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COSTAR MODULES, R-COSTAR MODULES OVER RING EXTENSIONS

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ABSTRACT. For a ring A, an extension ring B, and a fixed right A-module M, we prove that M is costar module, r-costar module if and only if the induced module $Hom_A(B, M)$ is costar module, r-costar module respectively.

1. Introduction

It has been showed by many researchers that some properties related to Morita equivalence pass from a right A-module M to the induced B-module $M \otimes_A B$, where B is a ring extension of the ring A. For example projective, generator, finitely generated, tilting module quasi-progenerator and $*$-module. On the other side, dual concepts as injective cogenerator, Morita duality, quasi-duality module and cotilting module are found to be inherited by the induced B-module $\mathrm{Hom}_A(B, M)$ from a right A-module M. In this work, under some conditions, we prove that M is costar module, r-costar module if and only if $\mathrm{Hom}_A(B, M)$ is costar module, r-costar module respectively.

2. Preliminaries

Let A be any ring, M a fixed right A-module and $D = \mathrm{End}_A(M)$. Let B be another ring and $\alpha\colon A \longrightarrow B$ a ring homomorphism. Suppose that $K = \mathrm{Hom}_A(B, M)$ and $E = \mathrm{End}_B(K)$, then the ring homomorphism $\alpha\colon D \longrightarrow E$ defined via $\alpha(d)(f) = d \circ f$, for all $d \in D$ and $f \in K$, is clearly identity

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preserving. If $B = A \oplus Q$, for a right A-module $Q \subseteq A$, we say that B is a split extension of A and we write $B = A \propto Q$. Hence A and B are extension of each other.

We will denote the functors $\mathrm{Hom}_A(-, M)$ and $\mathrm{Hom}_D(-, M)$ by Δ_M and the functors $\mathrm{Hom}_B(-, K)$ and $\mathrm{Hom}_E(-, K)$ by Δ_K. Let $\mathcal{D}$ denote the class of torsionless right R-modules whose M-dual are finitely generated over D and $\mathcal{C}$ denote the class of finitely generated torsionless left D-modules. M is called costar module if $\Delta_M : \mathcal{D} \rightleftarrows \mathcal{C} : \Delta_M$ is a duality. Let

$$\mathrm{scopres}(M_A) = \left\{V \in \mathrm{Mod}\text{-}A : \ 0 \longrightarrow V \xrightarrow{f} M^n \longrightarrow M^\Lambda \longrightarrow 0, \ \text{with } \ n \in \mathbb{N}\right\}.$$

If $M \in \mathrm{mod}\text{-}A$, then M is said to be an r-costar module provided that any exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ such that X and Y are M-reflexive, remains exact after applying the functor Δ_M if and only if Z is M-reflexive.

The following results would be needed in our proofs.

Lemma 2.1. ([3: Lemma 1.2]) *Let B be a ring extension of A and $K = \mathrm{Hom}_A(B, M)$, where M is a right A-module. Then M cogenerates K if and only if* $\mathrm{Cogen}(K_B) = \left\{V_B : \ V_A \in \mathrm{Cogen}(M_A)\right\}$.

Colby and Fuller obtain the following theorem.

Theorem 2.1. ([1: Theorem 2.7, Proposition 2.8]) *Let A be a ring, $M \in \mathrm{Mod}\text{-}A$ and $D = \mathrm{End}(M_A)$. The following are equivalent.*

(a) $\Delta_M : \mathcal{D} \rightleftarrows \mathcal{C} : \Delta_M$ *is a duality. That is, M_A is a costar module,*

(b) $\Delta_M :$ $\mathrm{scopres}(M_A) \rightleftarrows \mathcal{C} : \Delta_M$ *is a duality and* $\mathcal{D} = \mathrm{scopres}(M_A)$,

(c) δ_M *is an epimorphism if* $\Delta M \in D\text{-}\mathrm{mod}$, *and* δ_N *is an epimorphism if* $N \in D\text{-}\mathrm{mod}$,

