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ON THE CARATHÉODORY SUPERPOSITION OF MULTIFUNCTIONS AND AN EXISTENCE THEOREM

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ABSTRACT. Let $I \subset \mathbb{R}$ be an interval and Y a reflexive Banach space. We introduce the (H) property of a multifunction F from $I \times Y$ to Y and prove that the Carathéodory superposition of F with each continuous function f from I to Y is a derivative provided that F has the (H) property. Some application of this theorem to the existence of solutions of differential inclusions $f'(x) \in F(x, f(x))$ is given.

1. Introduction

Many problems of applied mathematics lead us to the study of dynamical systems having velocities not uniquely determined by the state of the systems, but depending only loosely upon it, i.e. dynamical systems

$$f'(x) = g(x, f(x), u(x)), \qquad f(x_0) = y_0,$$

"controled" by parameters $u(x) \in U(f(x))$, "the controls". If we introduce the set-valued map

$$F(x, f(x)) = \{g(x, f(x), u(x))\}_{u(x) \in U(f(x))},$$

the solutions of the above differential equation are solutions of the "differential inclusion"

$$f'(x) \in F(x, f(x)), \qquad f(x_0) = y_0,$$

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in which the controls do not appear explicitly. Such a differential inclusion is also called set-valued differential equation or a differential equation with multivalued right-hand said.

Let Y be a reflexive Banach space, $I \subset \mathbb{R}$ an interval, $(x_0, y_0) \in I \times Y$, and let $F\colon I \times Y \rightsquigarrow Y$ be a multifunction. We will be concerned with the initial value problem for the differential inclusion

$$(1)\quad f'(x) \in F(x, f(x)), \qquad f(x_0) = y_0.$$

By a solution of (1) we mean any absolutely continuous function $f\colon [x_0, b] \to Y$, where $x_0 < b$ and $b \in I$, such that $f(x_0) = y_0$ and $f'(x) \in F(x, f(x))$ for almost all $x \in [x_0, b]$.

The existence of solution of (1) my be shown in many ways. The conditions which to be imposed on the multifunction F in order to have solutions are mainly of two kinds: continuity or semicontinuity of F with compactness or convexity of values of F and some conditions type of Lipschitz or integrable boundedness of F (see e.g. [21], [8], [10], [13]). This was extended to the Caratheodory case, i.e. $F(x, \cdot)$ is continuous or semicontinuous, $F(\cdot, y)$ is measurable and F is integrably bounded (see e.g. [1], [5], [8], [16]).

We obtain a solution of (1) when F has the following property:

(H) • $F(\cdot, y)$ is a derivative for each $y \in Y$,
• the family $\{F(x, \cdot)\}_{x \in I}$ is equicontinuous and
• the family $\{F_f\}_{f \in \mathcal{C}(I,Y)}$ is uniformly integrably bounded,
where $F_f(x) = F(x, f(x))$ for $x \in I$, and
$\mathcal{C}(I, Y)$ denotes the family of all continuous vector functions on I.

In order to give effect we prove that the superposition F_f is a derivative whenever F has the (H) property and $f \in \mathcal{C}(I, Y)$.

2. Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively, $I \subset \mathbb{R}$ an interval and $\mathcal{L}(\mathbb{R})$ the σ-field of Lebesgue measurable subsets of $\mathbb{R}$.

Let X and Y be two nonempty sets. We assume that for every point $x \in X$ a non-empty subset $\Phi(x)$ of Y is given. In this case we say a multifunction Φ from X to Y is defined and we will write $\Phi\colon X \rightsquigarrow Y$.

If $\phi\colon X \to Y$ is a function such that $\phi(x) \in \Phi(x)$ for every $x \in X$, then ϕ is called a selection of Φ; ϕ may be considered as a multifunction assigning to $x \in X$ the singleton $\{\phi(x)\}$. Usually we identify $\{\phi(x)\}$ with $\phi(x)$.

If $\Phi\colon X \rightsquigarrow Y$, $G \subset Y$ and $A \subset X$, then we define

$$\Phi^{-}(G) = \{x \in X : \Phi(x) \cap G \neq \emptyset\} \quad \text{and} \quad \Phi(A) = \bigcup_{x \in A} \Phi(x).$$

If (Y, d) is a metric space, $y_0 \in Y$ and $M \subset Y$, then $B(y_0, r)$ will denote an open ball centered at y_0 with radius $r > 0$, $B(M, r) = \bigcup\{B(y, r) : y \in M\}$ and $d(y_0, M) = \inf\{d(y, y_0) : y \in M\}$. Moreover $\mathrm{Fr}(M)$ will denote the boundary of M and $\mathcal{B}(Y)$ the σ-field of Borel subsets of Y.

Let $\mathcal{P}(Y)$ be the power set of Y and let $\mathcal{P}_0(Y) = \mathcal{P}(Y) \setminus \emptyset$. We put

$$\mathcal{C}_b(Y) = \{A \in \mathcal{P}_0(Y) : A \text{ is closed and bounded}\}$$

and

$$\mathcal{K}(Y) = \{A \in \mathcal{P}_0(Y) : A \text{ is compact}\}.$$

In the sequel, convergence in the space $\mathcal{C}_b(Y)$ will denote the convergence in the Hausdorff metric in $\mathcal{C}_b(Y)$ denoted by d_H.

We also introduce a hemimetric h in $\mathcal{C}_b(Y)$ defined by

$$h(A, B) = \sup\{d(y, B) : y \in A\}.$$

Then $h(A, B) \leq d_H(A, B)$, $h(A, B) \leq h(A, C) + h(C, B)$ and $h(A, B) = 0$ if and only if $A \subset B$.

Let $\Phi_n\colon X \rightsquigarrow Y$ for $n \in \mathbb{N}$ and $\Phi\colon X \rightsquigarrow Y$ be multifunctions with values in $\mathcal{C}_b(Y)$ and let $x \in X$.

(2) If $d_H - \lim\limits_{n\to\infty} \Phi_n(x) = \Phi(x)$, then $\lim\limits_{n\to\infty} d(y, \Phi_n(x)) = d(y, \Phi(x))$ for each $y \in Y$.

Now we collect material that will be used in the next section: terminology, facts known in the literature and some properties which are new for multifunctions of one variable.

Let $(X, \mathcal{T}(X))$ be a topological space and let (Y, d) be a metric space. We will say that a multifunction $\Phi\colon X \rightsquigarrow Y$ is *upper* (resp. *lower*) *semicontinuous* at a point $x_0 \in X$ if, for each $\varepsilon > 0$ there exists a neighbourhood $U(x_0)$ of x_0 such that $\Phi(x) \subset B(\Phi(x_0), \varepsilon)$ (resp. $\Phi(x_0) \subset B(\Phi(x), \varepsilon)$) for all $x \in U(x_0)$.

