VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0207-9
Math. Slovaca **64** (2014), No. 2, 333–338

UNIVALENCE CONDITIONS OF CERTAIN INTEGRAL OPERATORS

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(*Communicated by Gregor Dolinar*)

ABSTRACT. The purpose of the present paper is to determine univalence conditions of certain integral operators.

1. Introduction

Let A be the class of functions f which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in U.

We consider the integral operators

$$G_\gamma(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\gamma}} \mathrm{d}u, \tag{1.1}$$

$$J_\gamma(z) = \left[\frac{1}{\gamma}\int_0^z u^{-1}(f(u))^{1/\gamma}\,\mathrm{d}u\right]^\gamma, \tag{1.2}$$

$$J_{\gamma_1,\gamma_2,\dots,\gamma_n}(z) = \left[\left(\sum_{j=1}^n \frac{1}{\gamma_j}\right)\int_0^z u^{-1}\prod_{j=1}^n (f_j(u))^{1/\gamma_j}\,\mathrm{d}u\right]^{1/\sum\limits_{j=1}^n 1/\gamma_j}, \tag{1.3}$$

$$T_{\alpha,\beta}(z) = \left[\beta\int_0^z u^{\beta-1}\left(\frac{f(u)}{u}\right)^{1/\alpha}\mathrm{d}u\right]^{1/\beta}, \tag{1.4}$$

2010 Mathematics Subject Classification: Primary 30C45.
Keywords: integral operator, univalence, analytic function.

$$L_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{1/\alpha} \mathrm{d}u\right]^{1/\beta}, \tag{1.5}$$

$$K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{1/\gamma_j} \mathrm{d}u, \tag{1.6}$$

for $f \in A$, α, β, γ complex numbers, $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ and $f_j \in A$, γ_j complex numbers, $\gamma_j \neq 0$ $(j = 1, 2, \ldots, n)$.

The integral operator in (1.1) was considered by Kim and Merkes [2]. The integral operators in (1.2) and (1.3) were introduced and investigated by Pescar [7]. Also, the integral operators in (1.4), (1.5) and (1.6) were studied by Pescar and Breaz [8]. Many univalent conditions associated with these integral operators in (1.1) to (1.6) were obtained by several authors [1–8].

Recently, Pescar and Breaz [9] considered a general integral operator

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{1/\gamma_j} \mathrm{d}u\right]^{1/\beta} \tag{1.7}$$

for $f_j \in A$, β, γ_j complex numbers, $\beta \neq 0$, $\gamma_j \neq 0$ $(j = 1, 2, \ldots, n)$ and $n \in N - \{0\}$.

For $\beta, \gamma_j, n \in N - \{0\}$, $j = 1, 2, \ldots, n$, in the particular cases, from (1.7) we have the integral operators G_γ, J_γ, $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}$, $T_{\alpha,\beta}$, $L_{\alpha,\beta}$, $K_{\gamma_1,\gamma_2,\ldots,\gamma_n}$.

In [9] Pescar and Breaz obtained certain univalent conditions of the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$. In this note we shall further consider the univalent conditions of the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}$.

For our purpose we need the following lemmas.

Lemma 1.1. (see [4]) *Let α be a complex number, $\operatorname{Re}\alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{1.8}$$

for all $z \in U$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\,\mathrm{d}u\right]^{1/\beta} \tag{1.9}$$

is in the class S.

Lemma 1.2. (see [5]) *Let α, c be complex numbers, $\operatorname{Re}\alpha > 0$, $|c| \leq 1$, $c \neq -1$ and $f \in A$. If*

$$\left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right| \leq 1 \qquad \text{for all} \quad z \in U, \tag{1.10}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)\,\mathrm{d}u\right]^{1/\alpha} \tag{1.11}$$

is in the class S.

2. Main results

Theorem 2.1. *Let γ_j, α be complex numbers, $\gamma_j \neq 0$ $(j = 1, 2, \ldots, n)$ and $\operatorname{Re}\alpha > 0$. Suppose that each of the functions $f_j \in A$ $(j = 1, 2, \ldots, n)$ satisfies the condition*

$$\left|\frac{z f_j'(z)}{f_j(z)} - 1\right| \leq \frac{|\gamma_j|}{n} \operatorname{Re}\alpha \qquad (z \in U), \tag{2.1}$$

then for any complex number $\beta, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta}(z) = \left\{\beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u}\right)^{1/\gamma_1} \cdots \left(\frac{f_n(u)}{u}\right)^{1/\gamma_n} \mathrm{d}u\right\}^{1/\beta} \tag{2.2}$$

is in the class S.

P r o o f. We consider the function

$$p(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{1/\gamma_1} \cdots \left(\frac{f_n(u)}{u}\right)^{1/\gamma_n} \mathrm{d}u \qquad (f_j \in A;\ j = 1, 2, \ldots, n). \tag{2.3}$$

Then the function $p \in A$ and $p(0) = p'(0) - 1 = 0$. From (2.3) we have

$$p'(z) = \prod_{j=1}^{n} \left(\frac{f_j(z)}{z}\right)^{1/\gamma_j} \tag{2.4}$$

and

$$p''(z) = \sum_{j=1}^{n} \frac{1}{\gamma_j} \left(\frac{f_j'(z)}{f_j(z)} - \frac{1}{z}\right) \cdot \prod_{k=1}^{n} \left(\frac{f_k(z)}{z}\right)^{\frac{1}{\gamma_k}}. \tag{2.5}$$

