=n}^{\infty}64(2s)\mathrm{e}^{-s}\right) \\
&\quad \leq 256\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{\infty}s\mathrm{e}^{-s}\right) \\
&\quad = 256\left(\sum_{n=n_0}^{\infty}s\mathrm{e}^{-s} + \sum_{n=n_0+1}^{\infty}s\mathrm{e}^{-s} + \dots\right) \\
&\quad = 256\mathrm{e}^{-n_0}\left(n_0 + 2(n_0+1)\mathrm{e}^{-1} + 3(n_0+2)\mathrm{e}^{-2} + \dots\right) \\
&\quad = 256\mathrm{e}^{-n_0}\left[n_0 e\sum_{n=1}^{\infty}n\mathrm{e}^{-n} + \sum_{n=1}^{\infty}n(n+1)\mathrm{e}^{-n}\right] < \infty.
\end{aligned}$$

It is now easy to see that equation (4.2) satisfies all the conditions of Theorem 2.3. Hence, any solution of equation (4.2) oscillates or converges to 0 as $t \to \infty$. In particular, $y(t) = (-1)^n$ is a solution of equation (4.2).

Example 3. Let $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\Delta^2[n\Delta^2(y(n) + (1 + (-1)^n)y(n-4))] + 16(n + 1 + 2\mathrm{e}^{-n})y(n-5) \\ -64\mathrm{e}^{-n}\frac{y(n-1)}{1 + y^2(n-1)} = 32(n+1)(-1)^n, \quad n \geq 6, \qquad (4.3)$$

for $n \in [6, \infty)_{\mathbb{Z}}$. Here $r(n) = n$, $p(n) = (1 + (-1)^n)$, $q(t) = 16(n + 1 + 2\mathrm{e}^{-n})$, $h(t) = 64\mathrm{e}^{-n}$, $G(u) = u$ and $H(u) = \frac{u}{1+u^2}$, and $F(n) = (-1)^n$. We also have $\alpha_3(n) = n - 1$ and $\alpha_2(t) = n - 5 < n - 4 = \alpha_1(n)$. Condition ($\mathrm{H}_{11}$) becomes

$$\sum_{n=n_0}^{\infty} 16(n + 1 + 2\mathrm{e}^{-n}) = \infty.$$

It is now easy to see that equation (4.3) satisfies all the conditions of Theorem 3.1. Hence, any solution of equation (4.3) oscillates or converges to 0 as $n \to \infty$. In particular, $y(n) = (-1)^n$ is a solution of this equation.

REFERENCES

[1] BOHNER, M.—PETERSON, A.: *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
[2] BOHNER, M.—PETERSON, A.: *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[3] GRAEF, J. R.—GRAMMATIKOPOULOS, M. K.—SPIKES, P. W.: *Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order*, Nonlinear Anal. **21** (1993), 23–42.
[4] HILGER, S.: *Analysis on measure chains: a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.
[5] KARPUZ, B.—ÖCALAN, Ö.: *Necessary and sufficient conditions on asymptotic behaviour of solutions of forced neutral delay dynamic equations*, Nonlinear Anal. **71** (2009), 3063–3071.
[6] PANIGRAHI, S.—REDDY, P. R.: *On oscillatory fourth order nonlinear neutral delay dynamic equations*, Comput. Math. Appl. **62** (2011), 4258–4271.
[7] PARHI, N.—TRIPATHY, A. K.: *On oscillatory fourth order nonlinear neutral differential equations II*, Math. Slovaca **55** (2005), 183–202.
[8] THANDAPANI, E.—AROCKIASAMY, I. M.: *Oscillatory and asymptotic behaviour of fourth order non-linear delay difference equations*, Indian J. Pure Appl. Math. **32** (2001), 109–123.
[9] THANDAPANI, E.—SUNDARAM, P.—GRAEF, J. R.—SPIKES, P. W.: *Asymptotic behavior and oscillation of solutions of neutral delay difference equations of arbitrary order*, Math. Slovaca **47** (1997), 539–551.

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