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OPTIMAL AMBIGUOUS DISCRIMINATION BETWEEN STATES GIVEN BY TWO PROJECTIONS

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ABSTRACT. Let P_0 and P_1 be projections in a Hilbert space $\mathcal{H}$. We shall construct a class of optimal measurements for the problem of discrimination between quantum states $\rho_i = \frac{1}{\dim P_i} P_i$, $i = 0, 1$, with prior probabilities π_0 and π_1. The probabilities of failure for such measurements will also be derived.

Introduction

We shall briefly outline the problem of quantum discrimination in its simplest form which we need for our purposes. For a more detailed survey of this topic see [1, 3–5].

Throughout the paper we consider only finite-dimensional Hilbert spaces. Let ρ_0 and ρ_1 be two different quantum states in a Hilbert space $\mathcal{H}$, that is nonnegative operators on $\mathcal{H}$ such that $\operatorname{tr}\rho_0 = \operatorname{tr}\rho_1 = 1$. One of these states is the actual state of a quantum system, however we do not know which one. The probability that ρ_i is the actual state equals π_i, $i = 0, 1$. We assume that $\pi_i \in (0, 1)$. We call π_i, $i = 0, 1$, prior probabilities. The problem of distinguishing between these two states will be called the discrimination problem. In order to solve it we perform a measurement on the quantum system. In our setting, measurement is identified with a pair of nonnegative operators (M_0, M_1) on $\mathcal{H}$ such that $M_0 + M_1 = \mathbf{1}_{\mathcal{H}}$. The result of the measurement is 0 or 1. If the result of the measurement equals i we decide that ρ_i is the state of the considered quantum system, $i = 0, 1$. If ρ_j is the state of the system, then the probability

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that measurement (M_0, M_1) gives result i equals $\operatorname{tr}(\rho_j M_i)$ for $i, j \in \{0,1\}$. In this situation the probability of a wrong decision is

$$P_e = \pi_0 \operatorname{tr}(\rho_0 M_1) + \pi_1 \operatorname{tr}(\rho_1 M_0) = 1 - [\pi_0 \operatorname{tr}(\rho_0 M_0) + \pi_1 \operatorname{tr}(\rho_1 M_1)] \qquad (1)$$

We call P_e the failure probability. Our aim is to find a measurement that minimizes the failure probability. We shall call it an optimal measurement. It has been shown (see [5]) that such a measurement always exists. The following theorem gives necessary and sufficient conditions for (M_0, M_1) to be an optimal measurement for the discrimination problem.

Theorem 1. *Consider the discrimination problem given by the states ρ_0 and ρ_1 with prior probabilities π_0, π_1. Measurement (M_0, M_1) is optimal if and only if it satisfies the following conditions*

$$M_0(\pi_0\rho_0 - \pi_1\rho_1)M_1 = \mathbf{0}, \qquad (2)$$

$$(\pi_0\rho_0 - \pi_1\rho_1)M_0 \geqslant \mathbf{0}, \qquad (3)$$

$$(\pi_1\rho_1 - \pi_0\rho_0)M_1 \geqslant \mathbf{0}. \qquad (4)$$

P r o o f. See [6]. □

It has been proved by Helstrom and Holevo (see [5,6]) that optimal measurement is given by the support of the nonnegative part of $(\pi_0\rho_0 - \pi_1\rho_1)$ and its orthogonal complement.

The aim of this paper is to provide a method of construction of an optimal measurement for the discrimination between two states of the form $\rho_0 = \frac{1}{\dim P_0} P_0$ and $\rho_1 = \frac{1}{\dim P_1} P_1$, where P_0 and P_1 are nontrivial projections in $\mathcal{H}$ with given canonical representation (here and throughout by a *projection* we mean an *orthogonal projection*, i.e., a selfadjoint operator P on $\mathcal{H}$ such that $P = P^2$). In that situation we can give an explict formula for the optimal measurement which is similar to the one for the two-dimensional case. Section 3 contains the main results of the paper. The remaining sections provide some auxiliary facts and examples.

1. The two-dimensional case

Let $\mathcal{H}$ be a two-dimensional Hilbert space, and let ψ_0 and ψ_1 be unit vectors from $\mathcal{H}$. Consider the pure quantum states on $\mathcal{H}$ given by $\rho_i = |\psi_i\rangle\langle\psi_i|$, $i = 0, 1$, with prior probabilities $\pi_0, \pi_1 \in (0,1)$. Assume that $\rho_0 \neq \rho_1$. One can choose an orthonormal basis $\{\varphi_0, \varphi_1\}$ in $\mathcal{H}$ such that

$$\psi_0 = c\varphi_0 + s\varphi_1,$$
$$e^{it}\psi_1 = c\varphi_0 - s\varphi_1,$$

for some positive real numbers c and s such that $c^2 + s^2 = 1$, and for some $t \in \mathbb{R}$. Let us observe that states ρ_0 and ρ_1 can be represented in this basis by the following matrices

$$\rho_0 = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}, \qquad \rho_1 = \begin{bmatrix} c^2 & -cs \\ -cs & s^2 \end{bmatrix}.$$