Φ is *continuous* at x_0 if it is simultaneously upper and lower semicontinuous at x_0; Φ is *continuous* if it is continuous at each point $x \in X$.

It is obvious that if Φ is closed and bounded valued, then Φ is continuous if and only if the function $\Phi\colon X \to \mathcal{C}_b(Y)$ is continuous (with respect to d_H).

Let us note that

(3) If a multifunction $\Phi\colon X \rightsquigarrow Y$ is lower semicontinuous, then for each open set $G \subset Y$ the set $\Phi^{-}(G)$ is open.

Indeed, if $x_0 \in \Phi^-(G)$, then $\Phi(x_0) \cap G \neq \emptyset$. Let $y_0 \in \Phi(x_0) \cap G$ and let $\varepsilon > 0$ be such that $B(y_0, \varepsilon) \subset G$. By lower semicontinuity of Φ at x_0 there is $U(x_0)$ such that $\Phi(x_0) \subset B(\Phi(x), \varepsilon)$ for $x \in U(x_0)$. Then $\Phi(x) \cap B(y_0, \varepsilon) \neq \emptyset$ for any $x \in U(x_0)$. Thus $\Phi(x) \cap G \neq \emptyset$ for $x \in U(x_0)$ and $\Phi^-(G)$ is open.

Let $\mathcal{I}$ be a set of indices and $\{\Phi_i\}_{i \in \mathcal{I}}$ a collection of multifunctions from X to Y. We will say $\{\Phi_i\}_{i \in \mathcal{I}}$ is *upper* (resp. *lower*) *equisemicontinuous* at $x_0 \in X$, if for each $\varepsilon > 0$ there exists an open neighbourhood $U(x_0)$ of x_0 such that $x \in U(x_0)$ implies $\Phi_i(x) \subset B(\Phi_i(x_0), \varepsilon)$ (resp. $\Phi_i(x_0) \subset B(\Phi_i(x), \varepsilon)$) for each $i \in \mathcal{I}$; $\{\Phi_i\}_{i \in \mathcal{I}}$ is *equicontinuous* if it is both upper and lower equisemicontinuous at each point $x \in X$.

Now let $(X, \mathcal{M}(X), \mu)$ be a measure space.

A multifunction $\Phi\colon X \rightsquigarrow Y$ is called $\mathcal{M}(X)$-*measurable* if $\Phi^-(G) \in \mathcal{M}(X)$ for every open set $G \subset Y$.

Let $y \in Y$ be given and let $g_y\colon X \to \mathbb{R}$ be a function given by

$$g_y(x) = d(y, \Phi(x)).$$

Then

(4) If (Y, d) is a separable metric space and $\Phi\colon X \rightsquigarrow Y$ is a complete valued multifunction, then Φ is $\mathcal{M}(X)$-measurable if and only if g_y is $\mathcal{M}(X)$-measurable for each $y \in Y$ [6: Theorem III.9].

Furthermore (see [17: Theorem 1])

(5) If (Y, d) is a Polish space and Φ is a closed valued $\mathcal{M}(X)$-measurable multifunction, then there exists an $\mathcal{M}(X)$-measurable selection of Φ.

Let $\mathcal{M}(X) \otimes \mathcal{B}(Y)$ be the σ-field in $X \times Y$ generated by the sets $A \times B$, where $A \in \mathcal{M}(X)$ and $B \in \mathcal{B}(Y)$. Given a multifunction $F\colon X \times Y \rightsquigarrow Z$ and a multifunction $\Phi\colon X \rightsquigarrow Y$ we define a multifunction $F_\Phi\colon X \rightsquigarrow Z$, called the Carathéodory superposition of F with Φ, by the following formula

(6) $$F_\Phi(x) = F(x, \Phi(x)) = \bigcup_{y \in \Phi(x)} F(x, y).$$

It is known (see [22: Theorem 2]) that

(7) If $(X, \mathcal{M}(X))$ is a complete measurable space, (Y, d) is a Polish space and (Z, ϱ) a metric one and if $F\colon X \times Y \rightsquigarrow Z$ is a compact valued multifunction such that $F(\cdot, y)$ is $\mathcal{M}(X)$-measurable for each $y \in Y$ and $F(x, \cdot)$ is continuous for each $x \in X$, then F is $\mathcal{M}(X) \otimes \mathcal{B}(Y)$-measurable and its Carathéodory superposition $F_\Phi\colon X \rightsquigarrow Z$ is $\mathcal{M}(X)$-measurable for each closed valued $\mathcal{M}(X)$-measurable multifunction $\Phi\colon X \rightsquigarrow Y$.

Now we suppose $(Y, \|\cdot\|)$ to be a reflexive real normed linear space with metric d generated by the norm in Y; θ will denote the origin of Y.

We put

$$\mathcal{C}_{bc}(Y) = \{A \in \mathcal{C}_b(Y) : A \text{ is convex}\} \quad \text{and} \quad \mathcal{K}_c(Y) = \{A \in \mathcal{K}(Y) : A \text{ is convex}\}.$$

If $A \in \mathcal{C}_b(Y)$, then $\|A\| = d_H(A, \{\theta\})$.

Let $\Phi\colon X \rightsquigarrow Y$ be a multifunction with $\Phi(x) \in \mathcal{C}_{bc}(Y)$. Φ is called *countable-valued* if there are disjoint sets $A_i \in \mathcal{M}(X)$ with $\bigcup_{i\in\mathbb{N}} A_i = X$ and the sets $E_i \in \mathcal{C}_{bc}(Y)$, $i \in \mathbb{N}$, such that

$$\Phi(x) = \sum_{i\in\mathbb{N}} \kappa_{A_i}(x)\, E_i,$$

where κ_{A_i} means the characteristic function of A_i.

A multifunction $\Phi\colon X \rightsquigarrow Y$ is called *μ-measurable* or *strongly $\mathcal{M}(X)$-measurable* if there is a sequence of countable-valued multifunctions $(\Phi_n)_{n\in\mathbb{N}}$ such that for μ-almost every $x \in X$, $(\Phi_n(x))_{n\in\mathbb{N}}$ converges to $\Phi(x)$.

It is known that

(8) If $X = [a, b] \subset \mathbb{R}$, Y is a separable Banach space and $\Phi\colon X \rightsquigarrow Y$is a multifunction with $\Phi(x) \in \mathcal{K}_c(Y)$, then $\mathcal{L}(\mathbb{R})$-measurability of Φ and strong $\mathcal{L}(\mathbb{R})$-measurability of Φ are equivalent [8: Proposition 3.3].

Many steps have been taken toward diffrential calculus for multifunctions, among others one by Hukuhara [15]. We will use the notion of differentiability given in [18], which is a generalization of idea used by Hukuhara. We consider multifunctions from an interval to a real reflexive normed linear space. In this case, the derivative of a multifunction at a point is a closed convex and bounded set.

