

NON-CLT GROUPS OF ORDER pq^3

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ABSTRACT. In this note we give a characterization of finite groups of order pq^3 (p, q primes) that fail to satisfy the Converse of Lagrange's Theorem.

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1. Introduction

All groups considered in this note are finite. A group is said to be CLT if it possesses subgroups of every possible order (that is, it satisfies the Converse of Lagrange's Theorem) and non-CLT otherwise. It is well-known that CLT groups are solvable (see [7]) and that supersolvable groups are CLT (see [6]). Recall also that the inclusion between the classes of CLT groups and solvable groups, as well as the inclusion between the classes of supersolvable groups and CLT groups are proper (see, for example, [3]).

An important class of solvable groups, which are not necessarily supersolvable, consists of the groups of order $p^\alpha q^\beta$ with p, q primes and $\alpha, \beta \in \mathbb{N}$. So, a natural question is whether such a group is CLT. The Baskaran's papers [1] and [2] answer this question for the particular cases $\alpha = 1, \beta = 2$ and $\alpha = \beta = 2$, respectively. In the current paper we study the case $\alpha = 1, \beta = 3$. Our main theorem gives necessary and sufficient conditions to exist non-CLT groups of order pq^3 and describes the structure of these groups.

THEOREM 1.1. *Let p and q be two primes. Then there exists a non-CLT group of order pq^3 if and only if either p divides $q+1$ or p divides q^2+q+1 . Moreover, excepting the case $p = 3, q = 2$ in which one obtains a unique non-CLT group, namely $SL(2, 3)$, all non-CLT groups of order pq^3 are nontrivial semidirect products of a normal subgroup $H \cong \mathbb{Z}_q^3$ or $H \cong E(q^3)$ by a subgroup $K \cong \mathbb{Z}_p$.*

Most of our notation is standard and will usually not be repeated here. For elementary concepts and results on group theory we refer to [4] and [5].

2. Proofs of the main results

We prove first Theorem 1.1.

Proof. Let G be a non-CLT group of order pq^3 . Since supersolvable groups are CLT, we easily infer that:

- $p \neq q$;
- G has no normal Sylow p -subgroup;
- $q \not\equiv 1 \pmod{p}$.

Then the number n_p of Sylow p -subgroups of G must be q^2 or q^3 . On the other hand, [5: Theorem 1.32] shows that G possesses a normal Sylow q -subgroup, except when $|G| = 24$.

Case 1. $|G| = 24$

In this case, by investigating the 15 types of groups of order 24, we deduce that the only possibility is $G \cong SL(2, 3)$ (this is non-CLT because it has no subgroup of order 12).

Case 2. $|G| \neq 24$

In this case G is a nontrivial semidirect product of a normal subgroup H of order q^3 by a subgroup K of order p . Since the classification of groups of order q^3 depends on the parity of q , we distinguish the following two subcases.

Subcase 2.1. $q = 2$

The conditions $n_p \in \{4, 8\}$ and $n_p \equiv 1 \pmod{p}$ lead to $p = 7$, $n_p = 8$ and $|G| = 56$. Up to isomorphism, there is a unique group of order 56 without normal Sylow 7-subgroups, and the Sylow 2-subgroup of this group is elementary abelian. Moreover, we can easily check that it does not possess subgroups of order 28, i.e. it is indeed non-CLT.

Subcase 2.2. $q \neq 2$

The groups of order q^3 for q odd are either abelian, namely $\mathbb{Z}_{q^3}$, $\mathbb{Z}_q \times \mathbb{Z}_{q^2}$ and $\mathbb{Z}_q^3$, or nonabelian, namely $M(q^3) = \langle x, y \mid x^{q^2} = y^q = 1, y^{-1}xy = x^{q+1} \rangle$ and $E(q^3) = \langle x, y \mid x^q = y^q = [x, y]^q = 1, [x, y] \in Z(E(q^3)) \rangle$. We observe that we cannot have $H \cong \mathbb{Z}_{q^3}$, because in this case G would be metacyclic and therefore CLT. The number of automorphisms of the other four groups is:

- $|\text{Aut}(\mathbb{Z}_q \times \mathbb{Z}_{q^2})| = q^3(q-1)^2$,
- $|\text{Aut}(\mathbb{Z}_q^3)| = q^3(q-1)(q^2-1)(q^3-1)$,
- $|\text{Aut}(M(q^3))| = q^3(q-1)^2$,
- $|\text{Aut}(E(q^3))| = q^3(q-1)^2(q+1)$.