Theorem 2. *The optimal measurement for the above discrimination problem is given by the following projections*

$$\Pi_0 = |\omega_0\rangle\langle\omega_0|, \qquad \Pi_1 = \mathbf{1} - |\omega_0\rangle\langle\omega_0|,$$

where

$$\begin{aligned} \omega_0 &= \frac{1}{\sqrt{2}} \left[\sqrt{1+a}\,\varphi_0 + \sqrt{1-a}\,\varphi_1 \right], \\ a &= \delta(c^2 - s^2) \left(1 - (c^2 - s^2)^2(1 - \delta^2)\right)^{-\frac{1}{2}}, \\ \delta &= \pi_0 - \pi_1. \end{aligned}$$

Moreover the probability of failure for this measurement equals

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\pi_0\pi_1 |\langle\psi_0|\psi_1\rangle|^2}.$$

P r o o f. See [4,5]. □

Let us observe now that projections Π_i, $i = 0, 1$, have the following matrix representations in the basis $\{\varphi_0, \varphi_1\}$

$$\Pi_0 = \frac{1}{2} \begin{bmatrix} 1+a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & 1-a \end{bmatrix}, \tag{5}$$

$$\Pi_1 = \frac{1}{2} \begin{bmatrix} 1-a & -\sqrt{1-a^2} \\ -\sqrt{1-a^2} & 1+a \end{bmatrix}. \tag{6}$$

By Theorem 1 we have

$$\Pi_0(\pi_0\rho_0 - \pi_1\rho_1)\Pi_1 = \mathbf{0}, \tag{7}$$

$$(\pi_0\rho_0 - \pi_1\rho_1)\Pi_0 \geqslant \mathbf{0}, \tag{8}$$

$$(\pi_1\rho_1 - \pi_0\rho_0)\Pi_1 \geqslant \mathbf{0}. \tag{9}$$

Suppose now that basis vectors φ_0, φ_1 are fixed, whereas c and s are arbitrarily chosen nonnegative real numbers such that $c^2+s^2 = 1$. All entries of the matrices representing operators $\Pi_0(\pi_0\rho_0 - \pi_1\rho_1)\Pi_1$, $(\pi_0\rho_0 - \pi_1\rho_1)\Pi_0$ and $(\pi_1\rho_1 - \pi_0\rho_0)\Pi_1$ in the basis $\{\varphi_0, \varphi_1\}$ can be treated now as functions of c and s. (Actually, since $s = \sqrt{1-c^2}$ we can treat them as functions of c only.) Let $f_{i,j}(c)$, $i, j = 1, 2$, denote the entries of the matrix of $\Pi_0(\pi_0\rho_0 - \pi_1\rho_1)\Pi_1$. By $g_1^{(1)}(c)$ and $g_2^{(1)}(c)$ we denote principal minors of the matrix of $(\pi_0\rho_0 - \pi_1\rho_1)\Pi_0$, similarly, by $g_1^{(2)}(c)$ and $g_2^{(2)}(c)$ we denote principal minors of the matrix of $(\pi_1\rho_1 - \pi_0\rho_0)\Pi_1$. The domain of all defined functions is $[0,1]$. Conditions (2), (3) and (4) can now be expressed as follows

$$f_{i,j}(c) = 0, \qquad \text{for} \quad i,j \in \{1,2\} \quad \text{and} \quad c \in [0,1], \tag{10}$$

$$g_l^{(k)}(c) \geqslant 0, \qquad \text{for} \quad k,l \in \{1,2\} \quad \text{and} \quad c \in [0,1]. \tag{11}$$

Remark 1. Let us observe that all defined functions are compositions of three operations: multiplication, taking square root and taking inverse of the argument. This fact will be utilized later.

2. Canonical representation of two projections

Let P_0 and P_1 be projections in a Hilbert space $\mathcal{H}$. The following theorem describes the relative position of P_0 and P_1. This result turns out to be crucial for our purposes.

Theorem 3. *Let P_0 and P_1 be two projections in a Hilbert space $\mathcal{H}$. Then there exist Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{K}$ and commuting operators S and C defined on $\mathcal{K}$ satisfying $\mathbf{0} \leqslant S \leqslant \mathbf{1}$, $S^2 + C^2 = \mathbf{1}$, $\operatorname{Ker} S = \operatorname{Ker} C = \{0\}$, such that*

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{K} \oplus \mathcal{K} \tag{12}$$

and

$$P_0 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus P_0',$$
$$P_1 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{1}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus P_1',$$

where P_0' and P_1' are projections in $\mathcal{K} \oplus \mathcal{K}$ with the following matrix representations

$$P_0' = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix}, \qquad P_1' = \begin{bmatrix} C^2 & -CS \\ -CS & S^2 \end{bmatrix}.$$

P r o o f. From the considerations of [7: Chapter V.1] we conclude that there exist Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{L}$ and commuting operators X and Y defined on $\mathcal{L}$ satisfying $\mathbf{0} \leqslant X \leqslant \mathbf{1}$, $\mathbf{0} \leqslant Y \leqslant \mathbf{1}$, $X^2 + Y^2 = \mathbf{1}$, $\operatorname{Ker} X = \operatorname{Ker} Y = \{0\}$, such that