If $A \subset Y$, $B \subset Y$ and $\lambda \in \mathbb{R}$, then (as usually)

$$A + B = \{a + b : a \in A, b \in B\}, \;\; \lambda A = \{\lambda a : a \in A\}, \;\; A - B = A + (-1)B.$$

We note that the space $(\mathcal{C}_{bc}(Y), d_H)$ is complete, when (Y, d) is complete. By reflexivity of Y, $A + B \in \mathcal{C}_{bc}(Y)$ if $A, B, \in \mathcal{C}_{bc}(Y)$. Moreover ([20: Lemma 3])

(9) If $A, B, C \in \mathcal{C}_{bc}(Y)$, then $d_H(A, B) = d_H(A + C, B + C)$.

If we agree to deal only with the subcollection $\mathcal{K}_c(Y)$, the requirement that $(Y, \|\cdot\|)$ is a reflexive real normed linear space can be replaced by the assumption that $(Y, \|\cdot\|)$ is a Banach space (see [3: p. 247]).

Let $A, B \in \mathcal{C}_{bc}(Y)$. We will say the *difference $A \ominus B$ is defined* if there exists a set $C \in \mathcal{C}_{bc}(Y)$ such that either $A = B + C$ or $B = A - C$, and we define $A \ominus B$ to be the set C; $A \ominus B$ is uniquely determined.

Example 1.

(a) Let $P \in \mathcal{C}_{bc}(Y)$, $A = \alpha P$ and $B = \beta P$, where $\alpha \geq 0$ and $\beta \geq 0$. Put $C = (\alpha - \beta)P$. Then $A = B + C$ or $B = A - C$ depending on whether $\alpha \geq \beta$ or $\alpha < \beta$. Therefore $A \ominus B$ exists and is equal to C.

(b) If $A = [a, x] \subset \mathbb{R}$ and $B = [b, y] \subset \mathbb{R}$, then the difference $A \ominus B$ exists and $A \ominus B = [\min\{a - b, x - y\}, \max\{a - b, x - y\}]$.

In each case of the above example, with $Y = \mathbb{R}^n$, Hukuhara's differences of the relevvant sets do not exist, since Hukuhara's difference $A \overset{H}{-} B$ of $A, B \in \mathcal{K}_c(Z)$ exists only if $\operatorname{diam}(A) \geq \operatorname{diam}(B)$.

It is known ([18: Theorem 2]) that the difference $A \ominus B$ exists and is equel to a set $C \in \mathcal{C}_{bc}(Y)$ if and only if either for each $a \in \operatorname{Fr}(A)$ there is $y \in Y$ such that $a \in B + \{y\} \subset A$ or for each $b \in \operatorname{Fr}(B)$ there is $y \in Y$ such that $b \in A + \{y\} \subset B$, and C is a set such that either $A = B + C$ or $B = A - C$, respectively.

When $A, B \in \mathcal{K}_c(Y)$, we can replace the sets $\operatorname{Fr}(A)$ and $\operatorname{Fr}(B)$ (used in the above property) by the respective sets of extreme points, appeling to the Krein-Milman theorem.

Note that for $A \in \mathcal{C}_{bc}(Y)$ and $y \in Y$ we have $(A + \{y\}) \ominus A = \{y\}$. In particular $A \ominus A = \{\theta\}$.

Let $I = [a, b] \subset \mathbb{R}$. From now on we assume that $\Phi\colon I \rightsquigarrow Y$ is a multifunction with values in $\mathcal{C}_{bc}(Y)$.

A multifunction $\Phi\colon I \rightsquigarrow Y$ is said to be *differentiable at a point* $x_0 \in I$, if there exists a set $D\Phi(x_0) \in \mathcal{C}_{bc}(Y)$ such that

$$d_H - \lim_{x \to x_0} \frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}$$

exists and is equal to $D\Phi(x_0)$.

The set $D\Phi(x_0)$ is called the *derivative* of Φ at x_0; Φ is called differentiable if it is differentiable at each $x \in I$ [18: Definition 3].

Of course, implicit in the definition of $D\Phi(x_0)$ is assumed the existence of the differences $\Phi(x) \ominus \Phi(x_0)$.

Example 2.

(a) Let B be the closed unit ball in Y and let us consider a multifunction $\Phi\colon (0, 2\pi) \rightsquigarrow Y$ defined by the formula $\Phi(\alpha) = (2 + \sin\alpha)B$. Then Φ is differentiable and $D\Phi(\alpha) = (\cos\alpha)B$.

(b) Let $\Phi\colon I \rightsquigarrow \mathbb{R}$ be a multifunction with $\Phi(x) \in \mathcal{C}_{bc}(\mathbb{R})$. Then $\Phi(x) = [i(x), s(x)]$, where $i(x) = \inf\Phi(x)$ and $s(x) = \sup\Phi(x)$. If the functions $i\colon I \to \mathbb{R}$ and $s\colon I \to \mathbb{R}$ are differentiable at $x_0 \in I$, then Φ is differentiable at x_0 and

$$D\Phi(x_0) = \begin{cases} [i'(x_0), s'(x_0)] & \text{if } \ i'(x_0) \le s'(x_0), \\ [s'(x_0), i'(x_0)] & \text{if } \ i'(x_0) > s'(x_0). \end{cases}$$

However, in general, differentiability of Φ does not imply differentiability of i and s, as the following example shows:

$$\Phi(x) = \begin{cases} [0, x] & \text{if } \ x \ge 0, \\ [x, 0] & \text{if } \ x < 0. \end{cases}$$

Proposition 1. *Let us suppose that a multifunction $\Phi\colon I \rightsquigarrow Y$ is differentiable at a point $x_0 \in I$ and let ϕ be a selection of Φ.*

If ϕ is differentiable at x_0, then $\phi'(x_0) \in D\Phi(x_0)$.