Since there is a nontrivial homomorphism from $K \cong \mathbb{Z}_p$ to $\text{Aut}(H)$, p must divide $|\text{Aut}(H)|$, which implies that either $H \cong \mathbb{Z}_q^3$ or $H \cong E(q^3)$. It is also clear that one of the conditions $p \mid q+1$ or $p \mid q^2+q+1$ is verified.

Conversely, suppose first that $p \mid q^2+q+1$. Then every nontrivial semidirect product G of a normal elementary abelian subgroup of order q^3 by a subgroup of order p is non-CLT. Indeed, if we assume that G possesses a subgroup of order pq^2 , say G_1 , then there is a Sylow p -subgroup S_p of G such that $S_p \subset G_1$. By applying the Sylow's theorems for G_1 , it follows that S_p is normal in G_1 , that is $G_1 \subseteq N_G(S_p)$. This leads to $N_G(S_p) = G_1$ and therefore $q^3 = n_p = [G : N_G(S_p)] = [G : G_1] = q$, a contradiction.

Suppose next that $p \mid q+1$ and let G be a nontrivial semidirect product of a normal subgroup isomorphic to $E(q^3)$ by a subgroup $\langle a \rangle$ of order p such that a commutes with $[x, y]$ (x and y denote the generators of $E(q^3)$, as above). We will prove that G is non-CLT by showing again that it does not possess subgroups of order pq^2 . If G_1 is such a subgroup and S_p is a Sylow p -subgroup of G contained in G_1 , then we can assume that $S_p = \langle a \rangle$. On the other hand, it is obvious that $\langle [x, y] \rangle = \Phi(\langle x, y \rangle) \subset G_1$. Then G_1 contains the commuting subgroups $\langle a \rangle \cong \mathbb{Z}_p$ and $\langle [x, y] \rangle \cong \mathbb{Z}_q$. Consequently, it has subgroups of order pq , i.e. it is CLT. By the main theorem of [1], we deduce that G_1 is necessarily abelian, which implies $G_1 \subseteq N_G(S_p)$. As in the first part of this implication, one obtains $N_G(S_p) = G_1$ and hence $q^2 = n_p = [G : N_G(S_p)] = [G : G_1] = q$, a contradiction. This completes the proof. $\square$

An immediate consequence of Theorem 1.1 is given by the following corollary.

COROLLARY 2.1. *The unique non-CLT groups of order $8p$ with p prime are $SL(2, 3)$ and the group of order 56 described above.*

Next, let G be a non-CLT group of order pq^3 (p, q primes), S_q be a Sylow q -subgroup of G and $\mathcal{M}$ be the set of subgroups of order q^2 of S_q . We consider the conjugation action of G on $\mathcal{M}$ and we choose a set of representatives $\{H_1, H_2, \dots, H_k\}$ for the conjugacy classes. Clearly, we have $S_q \subseteq N_G(H_i)$, for all $i = \overline{1, k}$. On the other hand, every H_i is not normal in G because G does not possess subgroups of order pq^2 . These prove that $N_G(H_i) = S_q$ and therefore $[G : N_G(H_i)] = p$, for all $i = \overline{1, k}$. Thus, we infer that p divides $|\mathcal{M}|$.

Since the numbers of subgroups of order q^2 of $E(q^3)$ and $\mathbb{Z}_q^3$ are $q+1$ and q^2+q+1 , respectively, the above remark shows that Theorem 1.1 can be reformulated in the following way for $q \neq 2$.

COROLLARY 2.2. *Given two primes p and q , with $q \neq 2$, the following statements are true:*

- a) *There exists a non-CLT group of order pq^3 having a Sylow q -subgroup isomorphic to $E(q^3)$ if and only if p divides $q + 1$.*
- b) *There exists a non-CLT group of order pq^3 having an elementary abelian Sylow q -subgroup if and only if p divides $q^2 + q + 1$.*

Finally, we remark that a result similar with Corollary 2.2 also holds in the case $q = 2$ (notice that the Sylow 2-subgroups of $SL(2, 3)$ are isomorphic to the well-known quaternion group Q_8).

COROLLARY 2.3. *Given a prime p , the following statements are true:*

- a) *There exists a non-CLT group of order $8p$ having a Sylow 2-subgroup isomorphic to Q_8 if and only if $p = 3$.*
- b) *There exists a non-CLT group of order $8p$ having an elementary abelian Sylow 2-subgroup if and only if $p = 7$.*

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