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{L} \oplus \mathcal{L}$$

and

$$P_0 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus P_0',$$
$$P_1 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{1}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus P_1',$$

where P_0' and P_1' are projections in $\mathcal{L} \oplus \mathcal{L}$ with the following matrix representations

$$P_0' = \begin{bmatrix} X^2 & XY \\ XY & Y^2 \end{bmatrix}, \qquad P_1' = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Let $\dim(\mathcal{L} \oplus \mathcal{L}) = 2n$ and $\{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n\}$ be an orthonormal basis of $\mathcal{L} \oplus \mathcal{L}$. We can assume that $\{\varphi_1, \ldots, \varphi_n\}$ is an orthonormal basis of $\mathcal{L}$ and that the subspace spanned by vectors $\psi_1, \ldots, \psi_n$ is an isomorphic copy of $\mathcal{L}$. Let $\widehat{X}$ and $\widehat{Y}$ be the matrix representations of X and Y in the orthonormal bases $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\psi_1, \ldots, \psi_n\}$, respectively. Then operators P_0' and P_1' are represented in the basis $\{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n\}$ by the following matrices

$$\widehat{P_0'} = \begin{bmatrix} \widehat{X}^2 & \widehat{X}\widehat{Y} \\ \widehat{X}\widehat{Y} & \widehat{Y}^2 \end{bmatrix}, \qquad \widehat{P_1'} = \begin{bmatrix} \widehat{1} & 0 \\ 0 & 0 \end{bmatrix},$$

$\widehat{1}$ being obviously the identity matrix. Let us consider the following unitary operator on $\mathcal{L} \oplus \mathcal{L}$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+X} & -\sqrt{1-X} \\ \sqrt{1-X} & \sqrt{1+X} \end{bmatrix}.$$

Let $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n, \widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$ be an orthonormal basis of $\mathcal{L} \oplus \mathcal{L}$ given by

$$\widetilde{\varphi}_i = U\varphi_i, \quad \widetilde{\psi}_i = U\psi_i, \qquad \text{for} \quad i = 1, \ldots, n.$$

Projections P_0' and P_1' have the following matrix representations in this new basis

$$\begin{aligned} \widehat{U^*}\widehat{P_0'}\widehat{U} = \frac{1}{2} &\begin{bmatrix} \sqrt{\widehat{1}+\widehat{X}} & \sqrt{\widehat{1}-\widehat{X}} \\ -\sqrt{\widehat{1}-\widehat{X}} & \sqrt{\widehat{1}+\widehat{X}} \end{bmatrix} \begin{bmatrix} \widehat{X}^2 & \widehat{X}\widehat{Y} \\ \widehat{X}\widehat{Y} & \widehat{Y}^2 \end{bmatrix} \\ &\times \begin{bmatrix} \sqrt{\widehat{1}+\widehat{X}} & -\sqrt{\widehat{1}-\widehat{X}} \\ \sqrt{\widehat{1}-\widehat{X}} & \sqrt{\widehat{1}+\widehat{X}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \widehat{1}+\widehat{X} & \widehat{Y} \\ \widehat{Y} & \widehat{1}-\widehat{X} \end{bmatrix}, \end{aligned} \tag{13}$$

$$\begin{aligned} \widehat{U^*}\widehat{P_1'}\widehat{U} = \frac{1}{2} &\begin{bmatrix} \sqrt{\widehat{1}+\widehat{X}} & \sqrt{\widehat{1}-\widehat{X}} \\ -\sqrt{\widehat{1}-\widehat{X}} & \sqrt{\widehat{1}+\widehat{X}} \end{bmatrix} \begin{bmatrix} \widehat{1} & 0 \\ 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} \sqrt{\widehat{1}+\widehat{X}} & -\sqrt{\widehat{1}-\widehat{X}} \\ \sqrt{\widehat{1}-\widehat{X}} & \sqrt{\widehat{1}+\widehat{X}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \widehat{1}+\widehat{X} & -\widehat{Y} \\ -\widehat{Y} & \widehat{1}-\widehat{X} \end{bmatrix}. \end{aligned} \tag{14}$$

Let us take now operators C and S on the subspaces $\mathrm{Lin}[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]$ and $\mathrm{Lin}[\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n]$ with the following matrix representations in the bases $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n\}$ and $\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n\}$

$$\widehat{C} = \sqrt{\frac{\widehat{1}+\widehat{X}}{2}}, \qquad \widehat{S} = \sqrt{\frac{\widehat{1}-\widehat{X}}{2}}.$$

Assumptions on X and Y imply that $\mathbf{0} \leqslant C \leqslant \mathbf{1}$, $\mathbf{0} \leqslant S \leqslant \mathbf{1}$, $S^2 + C^2 = \mathbf{1}$ and $\operatorname{Ker} C = \operatorname{Ker} S = \{0\}$. Since we can identify $\mathcal{L} \oplus \mathcal{L}$ with $\operatorname{Lin}[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n] \oplus \operatorname{Lin}[\widetilde{\psi}_1, \ldots, \widetilde{\psi}_n]$, from (13), (14) we conclude that

$$P_0' = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix}, \qquad P_1' = \begin{bmatrix} C^2 & -CS \\ -CS & S^2 \end{bmatrix}.$$

To finish the proof we only have to put $\mathcal{K} = \operatorname{Lin}[\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n]$. □

Remark 2. Projections P_0' and P_1' have the same dimension.