Proof. Note that $\Phi(x) \ominus \Phi(x_0)$ exists. Let $C \in \mathcal{C}_{bc}(Y)$ be such that $\Phi(x) \ominus \Phi(x_0) = C$. If $\Phi(x) = \Phi(x_0) + C$, then there is $c \in C$ such that $\phi(x) = \phi(x_0) + c$. Thus $c = \phi(x) - \phi(x_0) \in \Phi(x) \ominus \Phi(x_0)$. Similarly $\phi(x) - \phi(x_0) \in \Phi(x) \ominus \Phi(x_0)$ if $\Phi(x_0) = \Phi(x) - C$. Therefore $h(\{\phi(x) - \phi(x_0)\}, \Phi(x) \ominus \Phi(x_0)) = 0$. It is clear that for $x \neq x_0$ we have

$$\begin{aligned} h(\{\phi'(x_0)\}, D\Phi(x_0)) \le{}& h\left(\{\phi'(x_0)\}, \frac{\{\phi(x) - \phi(x_0)\}}{x - x_0}\right) \\ &+ h\left(\frac{\{\phi(x) - \phi(x_0)\}}{x - x_0}, \frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}\right) \\ &+ h\left(\frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}, D\Phi(x_0)\right) \\ \le{}& d_H\left(\{\phi'(x_0)\}, \frac{\{\phi(x) - \phi(x_0)\}}{x - x_0}\right) \\ &+ h\left(\frac{\{\phi(x) - \phi(x_0)\}}{x - x_0}, \frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}\right) \\ &+ d_H\left(\frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}, D\Phi(x_0)\right). \end{aligned}$$

Let $\varepsilon > 0$. By differentiability of Φ and ϕ at x_0 there exists a neighbourhood $U(x_0)$ of x_0 such that for any $x \in U(x_0)$ and $x \neq x_0$

$$d_H\left(\frac{\Phi(x) \ominus \Phi(x_0)}{x - x_0}, D\Phi(x_0)\right) < \frac{\varepsilon}{2} \quad \text{and} \quad d_H\left(\frac{\{\phi(x) - \phi(x_0)\}}{x - x_0}, \{\phi'(x_0)\}\right) < \frac{\varepsilon}{2}.$$

Therefore

$$h(\{\phi'(x_0)\}, D\Phi(x_0)) < \frac{\varepsilon}{2} + \frac{1}{|x - x_0|} h(\{\phi(x) - \phi(x_0)\}, \Phi(x) \ominus \Phi(x_0)) + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\{\phi'(x_0)\} \subset D\Phi(x_0)$, which finishes the proof of Proposition 1. □

A notion of a derivative multifunction we introduce making use of the notion of integral given by Banks and Jacobs in [3]. They discuss integral of a multifunction $F\colon I \rightsquigarrow Y$, where $(Y, \|\cdot\|)$ is a reflexive Banach space. They gave it by taking an advantage of Rådström's embedding theorem ([20: Theorem 2]).

From now on we make the assumption that $(Y, \|\cdot\|)$ is a reflexive Banach space. Let $\sim$ be an equivalence relation defined on $\mathcal{C}_{bc}(Y) \times \mathcal{C}_{bc}(Y)$ by taking that

$$(A, B) \sim (C, D) \qquad \text{iff} \qquad A + D = B + C.$$

Let $\langle A, B\rangle$ denote the equivalence class of $\sim$ containing (A, B), and let $\mathcal{V}(Y)$ be the quotient space $\mathcal{C}_{bc}(Y) \times \mathcal{C}_{bc}(Y)/_{\sim}$, whith addition and scalar multiplication defined by

$$\langle A, B\rangle + \langle C, D\rangle = \langle A + C, B + D\rangle$$

and

$$\alpha\langle A, B\rangle = \begin{cases} \langle \alpha A, \alpha B\rangle & \text{if } \ \alpha \geq 0, \\ \langle |\alpha| B, |\alpha| A\rangle & \text{if } \ \alpha < 0, \end{cases}$$

respectively. The neutral element $\langle\theta, \theta\rangle$ of $\mathcal{V}(Y)$ is the equivalence class $\{(A, A) : A \in \mathcal{C}_{bc}(Y)\}$.

A metric δ on $\mathcal{V}(Y) \times \mathcal{V}(Y)$ is defined by

(10) $\delta(\langle A, B\rangle, \langle C, D\rangle) = d_H(A + D, B + C)$,

and $\|\langle A, B\rangle\| = \delta(\langle A, B\rangle, \langle\theta, \theta\rangle)$ defines a norm in $\mathcal{V}(Y)$ such that

(11) $\delta(\langle A, B\rangle, \langle C, D\rangle) = \|\langle A, B\rangle - \langle C, D\rangle\|$.

The space $(\mathcal{V}(Y), \delta)$ need not be complete, if (Y, d) is complete [7: p. 363].

Let $\overline{\mathcal{V}}(Y)$ be δ-completion of $\mathcal{V}(Y)$, which is a Banach space.

Let $\pi\colon \mathcal{C}_{bc}(Y) \to \mathcal{V}(Y)$ be an embedding given by the equality $\pi(A) = \langle A, \theta\rangle$: π is an isometric mapping.

We will denote $\pi(A)$ by $\widehat{A}$ and the convex cone $\pi(\mathcal{C}_{bc}(Y))$ by $\widehat{\mathcal{C}}_{bc}(Y)$.

Following Banks and Jacobs, a multifunction $\Phi\colon I \rightsquigarrow Y$ is called *integrable* (on I) if the function $\widehat{\Phi}\colon I \to \overline{\mathcal{V}}(Y)$ is Bochner integrable (on I) in the sense of [9: Definition 17, p. 112] (Lebesgue measure m on I is understood), and the integral of $\widehat{\Phi}$ is denoted by $\int\limits_I \widehat{\Phi}(x)\, \mathrm{d}x x$ or $\int\limits_a^b \widehat{\Phi}(x)\, \mathrm{d}x x$.

A multifunction $\Phi\colon I \rightsquigarrow Y$ is called *integrably bounded* if there exists a Lebesgue integrable function $g\colon I \to \mathbb{R}$ such that $\|\Phi(x)\| \leq g(x)$ for $x \in I$.

Note that, by [9: Theorem 22, p. 117], the following is true.

(12) If a multifunction $\Phi\colon I \rightsquigarrow Y$ is strongly $\mathcal{L}(\mathbb{R})$-measurable and integrably bounded, then Φ is integrable.

It is known [3: Lemma 5.4], that

(13) If a multifunction $\Phi\colon I \rightsquigarrow Y$ is integrable, then $\int_I \widehat{\Phi}(x)\,\mathrm{d}x \in \widehat{\mathcal{C}}_{bc}(Y)$.

If $\Phi\colon I \rightsquigarrow Y$ is integrable, then we define

$$\int_I \Phi(x)\ \mathrm{d}x = A \in \mathcal{C}_{bc}(Y) \qquad \text{whenever} \quad \int_I \widehat{\Phi}(x)\,\mathrm{d}x = \widehat{A}.$$

Let $\Phi_i\colon I \rightsquigarrow Y$, $i = 1, 2$, be integrable multifunctions. By (10) and (11),

$$\Big\|\Big\langle \int_I \Phi_1(x)\,\mathrm{d}x, \theta\Big\rangle - \Big\langle \int_I \Phi_2(x)\,\mathrm{d}x, \theta\Big\rangle\Big\| = d_H\Big(\int_I \Phi_1(x)\,\mathrm{d}x, \int_I \Phi_2(x)\,\mathrm{d}x\Big)$$

and

$$\|\langle \Phi_1(x), \theta\rangle - \langle \Phi_2(x), \theta\rangle\| = d_H(\Phi_1(x), \Phi_2(x)).$$

Therefore (see [14: Theorem 3.7.6]), we have

(14) $$d_H\Big(\int_I \Phi_1(x)\,\mathrm{d}x, \int_I \Phi_2(x)\,\mathrm{d}x\Big) \le \int_I d_H(\Phi_1(x), \Phi_2(x))\,\mathrm{d}x.$$

In particular, $\|\int_I \Phi(x)\,\mathrm{d}x\| \le \int_I \|\Phi(x)\|\,\mathrm{d}x$.