(d) $\mathcal{D} \subseteq \mathrm{scopres}(M_A)$ *and if* $0 \longrightarrow V \xrightarrow{f} M^n \longrightarrow L \longrightarrow 0$ *is exact with* $L \in \mathrm{Cogen}(M)$, *then* $\Delta_M f$ *is an epimorphism,*

(e) *If* $0 \longrightarrow V \xrightarrow{f} M^n \longrightarrow L \longrightarrow 0$ *is exact then* $L \in \mathrm{Cogen}(M)$ *if and only if* $\Delta_M f$ *is an epimorphism.*

(f) *If* $0 \longrightarrow V \xrightarrow{f} X \longrightarrow L \longrightarrow 0$ *is exact with* $X \in D$, *then* $L \in \mathrm{Cogen}(M)$ *if and only if* $\Delta_M f$ *is an epimorphism.*

Proposition 2.2. ([3: Proposition 3.3]) *Let M be a right A-module. If $K = \mathrm{Hom}_A(B, M)$ is M-reflexive, then*

(1) *V_B is K-reflexive if and only if V_A is M-reflexive;*

(2) *${}_EW$ is K-reflexive if and only if ${}_DW$ is M-reflexive.*

We say that N weakly divide M in $\mathrm{Mod}\text{-}A$ if $M^n \cong N \oplus Q$, for some positive integer n and for some $Q \in \mathrm{Mod}\text{-}A$ and we write $N|M$.

We assume that the rings are associative with identity, the ring homomorphisms are identity preserving, all (left, right) modules are unital, and all subcategories are full and additive.

3. Costar modules and r-Costar modules

We will fix all the notations and terminologies used in previous section.

Remark 1.

1. For any $V \in \text{Mod-}B$, by the adjoint associativity theorem we have the following isomorphisms

$$\operatorname{Hom}_B(V, \operatorname{Hom}_A(B, M)) \cong \operatorname{Hom}_A(V \otimes_B B, M) \cong \operatorname{Hom}_A(V, M).$$

As a special case, for $K = \operatorname{Hom}_A(B, M)$ we have $E = \operatorname{Hom}_B(K, K) \cong \operatorname{Hom}_A(K, M)$.

2. For the following you can see [3: Lemma 3.2]. From 1 we have

$$\operatorname{Hom}_A(\operatorname{Hom}_A(B, M), M) \cong E.$$

Applying the functor $\operatorname{Hom}_D(-, M)$, we get

$$\operatorname{Hom}_D\big(\operatorname{Hom}_A\left(\operatorname{Hom}_A(B, M), M\right), M\big) \cong \operatorname{Hom}_D(E, M).$$

So if $\operatorname{Hom}_A(B, M)$ *is* M*-reflexive*, then

$$\operatorname{Hom}_A(B, M) \cong \operatorname{Hom}_D(E, M).$$

Hence for any $W \in E\text{-}\operatorname{Mod}$, we get

$$\operatorname{Hom}_E(W, \operatorname{Hom}_A(B, M)) \cong \operatorname{Hom}_E(W, \operatorname{Hom}_D(E, M)).$$

On the other hand, by the adjoint associativity theorem we have

$$\operatorname{Hom}_E(W, \operatorname{Hom}_D(E, M)) \cong \operatorname{Hom}_D(W, M),$$

thus

$$\operatorname{Hom}_E(W, \operatorname{Hom}_A(B, M)) \cong \operatorname{Hom}_D(W, M).$$

3. Let $K = \operatorname{Hom}_A(B, M)$ be M-reflexive in $\text{Mod-}A$. If M has the structure of a right B-module, then by 1 and 2

$$\begin{aligned} \operatorname{Hom}_E(\operatorname{Hom}_B(M, K), K) &\cong \operatorname{Hom}_D(\operatorname{Hom}_B(M, K), M) \\ &\cong \operatorname{Hom}_D(\operatorname{Hom}_A(M, M), M) \\ &\cong \operatorname{Hom}_D(D, M) \\ &\cong M. \end{aligned}$$

Thus M is K-reflexive in $\text{Mod-}B$.

Lemma 3.1. *If $K|M$ in* Mod-A, *then E is finitely generated in D-*Mod *and if $M|K$ in* Mod-B, *then D is finitely generated in E-*Mod.