Indeed, take $V = \left[\begin{smallmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{smallmatrix}\right]$. Then $P_0' = V P_1' V$, and since V is unitary this implies that $\dim P_0' = \dim P_1'$.

Remark 3. Let $\mathcal{R}_0$ and $\mathcal{R}_1$ be the ranges of projections P_0 and P_1, respectively. Then we have

$$\mathcal{H}_1 = \mathcal{R}_0 \cap \mathcal{R}_1, \quad \mathcal{H}_2 = \mathcal{R}_0 \cap \mathcal{R}_1^\perp, \quad \mathcal{H}_3 = \mathcal{R}_0^\perp \cap \mathcal{R}_1, \quad \mathcal{H}_4 = \mathcal{R}_0^\perp \cap \mathcal{R}_1^\perp,$$
$$\mathcal{K} \oplus \mathcal{K} = (\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4)^\perp$$

(cf. [7: Chapter V.1]).

3. Main results

Let P_0 and P_1 be nontrivial projections in $\mathcal{H}$ and $\pi_i \in (0,1)$, $i = 1, 2$. Let us consider the representations of P_0 and P_1 given by Theorem 3. Put $m_i = \dim P_i$, $i = 0, 1$, and $k = \dim P_0' = \dim P_1'$. Set $\widetilde{\pi}_i = \theta \frac{k}{m_i} \pi_i$, $i = 0, 1$, where $\theta = \frac{1}{\frac{k}{m_0}\pi_0 + \frac{k}{m_1}\pi_1}$. The following theorem shows that the problem of discrimination between states $\rho_0 = \frac{1}{\dim P_0} P_0$ and $\rho_1 = \frac{1}{\dim P_1} P_1$ can be reduced to the discrimination problem in $\mathcal{K} \oplus \mathcal{K}$.

Theorem 4. *Suppose that $\mathcal{K} \oplus \mathcal{K}$ is nontrivial. Let $(\widetilde{M}_0, \widetilde{M}_1)$ be an optimal measurement in $\mathcal{K} \oplus \mathcal{K}$ for the discrimination problem given by $(\widetilde{\rho}_0, \widetilde{\rho}_1)$ and $(\widetilde{\pi}_0, \widetilde{\pi}_1)$, where $\widetilde{\rho}_0 = \frac{1}{\dim P_0'} P_0'$, $\widetilde{\rho}_1 = \frac{1}{\dim P_1'} P_1'$. Let $\alpha \in [0,1]$ be arbitrary. Measurement (M_0, M_1) in $\mathcal{H}$ given by*

$$\begin{aligned} M_0 &= \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus (\alpha \mathbf{1}_{\mathcal{H}_4}) \oplus \widetilde{M}_0, \\ M_1 &= \mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{1}_{\mathcal{H}_3} \oplus ((1-\alpha)\mathbf{1}_{\mathcal{H}_4}) \oplus \widetilde{M}_1, \qquad \text{when} \quad \frac{\pi_0}{m_0} \geqslant \frac{\pi_1}{m_1}, \end{aligned} \tag{15}$$

or

$$\begin{aligned} M_0 &= \mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus (\alpha \mathbf{1}_{\mathcal{H}_4}) \oplus \widetilde{M}_0, \\ M_1 &= \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{1}_{\mathcal{H}_3} \oplus ((1-\alpha)\mathbf{1}_{\mathcal{H}_4}) \oplus \widetilde{M}_1, \qquad \text{when} \quad \frac{\pi_0}{m_0} < \frac{\pi_1}{m_1} \end{aligned} \tag{16}$$

is an optimal measurement for the discrimination problem given by (ρ_0, ρ_1) and (π_0, π_1), where $\rho_0 = \frac{1}{\dim P_0} P_0$ and $\rho_1 = \frac{1}{\dim P_1} P_1$. In case when $\mathcal{K} \oplus \mathcal{K}$ is trivial we omit the last summand in formulas (15), (16).