Lemma 1. *If a multifunction $\Phi\colon I \rightsquigarrow Y$ is integrable and $\delta > 0$, then a multifunction $\Phi_\delta\colon I \rightsquigarrow Y$ given by the formula*

$$\Phi_\delta(x) = \begin{cases} \int_x^{x+\delta} \Phi(t)\,\mathrm{d}t & \text{if } a \le x < b - \delta, \\ \int_{b-\delta}^{b} \Phi(t)\,\mathrm{d}t & \text{if } b - \delta \le x \le b, \end{cases}$$

is continuous.

P r o o f. Let $x_0 \in I$ be fixed. Let us suppose that $x_0 < b - \delta$ and $x_0 < x < b - \delta$. Then

$$\begin{aligned} d_H(\Phi_\delta(x_0), \Phi_\delta(x)) &= d_H\Big(\int_{x_0}^{x_0+\delta} \Phi(t)\,\mathrm{d}t, \int_x^{x+\delta} \Phi(t)\mathrm{d}t\Big) \\ &= d_H\Big(\int_{x_0}^{x} \Phi(t)\,\mathrm{d}t + \int_x^{x_0+\delta} \Phi(t)\,\mathrm{d}t, \int_x^{x_0+\delta} \Phi(t)\,\mathrm{d}t + \int_{x_0+\delta}^{x+\delta} \Phi(t)\,\mathrm{d}t\Big) \\ &= d_H\Big(\int_{x_0}^{x} \Phi(t)\,\mathrm{d}t, \int_{x_0+\delta}^{x+\delta} \Phi(t)\mathrm{d}t\Big), \qquad \text{by (9).} \end{aligned}$$

Thus, by (14),

$$d_H(\Phi_\delta(x_0),\Phi_\delta(x)) = d_H\Big(\int_{x_0}^{x}\Phi(t)\,\mathrm{d}t, \int_{x_0+\delta}^{x+\delta}\Phi(t)\,\mathrm{d}t\Big)$$
$$\le \Big\|\int_{x_0}^{x}\Phi(t)\,\mathrm{d}t\Big\| + \Big\|\int_{x_0+\delta}^{x+\delta}\Phi(t)\,\mathrm{d}t\Big\| \to 0 \qquad \text{as} \quad x\to x_0.$$

If $x_0-\delta<x<x_0$, then

$$d_H(\Phi_\delta(x_0),\Phi_\delta(x)) = d_H\Big(\int_{x_0}^{x_0+\delta}\Phi(t)\,\mathrm{d}t, \int_{x}^{x+\delta}\Phi(t)\mathrm{d}t\Big)$$
$$= d_H\Big(\int_{x_0}^{x+\delta}\Phi(t)\,\mathrm{d}t + \int_{x+\delta}^{x_0+\delta}\Phi(t)\,\mathrm{d}t, \int_{x}^{x_0}\Phi(t)\,\mathrm{d}t + \int_{x_0}^{x+\delta}\Phi(t)\,\mathrm{d}t\Big)$$
$$= d_H\Big(\int_{x+\delta}^{x_0+\delta}\Phi(t)\,\mathrm{d}t, \int_{x}^{x_0}\Phi(t)\,\mathrm{d}t\Big) \to 0 \qquad \text{as} \quad x\to x_0.$$

Now let us suppose that $x_0\ge b-\delta$. Since Φ_δ is constant for $b-\delta\le x\le b$, it is enough to consider only the case $x_0=b-\delta$ and $x_0-\delta<x<x_0$. Then

$$d_H(\Phi_\delta(x_0),\Phi_\delta(x)) = d_H\Big(\int_{x_0}^{b}\Phi(t)\,\mathrm{d}t, \int_{x}^{x+\delta}\Phi(t)\mathrm{d}t\Big)$$
$$= d_H\Big(\int_{x_0}^{x+\delta}\Phi(t)\,\mathrm{d}t + \int_{x+\delta}^{b}\Phi(t)\,\mathrm{d}t, \int_{x}^{x_0}\Phi(t)\,\mathrm{d}t + \int_{x_0}^{x+\delta}\Phi(t)\,\mathrm{d}t\Big)$$
$$= d_H\Big(\int_{x+\delta}^{x_0+\delta}\Phi(t)\,\mathrm{d}t, \int_{x}^{x_0}\Phi(t)\,\mathrm{d}t\Big) \to 0 \qquad \text{as} \quad x\to x_0,$$

which proves Lemma 1. □

Let $I=[a,b]$ and let $\Phi\colon I\rightsquigarrow Y$ be integrable. We define a multifunction $\Psi\colon I\rightsquigarrow Y$ by

$$x\to\Psi(x)=\int_a^x\Phi(t)\,\mathrm{d}t.$$

Then, by (13), $\Psi(x)\in\mathcal{C}_{bc}(Y)$ for $x\in I$. A simple computation shows that

(15) If Φ is integrable and $x_0 \in I$, then the differences $\Psi(x) \ominus \Psi(x_0)$ exist for $x \in I$, and $\int\limits_{x_0}^{x} \Phi(t)\,\mathrm{d}t = \Psi(x) \ominus \Psi(x_0)$.

If $x < x_0$, then we put $\int\limits_{x_0}^{x} \Phi(t)\,\mathrm{d}t = -\int\limits_{x}^{x_0} \Phi(t)\,\mathrm{d}t$.

It is known ([19: Proposition 1.39]) that

(16) If a multifunction $\Phi\colon I \rightsquigarrow Y$ is continuous and $x_0 \in I$, then $D\Psi(x_0) = \Phi(x_0)$.

Now we can define a derivative multifunction. Let $\Phi\colon I \rightsquigarrow Y$ be an integrable multifunction and $x_0 \in I$.

The statement that Φ is a derivative at x_0 means, that $\Phi(x_0) = D\Psi(x_0)$, i.e.

$$\Phi(x_0) = d_H - \lim_{x\to x_0} \frac{\Psi(x) \ominus \Psi(x_0)}{x - x_0}.$$

By (15), Φ is a derivative at x_0 if and only if

$$\Phi(x_0) = d_H - \lim_{x\to x_0} \frac{1}{x - x_0} \int\limits_{x_0}^{x} \Phi(t)\,\mathrm{d}t.$$

Φ is a derivative if it is a derivative at each point $x \in I$.