Proof. Since $K|M$ in Mod-A, then $M^n \cong K \oplus Q$, for some positive integer n and for some $Q \in$ Mod-A. Applying the functor $\mathrm{Hom}_A(-, M)$ we get

$$\begin{aligned} \mathrm{Hom}_A(M^n, M) &\cong \mathrm{Hom}_A(K, M) \oplus \mathrm{Hom}_A(Q, M) \\ D^n &\cong \mathrm{Hom}_B(K, K) \oplus \mathrm{Hom}_A(Q, M) \\ D^n &\cong E \oplus \mathrm{Hom}_A(Q, M), \end{aligned}$$

so E is finitely generated in D-Mod.

If $M|K$ in Mod-B, then $K^n \cong M \oplus F$, for some positive integer n and for some $F \in$ Mod-B. Similarly applying the functor $\mathrm{Hom}_B(-, K)$ we get

$$\mathrm{Hom}_B(K^n, K) \cong E^n \cong D \oplus \mathrm{Hom}_B(F, K),$$

so D is finitely generated in E-Mod. □

Theorem 3.1. *Let M be a right A-module. Let K be M-torsionless and $K|M$ in* Mod-A. *If M is costar, then K is costar.*

Proof. We just need to prove the following statement, if $0 \longrightarrow V \xrightarrow{f} K^n \longrightarrow L \longrightarrow 0$ is an exact sequence in Mod-B, then $L \in \mathrm{Cogen}(K)$ if and only if Δf is an epimorphism. Then the desired result follows from Theorem 2.1. Now suppose we have the exact sequence $0 \longrightarrow V \xrightarrow{f} K^n \longrightarrow L \longrightarrow 0$ in Mod-B, hence its restriction to Mod-A, $0 \longrightarrow V \xrightarrow{f|_A} K^n \longrightarrow L \longrightarrow 0$ is exact. Suppose that $L \in \mathrm{Cogen}(K)$ in Mod-B. Since $K|M$, it is clear that M cogenerates K and hence by Lemma 2.1, $L|_A \in \mathrm{Cogen}(M)$. Since K is M-torsionless, is K^n. By Remark 1.1, $\Delta_M(K) \cong E$. Since $K|M$, by Lemma 3.1, $E \in D$-mod. So $\Delta_M K \in D$-mod and hence $\Delta_M(K^n) \in D$-mod. By definition of $\mathcal{D}$, $K^n \in \mathcal{D}$. Since M is costar, applying Theorem 2.1(f), $\Delta f|_A$ is an epimorphism, and hence, the following commutative diagram

$$\begin{array}{ccc} \Delta_M K^n & \xrightarrow{\Delta(f|_A)} & \Delta_M V \\ \downarrow{\cong} & & \downarrow{\cong} \\ \Delta_K K^n & \xrightarrow{\Delta f} & \Delta_K V \end{array}$$

induce that Δf is an epimorphism (note that the isomorphisms in the diagram come from Remark 1.1).

Conversely Let us suppose that Δf is an epimorphism. Hence by the same commutative diagram above $\Delta(f|_A)$ is an epimorphism. Thus again since M is costar, applying Theorem 2.1(e), $L|_A \in \mathrm{Cogen}(M)$ and hence by Lemma 2.1, $L \in \mathrm{Cogen}(K)$. □

Theorem 3.2. *Let $B = A \propto Q$ be a split extension of A. Let M be a right A-module. Let $K = \mathrm{Hom}_A(B, M)$ be M-reflexive. If $K|M$ in $\mathrm{Mod}\text{-}A$, and $M|K$ in $\mathrm{Mod}\text{-}B$, then M is costar if and only if K is costar.*

P r o o f. The only if part is by Theorem 3.1.