Proof. Suppose that $\mathcal{K} \oplus \mathcal{K}$ is nontrivial and $\frac{\pi_0}{m_0} \geqslant \frac{\pi_1}{m_1}$. It is straightforward to check that (M_0, M_1) is a measurement in $\mathcal{H}$. By virtue of Theorem 1 we have

$$\widetilde{M_0}(\widetilde{\pi}_0\widetilde{\rho}_0 - \widetilde{\pi}_1\widetilde{\rho}_1)\widetilde{M_1} = \mathbf{0}, \tag{17}$$

$$(\widetilde{\pi}_0\widetilde{\rho}_0 - \widetilde{\pi}_1\widetilde{\rho}_1)\widetilde{M_0} \geqslant \mathbf{0}, \tag{18}$$

$$(\widetilde{\pi}_1\widetilde{\rho}_1 - \widetilde{\pi}_0\widetilde{\rho}_0)\widetilde{M_1} \geqslant \mathbf{0}. \tag{19}$$

We have to check that the same is true for the measurement (M_0, M_1) and states (ρ_0, ρ_1) with prior probabilities (π_0, π_1). By (17)–(19) we have

$$\begin{aligned}
&M_0(\pi_0\rho_0 - \pi_1\rho_1)M_1 \\
&= \frac{1}{\theta}\left[\mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus \widetilde{M_0}(\widetilde{\pi_0}\widetilde{\rho}_0 - \widetilde{\pi_1}\widetilde{\rho}_1)\widetilde{M_1}\right] = \mathbf{0}, \\
&(\pi_0\rho_0 - \pi_1\rho_1)M_0 \\
&= \left[\left(\frac{\pi_0}{m_0} - \frac{\pi_1}{m_1}\right)\mathbf{1}_{\mathcal{H}_1} \oplus \frac{\pi_0}{m_0}\mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus \frac{1}{\theta}(\widetilde{\pi_0}\widetilde{\rho}_0 - \widetilde{\pi_1}\widetilde{\rho}_1)\widetilde{M_1}\right] \geqslant \mathbf{0}, \\
&(\pi_1\rho_1 - \pi_0\rho_0)M_1 \\
&= \left[\mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \frac{\pi_1}{m_1}\mathbf{1}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus \frac{1}{\theta}\left(\widetilde{\pi_0}\widetilde{\rho}_0 - \widetilde{\pi_1}\widetilde{\rho}_1\right)\widetilde{M_0}\right] \geqslant \mathbf{0}.
\end{aligned}$$

Using again Theorem 1 we conclude that measurement (M_0, M_1) is optimal for the discrimination problem given by (ρ_0, ρ_1) and (π_0, π_1). The same proof works for the case when $\frac{\pi_0}{m_0} < \frac{\pi_1}{m_1}$ and when $\mathcal{K} \oplus \mathcal{K}$ is trivial. □

Remark 4. Later we shall show that we can always find an optimal measurement $(\widetilde{M_0}, \widetilde{M_1})$ which is simple, i.e., such that $\widetilde{M_0}$ and $\widetilde{M_1}$ are projections (see Theorem 6 and Remark 5 below). By taking $\alpha = 0$ or $\alpha = 1$ in (15), (16) we then obtain an optimal simple measurement.

Under the assumptions of the above theorem we have

Theorem 5. *The probability of failure for the optimal measurement given by* Theorem 4 *equals*

$$\begin{aligned}
&P_e = 1 - \frac{1}{\theta} + \frac{1}{\theta}\widetilde{P}_e - \frac{\pi_0}{m_0}(\dim\mathcal{H}_1 + \dim\mathcal{H}_2) - \frac{\pi_1}{m_1}\dim\mathcal{H}_3, \\
&\qquad \textit{when} \quad \frac{\pi_0}{m_0} \geqslant \frac{\pi_1}{m_1},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&P_e = 1 - \frac{1}{\theta} + \frac{1}{\theta}\widetilde{P}_e - \frac{\pi_0}{m_0}\dim\mathcal{H}_2 - \frac{\pi_1}{m_1}(\dim\mathcal{H}_1 + \dim\mathcal{H}_3), \\
&\qquad \textit{when} \quad \frac{\pi_0}{m_0} < \frac{\pi_1}{m_1},
\end{aligned} \tag{21}$$

where $\widetilde{P}_e$ denotes the failure probability for the discrimination problem given by $(\widetilde{\rho}_0, \widetilde{\rho}_1)$ and $(\widetilde{\pi}_0, \widetilde{\pi}_1)$. In case when $\mathcal{K} \oplus \mathcal{K}$ is trivial we take $\widetilde{P}_e = 1$.

P r o o f. Suppose that $\frac{\pi_0}{m_0} \geqslant \frac{\pi_1}{m_1}$ and $\mathcal{K} \oplus \mathcal{K}$ is nontrivial. Using formula (15) we obtain

$$\begin{aligned}\pi_0 \operatorname{tr}(\rho_0 M_0) &= \frac{\pi_0}{m_0} \operatorname{tr}(\mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_2} \oplus \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0} \oplus \widetilde{M}_0 P_0') \\ &= \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2) + \frac{\pi_0}{m_0} \operatorname{tr}(\widetilde{M}_0 P_0') \\ &= \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2) + \frac{1}{\theta}\widetilde{\pi}_0 \operatorname{tr}(\widetilde{M}_0 \widetilde{\rho}_0),\end{aligned}$$

$$\begin{aligned}\pi_1 \operatorname{tr}(\rho_1 M_1) &= \frac{\pi_1}{m_1} \operatorname{tr}(\mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_2} \oplus \mathbf{1}_{\mathcal{H}_3} \oplus \mathbf{0} \oplus \widetilde{M}_1 P_1') \\ &= \frac{\pi_1}{m_1} \dim \mathcal{H}_3 + \frac{\pi_1}{m_1} \operatorname{tr}(\widetilde{M}_1 P_1') \\ &= \frac{\pi_1}{m_1} \dim \mathcal{H}_3 + \frac{1}{\theta}\widetilde{\pi}_1 \operatorname{tr}(\widetilde{M}_1 \widetilde{\rho}_1).\end{aligned}$$