It follows from (16) that

Corollary 1. *If $\Phi\colon I \rightsquigarrow Y$ is a continuous multifunction, then Φ is a derivative.*

Another concept of integral of multifunction was used by Aumann in [2].

Let $L_1(I, m, Y)$ be the space of all Bochner integrable functions $\phi\colon I \to Y$ and let a multifunction $\Phi\colon I \rightsquigarrow Y$ be given. Then

$$\mathcal{F}_I(\Phi) = \bigl\{\phi \in L_1(I, m, Y) : \ \phi(x) \in \Phi(x) \ \text{ a.e. in } \ I\bigr\}.$$

Aumann defines the integral of Φ (on I) by

$$(A)\int\limits_I \Phi(x)\,\mathrm{d}x = \Bigl\{ \int\limits_I \phi(x)\,\mathrm{d}x : \ \phi \in \mathcal{F}_I(\Phi)\Bigr\}.$$

Φ is called *A-integrable* if $(A)\int\limits_I \Phi(x)\,\mathrm{d}x \neq \emptyset$.

We conclude from (5) that a multifunction $\Phi\colon I \rightsquigarrow Y$ is A-integrable when the space Y is separable and Φ is an $\mathcal{L}(\mathbb{R})$-measurable and integrably bouned multifunction.

It is known that

(17) If $\Phi\colon I \rightsquigarrow Y$ is an integrable multifunction, then Φ is A-integrable and
$$\int_I \Phi(x)\,\mathrm{d}x = (A)\int_I \Phi(x)\,\mathrm{d}x, \qquad \text{(see [3: Lema 5.6])}.$$

Let $\mathcal{S}_I(\Phi) = \{\psi_\phi : \phi \in \mathcal{F}_I(\Phi)\}$, where $\psi_\phi(x) = \int_a^x \phi(t)\,\mathrm{d}t$ for $x \in I = [a,b]$. Then the set $\mathcal{S}_I(\Phi)$ is a subset of the metric space of all absolutely continuous functions $f\colon I \to Y$ vanishing at the left point of I with the norm $\langle f\rangle = \int_I \|f'(t)\|\,\mathrm{d}t$. It is evident that $\mathcal{S}_I(\Phi)$ is an equicontinuous set. If Φ is integrably bounded, then $\mathcal{S}_I(\Phi)$ is bounded.

If $\Phi_i\colon I \rightsquigarrow Y$, $i \in \mathcal{I}$, are multifunctions, then the family $\{\Phi_i\}_{i\in\mathcal{I}}$ is called *uniformly integrably bounded* if there is a Lebesgue integrable function $g\colon I \to \mathbb{R}$ such that $\|\Phi_i(x)\| \le g(x)$ for all $i \in \mathcal{I}$ and $x \in I$.

Let ϱ_H be the Hausdorff metric in the space $\mathcal{C}_b(\mathcal{C}(I,Y))$, with metric generated by the supremum norm. Then the following is true (see [4: Theorem 3.2]):

(18) Let $Y = \mathbb{R}^k$ and let $\Phi, \Phi_n\colon I \rightsquigarrow Y$, for $n \in \mathbb{N}$, be a sequence of multifunctions with values in $\mathcal{K}_c(Y)$ such that $\lim_{n\to\infty} d_H(\Phi_n(x), \Phi(x)) = 0$ for $x \in I$. If $\{\Phi_n\}_{n\in\mathbb{N}}$ is uniformly integrably bounded and all multifunctions Φ_n are $\mathcal{L}(\mathbb{R})$-measurable, then $\mathcal{S}_I(\Phi_n) \in \mathcal{K}_c(\mathcal{C}(I,Y))$, $\mathcal{S}_I(\Phi) \in \mathcal{K}_c(\mathcal{C}(I,Y))$, and $\lim_{n\to\infty} \varrho_H(\mathcal{S}_I(\Phi_n), \mathcal{S}_I(\Phi)) = 0$.

3. Main results

Still let $I = [a,b] \subset \mathbb{R}$ be an interval and $(Y, \|\cdot\|)$ a reflexive Banach space.

Given a multifunction $F\colon I \times Y \rightsquigarrow Y$ and a function $f\colon I \to Y$, $F_f\colon I \rightsquigarrow Y$ will denote the Carathéodory superposition of F with f, i.e. in compliance with (6)
$$F_f(x) = F(x, f(x)) \qquad \text{for} \quad x \in I.$$

Definition 1. We say that a multifunction $F\colon I\times Y \rightsquigarrow Y$ with values in $\mathcal{C}_{cb}(Y)$ has the (H) property if

(i) for each $y \in Y$, $F(\cdot, y)$ is a derivative multifunction,

(ii) the collection $\{F(x,\cdot)\}_{x\in I}$ is equicontinuous,

(iii) the family $\{F_f\}_{f\in\mathcal{C}(I,Y)}$ is uniformly integrably bounded.

Now we can prove a theorem which is useful to consider the existence of solution of differential inclusion. From now on we assume that $(Y, \|\cdot\|)$ is a separable Banach space and $F\colon I\times Y \rightsquigarrow Y$ a multifunction with $F(x,y) \in \mathcal{K}_c(Y)$.

Theorem 1. *If a multifunction $F\colon I \times Y \rightsquigarrow Y$ has the* (H) *property, then for every function $f \in \mathcal{C}(I,Y)$, the Carathéodory superposition F_f is a derivative.*

P r o o f. Let $x_0 \in I$, $\varepsilon > 0$ and $f \in \mathcal{C}(I,Y)$ be given. Since $F(\cdot, f(x_0))$ is a derivative at x_0, we have

$$F(x_0, f(x_0)) = \lim_{x\to x_0} \frac{1}{x-x_0} \int_{x_0}^{x} F(t, f(x_0))\,\mathrm{d}t.$$

Note that

$$F(x_0, f(x_0)) = \frac{1}{x-x_0} \int_{x_0}^{x} F(x_0, f(x_0))\,\mathrm{d}t.$$

Therefore

$$\lim_{x\to x_0} d_H\left(\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(x_0))\,\mathrm{d}t, \frac{1}{x-x_0}\int_{x_0}^{x} F(x_0,f(x_0))\,\mathrm{d}t\right) = 0. \tag{19}$$

The collection $\{F(x,\cdot)\}_{x\in I}$ is equicontinuous at $f(x_0)$, therefore there exists $\delta > 0$ such that $d_H(F(x,y), F(x,f(x_0))) < \varepsilon$ for each $y \in B(f(x_0),\delta)$ and $x \in I$. By continuity of f at x_0, there exists $\eta > 0$ such that $|x - x_0| < \eta$ implies $d(f(x), f(x_0)) < \delta$. Thus

$$d_H(F(x,f(x)), F(x,f(x_0))) < \varepsilon \quad \text{whenever } |x-x_0| < \eta. \tag{20}$$