Thus we obtain

$$\left|\frac{zp''(z)}{p'(z)}\right| \leq \sum_{j=1}^{n} \frac{1}{|\gamma_j|} \left|\frac{z f_j'(z)}{f_j(z)} - 1\right|. \tag{2.6}$$

From (2.1) and (2.6) we have

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left|\frac{zp''(z)}{p'(z)}\right| \leq 1 \qquad \text{for all} \quad z \in U.$$

So, according to Lemma 1.1, the integral operator $H_{\gamma_1,\gamma_2,\dots,\gamma_n,\beta}$ belongs to the class S. $\square$

Corollary 2.1. *Let γ_j, α complex numbers, $\operatorname{Re}\gamma_j \neq 0$ $(j = 1, 2, \dots, n)$ and $\sum\limits_{j=1}^{n} \operatorname{Re} 1/\gamma_j \geq \operatorname{Re}\alpha > 0$. Suppose that each of the functions $f_j \in A$ $(j = 1, 2, \dots, n)$ satisfies the condition*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{|\gamma_j|}{n} \operatorname{Re}\alpha \qquad (z \in U),$$

then the integral operator $J_{\gamma_1,\gamma_2,\dots,\gamma_n}$ given by (1.3) *is in the class S.*

Proof. We take $\beta = \sum\limits_{j=1}^{n} 1/\gamma_j$ in Theorem 2.1. $\square$

Remark 2.1. For $n = 1, \gamma_1 = \gamma, f_1 = f$, from Corollary 2.1, we obtain that the integral operator J_γ given by (1.2) is in the class S.

Corollary 2.2. *Let γ_j, α be complex numbers, $\gamma_j \neq 0$ $(j = 1, 2, \dots, n)$ and $0 < \operatorname{Re}\alpha \leq 1$. Suppose that each of the functions $f_j \in A$ $(j = 1, 2, \dots, n)$ satisfies the condition*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{|\gamma_j|}{n} \operatorname{Re}\alpha \qquad (z \in U),$$

then the integral operator $K_{\gamma_1,\gamma_2,\dots,\gamma_n}$ given by (1.6) *is in the class S.*

Proof. We take $\beta = 1$ in Theorem 2.1. $\square$

Remark 2.2. For $n = 1$, $\gamma_1 = \gamma$, $f_1 = f$, from Corollary 2.2, we obtain that the integral operator G_γ defined by (1.1) is in the class S.

Corollary 2.3. *Let α, γ complex numbers, $\alpha \neq 0$ and $\operatorname{Re}\gamma > 0$. Suppose that each of the functions $f_j \in A$ $(j = 1, 2, \dots, n)$ satisfies the condition*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{|\gamma_j|}{n} \operatorname{Re}\alpha \qquad (z \in U),$$

then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\gamma$ the integral operator $L_{\alpha,\beta}$ given by (1.5) *is in the class S.*

Proof. For $\gamma_1 = \gamma_2 = \dots = \gamma_n = \alpha$, from Theorem 2.1, we have Corollary 2.3. $\square$

Remark 2.3. If we take $n = 1$, $f_1 = f$, $\gamma_1 = \alpha$ in Corollary 2.3, we obtain that the integral operator $T_{\alpha,\beta}$ given by (1.4) is in the class S.

Theorem 2.2. *Let γ_j, c, α be complex numbers, $\gamma_j \neq 0$ $(j = 1, 2, \dots, n)$, $|c| < 1$ and $\operatorname{Re}\alpha > 0$. Suppose that each of the functions $f_j \in A$ $(j = 1, 2, \dots, n)$ satisfies the condition*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{|\alpha| \cdot |\gamma_j|}{n}(1 - |c|), \tag{2.7}$$

then the function

$$H_{\gamma_1,\gamma_2,\dots,\gamma_n,\alpha}(z) = \left\{\alpha \int_0^z u^{\alpha-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}} \mathrm{d}u\right\}^{\frac{1}{\alpha}} \tag{2.8}$$

is in the class S.

P r o o f. By using the same method as in Theorem 2.1, we have

$$\frac{zp''(z)}{p'(z)} = \sum_{j=1}^{n} \frac{1}{\gamma_j}\left[\frac{zf_j'(z)}{f_j(z)} - 1\right]$$

where $p(z)$ is defined by (2.3). This readily shows that

$$\begin{aligned}\left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zp''(z)}{\alpha p'(z)}\right| &= \left|c|z|^{2\alpha} + \frac{(1 - |z|^{2\alpha})}{\alpha}\sum_{j=1}^{n}\frac{1}{\gamma_j}\left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right| \\ &\leq |c| + \frac{1}{|\alpha|}\sum_{j=1}^{n}\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|.\end{aligned}$$

From (2.7) we obtain

$$\left|c|z|^{2\alpha} + (1 - |z|^{2\alpha})\frac{zp''(z)}{\alpha p'(z)}\right| \leq 1 \qquad (z \in U).$$

Now by applying Lemma 1.2, we conclude that the function $H_{\gamma_1,\gamma_2,\dots,\gamma_n,\alpha}$ defined by (2.8) is in the class S. This evidently completes the proof of the theorem. □

Acknowledgement. The authors would like to express sincere thanks to the referee for careful reading and suggestions which helped us to improve the paper.

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Received 4. 10. 2011
Accepted 10. 3. 2012

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