From this and (1) we have

$$\begin{aligned}1 - P_e &= \pi_0 \operatorname{tr}(\rho_0 M_0) + \pi_1 \operatorname{tr}(\rho_1 M_1) \\ &= \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2) + \frac{\pi_1}{m_1} \dim \mathcal{H}_3 + \frac{1}{\theta}\widetilde{\pi}_0 \operatorname{tr}(\widetilde{M}_0 \widetilde{\rho}_0) + \frac{1}{\theta}\widetilde{\pi}_1 \operatorname{tr}(\widetilde{M}_1 \widetilde{\rho}_1) \\ &= \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2) + \frac{\pi_1}{m_1} \dim \mathcal{H}_3 + \frac{1}{\theta}(1 - \widetilde{P}_e),\end{aligned}$$

which gives us (20).

In case when $\mathcal{K} \oplus \mathcal{K}$ is trivial we have

$$\begin{aligned}\pi_0 \operatorname{tr}(\rho_0 M_0) &= \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2), \\ \pi_1 \operatorname{tr}(\rho_1 M_1) &= \frac{\pi_1}{m_1} \dim \mathcal{H}_3,\end{aligned}$$

which yields

$$P_e = 1 - \frac{\pi_0}{m_0}(\dim \mathcal{H}_1 + \dim \mathcal{H}_2) - \frac{\pi_1}{m_1} \dim \mathcal{H}_3. \tag{22}$$

It is easily seen that putting $\widetilde{P}_e = 1$ in (20) we obtain (22). The proof for the case when $\frac{\pi_0}{m_0} < \frac{\pi_1}{m_1}$ is similar. □

Let $\mathcal{K}$ be a Hilbert space and let P_0', P_1' be projections in $\mathcal{K} \oplus \mathcal{K}$ of the following form

$$P_0' = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix}, \qquad P_1' = \begin{bmatrix} C^2 & -CS \\ -CS & S^2 \end{bmatrix},$$

where S and C are nonnegative commuting operators on $\mathcal{K}$ satisfying $S^2+C^2 = \mathbf{1}$ and $\operatorname{Ker} S = \operatorname{Ker} C = \{0\}$. Let us consider the quantum discrimination problem given by the states $\widetilde{\rho}_i = \frac{1}{\dim P_i'} P_i'$, $i = 0, 1$, with prior probabilities $\widetilde{\pi}_i \in (0,1)$, $i = 0, 1$.

Theorem 6. *Measurement $(\widetilde{M}_0, \widetilde{M}_1)$ in $\mathcal{K} \oplus \mathcal{K}$ given by*

$$\widetilde{M}_0 = \frac{1}{2} \begin{bmatrix} \mathbf{1}+A & \sqrt{\mathbf{1}-A^2} \\ \sqrt{\mathbf{1}-A^2} & \mathbf{1}-A \end{bmatrix}, \tag{23}$$

$$\widetilde{M}_1 = \frac{1}{2} \begin{bmatrix} \mathbf{1}-A & -\sqrt{\mathbf{1}-A^2} \\ -\sqrt{\mathbf{1}-A^2} & \mathbf{1}+A \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned} A &= \delta(C^2 - S^2)\big(\mathbf{1} - (1-\delta^2)(C^2-S^2)^2\big)^{-\frac{1}{2}}, \\ \delta &= \widetilde{\pi}_0 - \widetilde{\pi}_1 \end{aligned} \tag{25}$$

is optimal for the discrimination problem given by $(\widetilde{\rho}_0, \widetilde{\rho}_1)$ and $(\widetilde{\pi}_0, \widetilde{\pi}_1)$.

In the proof we shall use the following lemma.

Lemma 7. *Let $A = \left[\begin{smallmatrix} B & C \\ D & E \end{smallmatrix}\right]$ be an operator on $\mathcal{K} \oplus \mathcal{K}$. Assume that B, C and D commute. Then $A \geqslant \mathbf{0}$ if and only if $B \geqslant \mathbf{0}$, $E \geqslant \mathbf{0}$, $D = C^*$ and $BE - CC^* \geqslant \mathbf{0}$.*