Let $y \in Y$ and $n \in \mathbb{N}$ be fixed. Then, by Lemma 1, a multifunction $\Phi_n\colon I \rightsquigarrow Y$ given by

$$\Phi_n(x) = \begin{cases} n \int\limits_{x}^{x+\frac{1}{n}} F(t,y)\,\mathrm{d}t & \text{if } a \le x < b - \frac{1}{n}, \\ n \int\limits_{b-\frac{1}{n}}^{b} F(t,y)\,\mathrm{d}t & \text{if } b-\frac{1}{n} \le x \le b, \end{cases}$$

is continuous and, by (3), $\mathcal{L}(\mathbb{R})$-measurable. Note that $F(x,y) = \lim\limits_{n\to\infty} \Phi_n(x)$ for $x \in I$, since $F(\cdot,y)$ is a derivative. Therefore, $F(\cdot,y)$ is $\mathcal{L}(\mathbb{R})$-measurable (see (2) and (4)). On the other hand $F(x,\cdot)$ is continuous for each $x \in I$. Therefore, F is $\mathcal{L}(\mathbb{R})\otimes\mathcal{B}(Y)$-measurable and its Carathéodory superposition F_f is $\mathcal{L}(\mathbb{R})$-measurable, by (7). So, by (8), F_f is strongly $\mathcal{L}(\mathbb{R})$-measurable. F_f is also integrably bounded. Thus, by (12),

(21) the multifunction F_f is integrable.

It follows from (14) and (20) that

$$d_H\Big(\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(t))\,\mathrm{d}t, \frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(x_0))\,\mathrm{d}t\Big) \le \frac{1}{|x-x_0|}\Big|\int_{x_0}^{x} d_H\left(F(t,f(t)),F(t,f(x_0))\right)\mathrm{d}t\Big| < \varepsilon \tag{22}$$

whenever $0<|x-x_0|<\eta$. Furthermore

$$\begin{aligned} &d_H\Big(\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(t))\,\mathrm{d}t, \frac{1}{x-x_0}\int_{x_0}^{x} F(x_0,f(x_0))\,\mathrm{d}t\Big)\\ &\le d_H\Big(\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(t))\,\mathrm{d}t, \frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(x_0))\,\mathrm{d}t\Big)\\ &\quad + d_H\Big(\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(x_0))\,\mathrm{d}t, \frac{1}{x-x_0}\int_{x_0}^{x} F(x_0,f(x_0))\,\mathrm{d}t\Big). \end{aligned}$$

We see from (19) and (22) that

$$\lim_{x\to x_0}\frac{1}{x-x_0}\int_{x_0}^{x} F(t,f(t))\,\mathrm{d}t = F(x_0,f(x_0)),$$

and the proof of Theorem 1 is finished. □

Theorem 1 is a generalization of Grande theorem [11: Theorem 1] into the case of multifunctions.

As a consequence of Theorem 1 and Proposition 1 we obtain:

Corollary 2. *Let $I=[a,b]$ and $(x_0,y_0)\in I\times Y$. If a multifunction $F\colon I\times Y\rightsquigarrow Y$ has the* (H) *property and $f\in\mathcal{C}(I,Y)$ is such that*

$$f(x)=y_0+\int_{x_0}^{x}\phi(t)\,\mathrm{d}t$$

for $x\in[x_0,b]$, where $\phi\colon[x_0,b]\to Y$ is an integrable selection of the Carathéodory superposition of F with f, then $f'(x)\in F(x,f(x))$ for almost every $x\in[x_0,b]$ and $f(x_0)=y_0$.

P r o o f. Let F_f be the Carathéodory superposition of F with f. Then F_f is integrable (see (21)) and, by (17), we have

$$\int_{x_0}^{x} F_f(t)\,\mathrm{d}t = (A)\int_{x_0}^{x} F_f(t)\,\mathrm{d}t.$$

Let us put $\Phi(x) = \{y_0\} + \int_{x_0}^{x} F_f(t)\,\mathrm{d}t$ for $x \in [x_0, b]$. Then $f(x) \in \Phi(x)$ for $x \in [x_0, b]$. Note that the multifunction Φ is differentiable and $D\Phi(x) = F_f(x)$ for $x \in [x_0, b]$, by Theorem 1. Moreover f is almost everywhere differentiable, so $f'(x) \in D\Phi(x)$ almost everywhere in $[x_0, b]$, by Proposition 1. Thus $f'(x) \in F(x, f(x))$ almost everywhere in $[x_0, b]$. Obviously we have $f(x_0) = y_0$, which completes the proof of Corollary 2. □

Theorem 2. *Let $I = [a, b] \subset \mathbb{R}$, $Y = \mathbb{R}^k$, $(x_0, y_0) \in I \times Y$ and $x_0 < b$. If a multifunction $F\colon I \times Y \rightsquigarrow Y$ has the* (H) *property, then there is a continuous function $f\colon [x_0, b] \to Y$ such that*

$$f(x) = y_0 + \int_{x_0}^{x} \phi(t)\,\mathrm{d}t,$$

where ϕ is an integrable selection of the Carathéodory superposition of F with f.

P r o o f. Let $\alpha > 0$ be such that $[x_0, x_0 + \alpha] \subset I$ and let $g\colon I \to \mathbb{R}$ be a Lebesgue integrable function such that $\|F_f(x)\| \le g(x)$ for $f \in \mathcal{C}(I, Y)$ and $x \in I$. Idea of the proof is based on that of Hartman [12: Theorem 2.1]. First let us take a multifunction $F_{y_0}(x) = F(x, y_0)$ for $x \in [x_0, x_0 + \alpha]$. Then F_{y_0} is integrable (see (21)) and, by (17), $\int_{x_0}^{x} F_{y_0}(t)\,\mathrm{d}t = (A)\int_{x_0}^{x} F_{y_0}(t)\,\mathrm{d}t$ for $x \in [x_0, x_0 + \alpha]$. Let ϕ_1 be an itegrable selection of F_{y_0} in $[x_0, x_0 + \alpha]$ and let $\psi_1(x) = \int_{x_0}^{x} \phi_1(t)\,\mathrm{d}t$ for $x \in [x_0, x_0 + \alpha]$. Then $\psi_1 \in \mathcal{S}_{[x_0, x_0+\alpha]}(F_{y_0})$. Let us put $f_\alpha(x) = y_0 + \psi_1(x)$, i.e.

$$f_\alpha(x) = y_0 + \int_{x_0}^{x} \phi_1(t)\,\mathrm{d}t \qquad \text{for} \quad x \in [x_0, x_0 + \alpha].$$

Then $f_\alpha \in \{y_0\} + \mathcal{S}_{[x_0, x_0+\alpha]}(F_{y_0})$. Since F_{y_0} is integrably bounded by g, we have

(23) $\|f_\alpha(x)\| \le \|y_0\| + \int_{x_0}^{x} g(t)\,\mathrm{d}t$ for $x \in [x_0, x_0+\alpha]$

and

$\|f_\alpha(x_1) - f_\alpha(x_2)\| \le \left|\int_{x_1}^{x_2} g(t)\,\mathrm{d}t\right|$ for $x_1, x_2 \in [x_0, x_0+\alpha]$.