It is clear that

$$\begin{aligned}\mathrm{Hom}_B(A, K) &\cong \mathrm{Hom}_B(A, \mathrm{Hom}_A(B, M)) \\ &\cong \mathrm{Hom}_A(A \otimes_B B, M) \\ &\cong \mathrm{Hom}_A(A, M) \\ &\cong M.\end{aligned}$$

Since $M|K$, K cogenerates M. By Remark 1.3, K is M-reflexive. Since $M|K$ in $\mathrm{Mod}\text{-}B$, by Lemma 3.1, D is finitely generated in $E\text{-}\mathrm{Mod}$. Now we have all the necessary requirements of the if part of the proof to be followed from Theorem 3.1. $\square$

Theorem 3.3. *Let M be a right A-module. Let $K = \mathrm{Hom}_A(B, M)$ be M-reflexive and $\mathrm{Hom}_A(B, M) \cong M \otimes_A B$. If M is r-costar, then K is r-costar.*

P r o o f. Let

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \tag{3.1}$$

be an exact sequence in $\mathrm{Mod}\text{-}B$, such that X and Y are K-reflexive.

Firstly suppose that the exact sequence (3.1) remains exact after applying the functor Δ_K i.e., we have the exact sequence

$$0 \longrightarrow \Delta_K Z \longrightarrow \Delta_K Y \longrightarrow \Delta_K X \longrightarrow 0.$$

By Remark 1, $\Delta_K Y \cong \Delta_M Y$ and $\Delta_K X \cong \Delta_M X$, so the following commutative diagram

$$\begin{array}{ccccc} \Delta_K Y & \longrightarrow & \Delta_K X & \longrightarrow & 0 \\ \downarrow{\cong} & & \downarrow{\cong} & & \\ \Delta_M Y & \longrightarrow & \Delta_M X & & \end{array}$$

induce that

$$\Delta_M Y \longrightarrow \Delta_M X \longrightarrow 0$$

is an epimorphism. Thus we have the exact sequence

$$0 \longrightarrow \Delta_M Z \longrightarrow \Delta_M Y \longrightarrow \Delta_M X \longrightarrow 0.$$

Since X and Y are K-reflexive, by Proposition 2.2, X and Y are M-reflexive. Since M is r-costar module, Z is M-reflexive and hence, again by Proposition 2.2, K-reflexive.

Conversely assume that Z is K-reflexive. Hence by Proposition 2.2, it is M-reflexive. Since M is r-costar module, we get the exact sequence

$$0 \longrightarrow \Delta_M Z \longrightarrow \Delta_M Y \longrightarrow \Delta_M X \longrightarrow 0.$$

The commutative diagram

$$\begin{array}{ccccc} \Delta_M Y & \longrightarrow & \Delta_M X & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \Delta_K Y & \longrightarrow & \Delta_K X & & \end{array}$$

induce that

$$\Delta_K Y \longrightarrow \Delta_K X \longrightarrow 0$$

is an epimorphism. Thus we have the exact sequence

$$0 \longrightarrow \Delta_K Z \longrightarrow \Delta_K Y \longrightarrow \Delta_K X \longrightarrow 0.$$

Finally, since $\mathrm{Hom}_A(B, M) \cong M \otimes_A B$ and $M \in \text{mod-}A$, it is clear that $K \in \text{mod-}B$. $\square$

Theorem 3.4. *Let $B = A \propto Q$ be a split extension of A. Let M be a right A-module. Let $K = \mathrm{Hom}_A(B, M)$ be M-reflexive and $\mathrm{Hom}_A(B, M) \cong M \otimes_A B$. Then M is r-costar if and only if K is r-costar.*

P r o o f. The only if part is by Theorem 3.3.

By Remark 1.3, M is K-reflexive. Since $K \in \text{mod-}B$, we have the exact sequence

$$B^n \longrightarrow M \otimes_A B \longrightarrow 0.$$

Applying the functor $- \otimes_B A$, we get the exact sequence

$$B^n \otimes_B A \longrightarrow M \otimes_A B \otimes_B A \longrightarrow 0,$$

and hence the exact sequence

$$(B \otimes_B A)^n \longrightarrow M \otimes_A A \longrightarrow 0.$$

So we have the exact sequence

$$A^n \longrightarrow M \longrightarrow 0,$$

which means that $M \in \text{mod-}A$. The if part of the proof now follows from Theorem 3.3. $\square$

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