P r o o f. See [2]. □

P r o o f o f T h e o r e m 6. First notice that operator A is well defined. Indeed, since $|1-\delta^2| \leqslant 1$ the assumptions on S and C imply that all eigenvalues of operator $\mathbf{1} - (1-\delta^2)(C^2-S^2)^2$ are positive. Thus $\big(\mathbf{1} - (1-\delta^2)(C^2-S^2)^2\big)^{-1}$ exists. Let us observe that formulas (23)–(25) can be obtained from formulas (5), (6) by substituting operators S and C in place of real numbers s and c. Moreover, as before, we have $S = \sqrt{\mathbf{1}-C^2}$. These facts together with Remark 1 imply that

$$\widetilde{M}_0(\pi_0\widetilde{\rho}_0 - \pi_1\widetilde{\rho}_1)\widetilde{M}_1 = [f_{i,j}(C)]_{i,j=1,2},$$

where $f_{i,j}$, $i,j = 1,2$, are the functions defined in Section 1. Now from (7) we have

$$\widetilde{M}_0(\pi_0\widetilde{\rho}_0 - \pi_1\widetilde{\rho}_1)\widetilde{M}_1 = \mathbf{0}.$$

Denote by $g^{(k)}(c) = [g^{(k)}_{i,j}]_{i,j=1,2}$, $k = 1,2$, the matrices in equations (8) and (9), respectively. Since the matrices are positive for all $c \in (0,1)$, the principal minors $g^{(k)}_{1,1}(c)g^{(k)}_{2,2}(c) - |g^{(k)}_{1,2}(c)|^2$ and $g^{(k)}_{2,2}(c)$, $k = 1,2$, must be nonnegative. Since the matrices given by $(\pi_0\widetilde{\rho}_0 - \pi_1\widetilde{\rho}_1)\widetilde{M}_0$ and $(\pi_1\widetilde{\rho}_1 - \pi_0\widetilde{\rho}_0)\widetilde{M}_1$ have the form $g^{(k)}(C)$, $k = 1,2$, the nonnegativity of minors together with Lemma 7 and Theorem 1 imply the result. □

Remark 5. It can be easily checked that the measurement $(\widetilde{M_0}, \widetilde{M_1})$ in the above theorem is simple.

Summarizing, Theorems 4 and 6 enable us to construct an optimal measurement for the quantum discrimination problem determined by arbitrary two finite dimensional projections.

4. Examples

Let P be an m-dimensional projection on a Hilbert space $\mathcal{H}$. Take $\varphi \in \mathcal{H}$. Put $P_0 = |\varphi\rangle\langle\varphi|$, $P_1 = P$ and consider the discrimination problem given by $\rho_0 = P_0$, $\rho_1 = \frac{1}{m}P_1$ and $\pi_0, \pi_1 \in (0,1)$.

Example 1. Suppose that $\varphi \in P(\mathcal{H})$. Remark 3 yields

$$\mathcal{H}_1 = \mathrm{Lin}[\varphi],\ \ \mathcal{H}_2 = \{0\},\ \ \mathcal{H}_3 = P(\mathcal{H}) \cap \mathrm{Lin}[\varphi]^\perp,\ \ \mathcal{H}_4 = P(\mathcal{H})^\perp,\ \ \mathcal{K} \oplus \mathcal{K} = \{0\}.$$

From this we have

$$\dim \mathcal{H}_1 = 1, \qquad \dim \mathcal{H}_3 = m - 1.$$

Theorem 4 now gives the following formulas for the optimal measurement

$$\begin{aligned}
&M_0 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_3},\\
&M_1 = \mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_3}, \qquad \text{when} \quad \pi_0 \geqslant \frac{\pi_1}{m},\\
&M_0 = \mathbf{0}_{\mathcal{H}_1} \oplus \mathbf{0}_{\mathcal{H}_3} = \mathbf{0},\\
&M_1 = \mathbf{1}_{\mathcal{H}_1} \oplus \mathbf{1}_{\mathcal{H}_3} = \mathbf{1}, \qquad \text{when} \quad \pi_0 < \frac{\pi_1}{m}.
\end{aligned}$$

The failure probabilities obtained by the use of Theorem 5 are given by

$$\begin{aligned}
&P_e = 1 - \pi_0 - \frac{\pi_1}{m}(m-1) = \frac{\pi_1}{m}, \qquad \text{when} \quad \pi_0 \geqslant \frac{\pi_1}{m},\\
&P_e = 1 - \frac{\pi_1}{m} m = \pi_0, \qquad \text{when} \quad \pi_0 < \frac{\pi_1}{m}.
\end{aligned}$$

Let us mention that this case can also be easily handled by the use of Theorem 1 alone.

Example 2. Suppose now that $\varphi \notin P(\mathcal{H})$ and $P\varphi \neq 0$. From Remark 3 we easily conclude that

$$\mathcal{H}_1 = \mathcal{H}_2 = \{0\}, \quad \mathcal{H}_3 = \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H}), \quad \mathcal{H}_4 = \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H})^\perp.$$

It can easily be checked that

$$\mathcal{H}_3 = \big\{\xi \in P(\mathcal{H}) :\ \langle \xi | P\varphi \rangle = 0\big\} = \mathrm{Lin}[P\varphi]^\perp \cap P(\mathcal{H}). \tag{26}$$