If $x_0 + 2\alpha < b$, then we put

$$F_{f_\alpha}(x) = \begin{cases} F(x, y_0), & \text{if } x \in [x_0, x_0+\alpha], \\ F(x, f_\alpha(x-\alpha)), & \text{if } x \in (x_0+\alpha, x_0+2\alpha]. \end{cases}$$

By continuity of f_α, F_{f_α} is integrably bounded (by g). So, F_{f_α} is integrable. Let ϕ_2 be an integrable selection of F_{f_α} and let $\psi_2(x) = \int_{x_0+\alpha}^{x} \phi_2(t)\,\mathrm{d}t$ for $x \in [x_0+\alpha, x_0+2\alpha]$. Then $\psi_2 \in \mathcal{S}_{[x_0+\alpha, x_0+2\alpha]} F(\cdot, f_\alpha(\cdot - \alpha))$. We can extend f_α to the interval $[x_0+\alpha, x_0+2\alpha]$ putting $f_\alpha(x) = f_\alpha(x_0+\alpha) + \psi_2(x)$ for $x \in [x_0+\alpha, x_0+2\alpha]$. Thus we have

$$f_\alpha(x) = \begin{cases} y_0 + \int_{x_0}^{x} \phi_1(t)\,\mathrm{d}t, & \text{if } x \in [x_0, x_0+\alpha], \\ f_\alpha(x_0+\alpha) + \int_{x_0+\alpha}^{x} \phi_2(t)\,\mathrm{d}t, & \text{if } x \in (x_0+\alpha, x_0+2\alpha], \end{cases}$$

and $f_\alpha \in \{y_0\} + \mathcal{S}_{[x_0, x_0+2\alpha]}(F_{f_\alpha})$.

Note that the extended function f_α fulfil (23) for $x \in [x_0, x_0+2\alpha]$.

If $x_0 + 3\alpha < b$, the process can be continued. Finally at most in a finite meny steps, f_α can be extended to $[x_0, b]$ such that $f_\alpha \in \{y_0\} + \mathcal{S}_{[x_0,b]}(F_{f_\alpha})$, where $F_{f_\alpha}\colon [x_0, b] \rightsquigarrow Y$ is given by

$$F_{f_\alpha}(x) = \begin{cases} F(x, y_0), & \text{if } x \in [x_0, x_0+\alpha], \\ F(x, f_\alpha(x-\alpha)), & \text{if } x \in (x_0+\alpha, b], \end{cases}$$

and (23) is true for $x \in [x_0, b]$. Let $\mathcal{I} \subset \mathbb{R}_+$ be such that $x_0 + \alpha \le b$ for $\alpha \in \mathcal{I}$. Note that

(24) the family $\{F_{f_\alpha}\}_{\alpha \in \mathcal{I}}$ is uniformly integrably bounded on $[x_0, b]$, f_α is continuous, F_{f_α} is $\mathcal{L}(\mathbb{R})$-measurable and $f_\alpha \in \{y_0\} + \mathcal{S}_{[x_0,b]}(F_{f_\alpha})$ for each $\alpha \in \mathcal{I}$.

Now let $(\alpha_n)_{n\in\mathbb{N}}$ be a decreasing sequence of numbers from $\mathcal{I}$ converging to 0. Then $\{f_{\alpha_n}\}_{n\in\mathbb{N}} \subset \{y_0\} + \mathcal{S}_{[x_0,b]}(F_{f_{\alpha_n}})$ is bounded and equicontinuous. By the Arzela-Ascoli theorem, we can find a subsequence of $(f_{\alpha_n})_{n\in\mathbb{N}}$ (still denoted by

$(f_{\alpha_n})_{n\in\mathbb{N}})$ converging uniformly on $[x_0, b]$ to a limit function f. Then

$$\begin{aligned}\|f(x_1) - f(x_2)\| &\le \|f(x_1) - f_{\alpha_n}(x_1)\| + \|f(x_2) - f_{\alpha_n}(x_2)\| \\ &\qquad + \|f_{\alpha_n}(x_1) - f_{\alpha_n}(x_2)\| \\ &< \varepsilon + \Big|\int_{x_1}^{x_2} g(x)\,\mathrm{d}x\Big|\end{aligned}$$

for $\varepsilon > 0$, $x_1, x_2 \in [x_0, b]$ and n sufficiently large. Thus f is absolutely continuous on $[x_0, b]$. Since $\{f_{\alpha_n}\}_{n\in\mathbb{N}}$ is uniformly convergent to f and f is continuous, $\lim\limits_{n\to\infty} f_{\alpha_n}(x - \alpha_n) = f(x)$ for $x \in [x_0, b]$. Furthermore, by equicontinuity of $\{F(x, \cdot)\}_{x\in I}$, we have

(25) $\lim\limits_{n\to\infty} d_H(F_{f_{\alpha_n}}(x), F_f(x)) = 0 \quad$ for $x \in [x_0, b]$,

where F_f is the Carathéodory superposition of F with f in $[x_0, b]$. Then by (25), (24) and (18), $\mathcal{S}_{[x_0,b]}(F_f)$ is a compact subset of $\mathcal{C}([x_0, b], Y)$ and

$$\lim_{n\to\infty} \varrho_H(\mathcal{S}_{[x_0,b]}(F_{f_{\alpha_n}}), \mathcal{S}_{[x_0,b]}(F_f)) = 0.$$

Thus $f \in \{y_0\} + \mathcal{S}_{[x_0,b]}(F_f)$, since $\lim\limits_{n\to\infty} f_{\alpha_n} = f$. Moreover f is continuous and the proof of Theorem 2 is finished. $\square$

Since the function f in the above Theorem is absolutely continuous, by Theorem 2 and Corollary2 we have

COROLLARY 3. *Let $I = [a, b]$ and $Y = \mathbb{R}^n$. If a multifunction $F\colon I \times Y \rightsquigarrow Y$ has the* (H) *property, then for $(x_0, y_0) \in I \times Y$, $x_0 < b$, the initial value problem for the differential inclusion* (1) *has a solution.*

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