Since $P\varphi \neq 0$ the above equality implies $\dim \mathcal{H}_3 = \dim P(\mathcal{H}) - 1 = m - 1$. Put $\mathcal{H}_5 = \mathrm{Lin}\,[\varphi, P\varphi]$. Obviously, $\mathcal{H}_5 \perp \mathcal{H}_4$. (26) implies that $\mathcal{H}_5 \perp \mathcal{H}_3$ as well. Thus $\mathcal{H}_5 \perp \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$. We shall show now that $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ is the orthogonal complement of $\mathcal{H}_5$ in $\mathcal{H}$. Take $\xi \in \mathcal{H}$ such that $\xi \perp \mathcal{H}_5$. Then $\xi \perp \varphi$ and $\xi \perp P\varphi$; moreover, $\xi = P\xi + (\mathbf{1} - P)\xi$. Put $\xi_1 = P\xi \in P(\mathcal{H})$ and $\xi_2 = (\mathbf{1} - P)\xi$. We have now

$$\langle \xi_1 | \varphi \rangle = \langle \xi | P\varphi \rangle = 0.$$

Thus $\xi_1 \in \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H})$. Let us observe now that $\xi_2 = \xi - \xi_1$ and $\xi, \xi_1 \in \mathrm{Lin}[\varphi]^\perp$. Thus $\xi_2 \in \mathrm{Lin}[\varphi]^\perp$. Since $\xi_2 = (\mathbf{1} - P)\xi \in P(\mathcal{H})^\perp$ we conclude that $\xi_2 \in \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H})^\perp$. We have shown that for any $\xi \perp \mathcal{H}_5$ there exist $\xi_1 \in \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H})$ and $\xi_2 \in \mathrm{Lin}[\varphi]^\perp \cap P(\mathcal{H})^\perp$ such that $\xi = \xi_1 + \xi_2$. This means that $\xi \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$. Therefore $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ is the orthogonal complement of $\mathcal{H}_5$ which proves that

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5 = \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5.$$

We have $\dim \mathcal{H}_5 = 2$. Taking any two orthogonal vectors $\beta_1, \beta_2 \in \mathcal{H}_5$ we put $\mathcal{K} = \mathrm{Lin}[\beta_1]$. Identifying $\mathcal{K}$ with $\mathrm{Lin}[\beta_2]$ we get $\mathcal{H}_5 = \mathcal{K} \oplus \mathcal{K}$. Put $\psi_0 = \varphi$, $\psi_1 = \frac{P\varphi}{\|P\varphi\|}$, and notice that

$$P = P(\mathbf{1} - |\psi_1\rangle\langle\psi_1|) + P(|\psi_1\rangle\langle\psi_1|) = P(\mathbf{1} - |\psi_1\rangle\langle\psi_1|) + |\psi_1\rangle\langle\psi_1|. \tag{27}$$

Projection $P(\mathbf{1} - |\psi_1\rangle\langle\psi_1|)$ is a composition of two commuting projections with ranges $P(\mathcal{H})$ and $\mathrm{Lin}[P\varphi]^\perp$, respectively. This implies that $P(\mathbf{1} - |\psi_1\rangle\langle\psi_1|)(\mathcal{H}) = \mathrm{Lin}[P\varphi]^\perp \cap P(\mathcal{H}) = \mathcal{H}_3$. Thus from (27) it follows that, according to the identification of $\mathcal{H}$ with $\mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \mathcal{H}_5$, we can write

$$P_1 = P = \mathbf{1}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus |\psi_1\rangle\langle\psi_1|,$$

and obviously

$$P_0 = |\varphi\rangle\langle\varphi| = \mathbf{0}_{\mathcal{H}_3} \oplus \mathbf{0}_{\mathcal{H}_4} \oplus |\psi_0\rangle\langle\psi_0|.$$

By virtue of Theorem 4 the problem of discrimination between states $\rho_0 = P_0$ and $\rho_1 = \frac{1}{m}P_1$ with prior probabilities $\pi_0, \pi_1 \in (0,1)$ reduces now to discrimination between pure states $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$ with prior probabilities $\widetilde{\pi}_0 = \frac{\pi_0 m}{\pi_1 + \pi_0 m}$ and $\widetilde{\pi}_1 = \frac{\pi_1}{\pi_1 + \pi_0 m}$. The failure probability for the optimal measurement equals in this case

$$\begin{aligned}\widetilde{P}_e &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\widetilde{\pi}_0\widetilde{\pi}_1|\langle\psi_0|\psi_1\rangle|^2} \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\frac{\pi_0\pi_1 m}{(\pi_1 + \pi_0 m)^2}\|P\varphi\|^2}.\end{aligned}$$

From Theorem 5 we conclude that

$$P_e = 1 - \frac{1}{\theta} + \frac{1}{\theta}\widetilde{P}_e - \frac{\pi_1}{m}\dim \mathcal{H}_3, \qquad \text{where} \quad \theta = \frac{m}{m\pi_0 + \pi_1}.$$

Easy computations give now

$$P_e = \frac{1}{2}\left(\pi_0 + \frac{\pi_1}{m}\right)\left[1 - \sqrt{1 - \frac{4\pi_0\pi_1 m}{(\pi_1+\pi_0 m)^2}\|P\varphi\|^2}\,\right].$$

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